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# Crank -Nicolson scheme for solving a system of singularly perturbed partial differential equations of parabolic type

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**Abstract**

A singularly perturbed boundary value problem (SPBVP) for a system of two linear parabolic second order differential equations of convection-diffusion type is considered. Since the second order space derivative of each equation is multiplied by distinct singular perturbation parameters, the components of the solution exhibit overlapping layers. In this work, a method which comprises the Crank- Nicolson scheme to discretise time variable on a uniform mesh and standard central difference scheme on a Shishkin piecewise uniform mesh to discretise space variable is suggested to obtain numerical approximations to the solution of the continuous problem. The numerical solution obtained using the suggested method is second order convergent in time and first order convergent in space.

*Keywords:* Singular perturbation problems, parabolic problems, boundary layers, Crank - Nicolson scheme, Shishkin mesh.

*2010 MSC:* 65M06, 65M12, 65M15. 

## 1 Introduction

A singularly perturbed boundary value problem for a system of two linear parabolic second order differential equations of convection-diffusion type is considered as follows

$$\vec{L}\vec{u}(x,t) = \frac{\partial \vec{u}}{\partial t}(x,t) - E\frac{\partial^2 \vec{u}}{\partial x^2}(x,t) + A(x,t)\frac{\partial \vec{u}}{\partial x}(x,t) + B(x,t)\vec{u}(x,t) = \vec{f}(x,t) \text{ on } \Omega, \vec{u} \text{ given on } \Gamma, \tag{1.1}$$

where $\vec{u}(x,t) = (u_1(x,t), u_2(x,t))^T, \vec{f}(x,t) = (f_1(x,t), f_2(x,t))^T, A(x,t) = diag(a_1(x,t), a_2(x,t))$,

$$E = diag(\varepsilon_1, \varepsilon_2) \text{ and } B(x,t) = \begin{pmatrix} b_{11}(x,t) & b_{12}(x,t) \\ b_{21}(x,t) & b_{22}(x,t) \end{pmatrix},$$

$\Omega = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$, $\overline{\Omega} = \Omega \cup \Gamma$, $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\vec{u}(0,t) = \vec{\phi}_L(t)$ on $\Gamma_L = \{(0,t) : 0 \leq t \leq T\}$, $\vec{u}(x,0) = \vec{\phi}_B(x)$ on $\Gamma_B = \{(x,0) : 0 \leq x \leq 1\}$, $\vec{u}(1,t) = \vec{\phi}_R(t)$ on $\Gamma_R = \{(1,t) : 0 \leq t \leq T\}$. The functions $\vec{\phi}_L$, $\vec{\phi}_B$, and $\vec{\phi}_R$ are assumed to be sufficiently smooth. Standard theoretical results on the solutions of (1.1) are stated, without proof, in this paper. See [1], [2], [7], [8] and [9] for more details. Without loss of generality we shall assume that

$$0 < \varepsilon_1 < \varepsilon_2 \leq 1. \tag{1.2}$$

For all $(x,t) \in \overline{\Omega}$ , it is assumed that the components $a_i(x,t)$ of $A(x,t)$ and $b_{ij}(x,t)$ of $B(x,t)$ satisfy the inequalities

$$b_{ij}(x,t) \leq 0 \text{ for } i \neq j, b_{ii}(x,t) > 0, a_i(x,t) > 0, B(x,t) - \frac{1}{2}\frac{\partial A}{\partial x}(x,t) > 0 \tag{1.3}$$

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and

$$\min_{(x,t)\in[0,1]\times[0,T]}\{b_{11}(x,t)+b_{12}(x,t)+a_1(x,t),b_{21}(x,t)+b_{22}(x,t)+a_2(x,t)\}>\alpha>0, \tag{1.4}$$

The problem (1.1) can also be written in the form

$$\vec{L}\vec{u}=\vec{f} \ \text{ on } \Omega, \vec{u} \text{ given on } \Gamma,$$

where the operator $\vec{L}$ is defined by

$$\vec{L}=\frac{\partial}{\partial t}-E\frac{\partial^2}{\partial x^2}+A\frac{\partial}{\partial x}+BI,$$

where $I$ is the identity matrix. The reduced problem corresponding to (1.1) is defined by

$$\frac{\partial\vec{u}_0}{\partial t}(x,t)+A(x,t)\frac{\partial\vec{u}_0}{\partial x}(x,t)+B(x,t)\vec{u}_0(x,t)=\vec{f}(x,t), \quad \vec{u}_0(x,0)=\phi_B(x),\ 0<x<1 \tag{1.5}$$

In general as $\vec{u}_0(x,t)$ need not satisfy $\vec{u}_0(1,t)=\vec{u}(1,t)$, the solution $\vec{u}(x,t)$ exhibit boundary layers at $x=1$.

For any function $\vec{y}$ on a domain $D$ the following norms are introduced: $\parallel \vec{y}(x,t) \parallel_D = \max_i |y_i(x,t)|$ and $\parallel \vec{y} \parallel = \sup\{\parallel \vec{y}(x,t) \parallel : (x,t)\in[0,1]\times[0,T]\}$. If $D=\overline{\Omega}$, the subscript is dropped.
In a compact domain $D$ a function is said to be Hölder continuous of degree $\lambda, 0<\lambda\le 1$, if, for all $(x_1,t_1),(x_2,t_2)\in D$,

$$|\vec{u}(x_1,t_1)-\vec{u}(x_2,t_2)|\le C(|x_1-x_2|^2+|t_1-t_2|)^{\lambda/2}.$$

The set of Hölder continuous functions forms a normed linear space $C^0_\lambda(D)$ with the norm

$$||\vec{u}||_{\lambda,D}=||\vec{u}||_D+\sup_{(x_1,t_1),(x_2,t_2)\in D}\frac{|\vec{u}(x_1,t_1)-\vec{u}(x_2,t_2)|}{(|x_1-x_2|^2+|t_1-t_2|)^{\lambda/2}}.$$

For each integer $k\ge 1$, the subspaces $C^k_\lambda(D)$ of $C^0_\lambda(D)$, which contain functions having Hölder continuous derivatives, are defined as follows

$$C^k_\lambda(D)=\{\vec{u}:\frac{\partial^{l+m}\vec{u}}{\partial x^l\partial t^m}\in C^0_\lambda(D) \text{ for } l,m\ge 0 \text{ and } 0\le l+2m\le k\}.$$

The norm on $C^0_\lambda(D)$ is taken to be $||\vec{u}||_{\lambda,k,D}=\max_{0\le l+2m\le k}||\frac{\partial^{l+m}\vec{u}}{\partial x^l\partial t^m}||_{\lambda,D}$.
It is to be noted that the domain of the operators $\vec{L}$ is $\mathcal{M}_\lambda(\Omega)=\{\psi:\frac{\partial\psi}{\partial t},\frac{\partial^2\psi}{\partial x^2} \text{ exist on } \Omega\}$ and that of $I+\frac{\Delta}{2}\vec{L}_x$ is $\mathcal{M}^*_\lambda(\Omega)=\{\psi:\frac{\partial^2\psi}{\partial x^2} \text{ exist on } \Omega\}$, where $\vec{L}_x=-E\frac{\partial^2}{\partial x^2}+A\frac{\partial}{\partial x}+BI$.
Sufficient conditions for the existence, uniqueness and regularity of solution of (1.1) are given in the following theorem.

**Theorem 1.1.** *Assume that* $A,B\ \vec{f}\in C^2_\lambda(\overline{\Omega})$, $\vec{\phi}_L\in C^1(\Gamma_L)$, $\vec{\phi}_B\in C^2(\Gamma_B)$, $\vec{\phi}_R\in C^1(\Gamma_R)$ *and that the following compatibility conditions are fulfilled at the corners* $(0,0)$ *and* $(1,0)$ *of* $\Gamma$

$$\vec{\phi}_B(0)=\vec{\phi}_L(0) \quad \textit{and} \quad \vec{\phi}_B(1)=\vec{\phi}_R(0), \tag{1.6}$$

$$\begin{aligned}\frac{d\vec{\phi}_L(0)}{dt}-E\frac{d^2\vec{\phi}_B(0)}{dx^2}+A(0,0)\frac{d\vec{\phi}_L(0)}{dx}+B(0,0)\vec{\phi}_B(0) &= \vec{f}(0,0),\\ \frac{d\vec{\phi}_R(0)}{dt}-E\frac{d^2\vec{\phi}_B(1)}{dx^2}+A(1,0)\frac{d\vec{\phi}_L(0)}{dx}+B(1,0)\vec{\phi}_B(1) &= \vec{f}(1,0),\end{aligned} \tag{1.7}$$

$$\begin{aligned}\frac{d^2}{dt^2}\vec{\phi}_L(0) = {} & E^2\frac{d^4}{dx^4}\vec{\phi}_B(0)-2EA(0,0)\frac{d^3}{dx^3}\vec{\phi}_B(0)-A^2(0,0)\frac{d^2}{dx^2}\vec{\phi}_B(0)\\ & -E(\frac{\partial^2 A}{\partial x^2}(0,0)\frac{d}{dx}\phi_B(0)+\frac{\partial^2 B}{\partial x^2}(0,0)\vec{\phi}_B(0)+B(0,0)\frac{d^2}{dx^2}\vec{\phi}_B(0)-\frac{\partial^2 f}{\partial x^2}(0,0))\\ & +A(0,0)(\frac{\partial A}{\partial x}(0,0)\frac{d}{dx}\vec{\phi}_B(0)+\frac{\partial B}{\partial x}(0,0)\vec{\phi}_B(0)+B(0,0)\frac{d}{dx}\vec{\phi}_B(0)-\frac{\partial f}{\partial x}(0,0))\\ & -(\frac{\partial A}{\partial t}(0,0)\frac{d}{dx}\vec{\phi}_B(0)+\frac{\partial B}{\partial t}(0,0)\vec{\phi}_B(0)+B(0,0)\frac{d}{dt}\vec{\phi}_B(0)-\frac{\partial f}{\partial t}(0,0))\end{aligned} \tag{1.8}$$

*and*

$$\begin{aligned}\frac{d^2}{dt^2}\vec{\phi}_L(0) &= E^2\frac{d^4}{dx^4}\vec{\phi}_B(1)-2EA(1,0)\frac{d^3}{dx^3}\vec{\phi}_B(1)-A^2(1,0)\frac{d^2}{dx^2}\vec{\phi}_B(1)\\ &-E(\frac{\partial^2 A}{\partial x^2}(1,0)\frac{d}{dx}\phi_B(1)+\frac{\partial^2 B}{\partial x^2}(1,0)\vec{\phi}_B(1)+B(1,0)\frac{d^2}{dx^2}\vec{\phi}_B(1)-\frac{\partial^2 f}{\partial x^2}(1,0))\\ &+A(1,0)(\frac{\partial A}{\partial x}(1,0)\frac{d}{dx}\vec{\phi}_B(1)+\frac{\partial B}{\partial x}(1,0)\vec{\phi}_B(1)+B(1,0)\frac{d}{dx}\vec{\phi}_B(1)-\frac{\partial f}{\partial x}(1,0))\\ &-(\frac{\partial A}{\partial t}(1,0)\frac{d}{dx}\vec{\phi}_B(1)+\frac{\partial B}{\partial t}(1,0)\vec{\phi}_B(1)+B(1,0)\frac{d}{dt}\vec{\phi}_B(1)-\frac{\partial f}{\partial t}(1,0)).\end{aligned} \tag{1.9}$$

*Then there exists a unique solution $\vec{u}$ of* (1.1) *such that $\vec{u}\in C^4_\lambda(\overline{\Omega})$.*

It is assumed throughout the paper that all of the assumptions (1.2), (1.3), (1.4), (1.6), (1.7), (1.8) and (1.9) of this section hold. Furthermore, $C$ denotes a generic positive constant, which is independent of $x, t$ and of all singular perturbation and discretization parameters. Inequalities between vectors are understood in the componentwise sense.

## 2 Analytical results

The operator $\vec{L}$ satisfies the following maximum principle.

**Theorem 2.2.** *Let $\vec{\psi}$ be any vector-valued function in the domain of $\vec{L}$ such that $\vec{\psi}\geq\vec{0}$ on $\Gamma$. Then $\vec{L}\vec{\psi}(x,t)\geq\vec{0}$ on $\Omega$ implies that $\vec{\psi}(x,t)\geq\vec{0}$ on $\overline{\Omega}$.*

An immediate consequence of this is the following stability result.

**Theorem 2.3.** *If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{L}$, then, for each $i$, $1\leq i\leq 2$ and $(x,t)\in\overline{\Omega}$,*

$$|\psi_i(x,t)|\leq \max\left\{\parallel\vec{\psi}\parallel_\Gamma,\frac{1}{\alpha}\parallel\vec{L}\vec{\psi}\parallel\right\}.$$

## 3 Crank-Nicolson semi-discretization in time

On $[0,T]$, a uniform mesh with $M$ mesh intervals, given by $\overline{\Omega}_t^M=\{k\Delta t, 0\leq k\leq M, \Delta t=T/M\}$ is considered. The following Crank-Nicolson scheme is applied on this mesh

$$\begin{aligned}&\vec{u}^0(x)=\vec{u}(x,0),\\ &(I+\frac{\Delta t}{2}\vec{L}_x)\vec{u}^{k+1}(x)=\frac{\Delta t}{2}(\vec{f}^k+\vec{f}^{k+1})(x)+(I-\frac{\Delta t}{2}\vec{L}_x)\vec{u}^k(x),\\ &\vec{u}^{k+1}(0)=\vec{u}(0,t_{k+1}),\vec{u}^{k+1}(1)=\vec{u}(1,t_{k+1}),k=0,...,M-1\end{aligned} \tag{3.10}$$

It is helpful to introduce the following artificial problem:

$$\begin{aligned}&(I+\frac{\Delta t}{2}\vec{L}_x)\vec{\hat{u}}^{k+1}(x)=\frac{\Delta t}{2}(\vec{f}^k+\vec{f}^{k+1})(x)+(I-\frac{\Delta t}{2}\vec{L}_x)\vec{u}(x,t_k),\\ &\vec{\hat{u}}^{k+1}(0)=\vec{u}(0,t_{k+1}),\vec{\hat{u}}^{k+1}(1)=\vec{u}(1,t_{k+1}),\end{aligned} \tag{3.11}$$

where $\vec{L}_x=-E\frac{\partial^2}{\partial x^2}+A\frac{\partial}{\partial x}+BI$, $\vec{f}^k=\vec{f}(x,t_k)$ and the solution $\vec{u}$ of (1.1) has replaced $\vec{u}^k$ in(3.10).
The operator $I+\frac{\Delta t}{2}\vec{L}_x$ satisfies the following maximum principle.

**Theorem 3.4.** *Let $\vec{\psi}$ be any vector-valued function in the domain of the operator $I+\frac{\Delta t}{2}\vec{L}_x$ such that $\vec{\psi}(0)\geq\vec{0}$ and $\vec{\psi}(1)\geq\vec{0}$. Then $(I+\frac{\Delta t}{2}\vec{L}_x)\vec{\psi}(x)\geq\vec{0}$ on $(0,1)$ implies that $\vec{\psi}(x)\geq\vec{0}$ on $[0,1]$.*

The stability of the operator $I+\frac{\Delta t}{2}\vec{L}_x$ is now established.

**Theorem 3.5.** *If $\vec{\psi}$ is any vector-valued function in the domain of the operator $I+\dfrac{\Delta t}{2}\vec{L}_x$ then, for each $i,\ 1\le i\le 2$ and $x\in[0,1]$,*

$$|\psi_i(x)|\le\ \max\left\{\|\ \vec{\psi}(0)\ \|,\|\ \vec{\psi}(1)\ \|,\frac{1}{\alpha}\ \|\ (I+\frac{\Delta t}{2}\vec{L}_x)\vec{\psi}(x)\ \|\right\}.$$

# 4 The Shishkin mesh

A piecewise uniform Shishkin mesh with $M\times N$ mesh-intervals is now constructed. Let $\Omega_t^M=\{t_k\}_{k=1}^M,\ \Omega_x^N=\{x_j\}_{j=1}^{N-1},\ \overline{\Omega}_t^M=\{t_k\}_{k=0}^M,\ \overline{\Omega}_x^N=\{x_j\}_{j=0}^N,\ \Omega^{M,N}=\Omega_t^M\times\Omega_x^N,\ \overline{\Omega}^{M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^N$ and $\Gamma^{M,N}=\Gamma\cap\overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with $M$ mesh-intervals on $[0,T]$. The mesh $\overline{\Omega}_x^N$ is a piecewise-uniform mesh on $[0,1]$ obtained by dividing $[0,1]$ into 3 mesh-intervals as follows

$$[0,1-\sigma_2]\cup(1-\sigma_2,1-\sigma_1]\cup(1-\sigma_1,1].$$

The parameters $\sigma_1,\ \sigma_2$ which determine the points separating the uniform meshes, are defined by,

$$\sigma_2=\min\left\{\frac{1}{2},\ 2\frac{\varepsilon_2}{\alpha}\ln N\right\}\tag{4.12}$$

and,

$$\sigma_1=\min\left\{\frac{\sigma_2}{2},\ 2\frac{\varepsilon_1}{\alpha}\ln N\right\}.\tag{4.13}$$

Clearly

$$0\ <\ \sigma_1\ <\ \sigma_2\ \le\ \frac{1}{2}.$$

Then, on the sub-interval $[0,1-\sigma_2]$ a uniform mesh with $\frac{N}{2}$ mesh-points is placed and on each of the sub-intervals $(1-\sigma_2,1-\sigma_1]$ and $(1-\sigma_1,1]$, a uniform mesh of $\dfrac{N}{4}$ mesh-points is placed. In practice, it is convenient to take

$$N=8q,\ \ q\ge 3.\tag{4.14}$$

In particular, when both the parameters $\sigma_r,\ r=1,2,$ are with the left choice, the Shishkin mesh $\overline{\Omega}^{M,N}$ becomes the classical uniform mesh with the step size $N^{-1}$ throughout from $0$ to $1$.
The Shishkin mesh suggested here has the features of an ideal Shishkin mesh that (i) when both the transition parameters have the left choice, it is the classical uniform mesh and (ii) it is coarse in the outer region and becomes finer and finer towards the right boundary. From the above construction it is clear that the transition points $\sigma_r, r=1,2$ on $[0,1]$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points.

# 5 The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1.1).
The discrete boundary value problem is now defined on any mesh by the finite difference method

$$D_t^-\vec{U}-E\delta_x^2\vec{U}+AD_x^-\vec{U}+B\vec{U}=\vec{f}\ \text{ on }\ \Omega^{M,N},\ \ \vec{U}=\vec{u}\ \text{ on }\ \Gamma^{M,N}.\tag{5.15}$$

This is used to compute numerical approximations to the solution of (1.1). It is assumed henceforth that the mesh is a Shishkin mesh, as defined in the previous section. Note that (5.15) can also be written in the operator form

$$\vec{L}^{M,N}\vec{U}\ =\ \vec{f}\ \text{ on }\ \Omega^{M,N},\ \ \vec{U}=\vec{u}\ \text{ on }\ \Gamma^{M,N},$$

where

$$\vec{L}^{M,N}\ =\ D_t^- - E\delta_x^2+AD_x^- +BI$$

and $D_t^-$, $\delta_x^2$, $D_x^+$ and $D_x^-$ are the difference operators.

For any function $\vec{Z}$ defined on the Shishkin mesh $\overline{\Omega}^{M,N}$, it is defined that $||\vec{Z}||=\max\limits_i\max\limits_{j,k}|Z_i(x_j,t_k)|$.
The following discrete results are analogous to those for the continuous case.

**Theorem 5.6.** *For any vector-valued mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi} \geq \vec{0}$ on $\Gamma^{M,N}$ and $\vec{L}^{M,N}\vec{\Psi} \geq \vec{0}$ on $\Omega^{M,N}$ imply that $\vec{\Psi} \geq \vec{0}$ on $\overline{\Omega}^{M,N}$.*

An immediate consequence of this is the following discrete stability result.

**Theorem 5.7.** *For any vector-valued mesh function $\vec{\Psi}$ on $\overline{\Omega}^{M,N}$,*

$$|\Psi_i(x_j, t_k)| \leq \max\left\{||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||\vec{L}^{M,N}\vec{\Psi}||\right\},$$

$i = 1, 2$ *and* $(x_j, t_k) \in \overline{\Omega}^{M,N}$.

## 6 The complete discretisation in time and space

The discrete boundary value problem is now defined on any mesh by

$$\left.\begin{array}{l} \vec{U}^0(x_j) = \vec{u}(x_j, 0) \text{ on } \Gamma_B^N, \\ (I + \frac{\Delta t}{2}\vec{L}_x^N)\vec{U}^{k+1}(x_j) = \frac{\Delta t}{2}(\vec{f}^k + \vec{f}^{k+1})(x_j) + (I - \frac{\Delta t}{2}\vec{L}_x^N)\vec{U}^k(x_j), \\ \vec{U}^{k+1}(x_j) = \vec{u}(x_j, t_{k+1}) \text{ on } \Gamma_L^M \cup \Gamma_R^M, \text{ for } k = 0, ..., M-1 \end{array}\right\} \quad (6.16)$$

where $\Gamma_L^M = \{(0,t) : 0 \leq t \leq T\}, \Gamma_B^N = \{(x,0) : 0 \leq x \leq 2\}, \Gamma_R^M = \{(2,t) : 0 \leq t \leq T\}$ and

$$\vec{L}_x^N = -E\delta_x^2 + AD_x^- + BI$$

and $\delta_x^2$, $D_x^+$ and $D_x^-$ are the standard finite difference operators.
The following results for the discrete operator $I + \frac{\Delta t}{2}\vec{L}_x^N$ are analogous to theorems 3.4 and 3.5. They are presented below without proof.

**Theorem 6.8.** *For any vector-valued mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi} \geq \vec{0}$ on $\Gamma^{M,N}$ and $(I + \frac{\Delta t}{2}\vec{L}_x^N)\vec{\Psi} \geq \vec{0}$ on $\Omega^{M,N}$ imply that $\vec{\Psi} \geq \vec{0}$ on $\overline{\Omega}^{M,N}$.*

An immediate consequence of this is the following discrete stability result.

**Theorem 6.9.** *For any vector-valued mesh function $\vec{\Psi}$ on $\overline{\Omega}^{M,N}$, for $k = 0, ..., M-1$ and $0 \leq j \leq N$,*

$$\| \vec{\Psi}(x_j, t_{k+1}) \| \leq \max\left\{||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||(I + \frac{\Delta t}{2}\vec{L}_x^N)\vec{\Psi}||\right\}.$$

## 7 Error Estimate

**Theorem 7.10.** *Let $\vec{u}$ denote the solution of (1.1) and $\vec{U}$ the solution of (5.15). Then*

$$||\vec{U} - \vec{u}|| \leq C(M^{-2} + N^{-1}\ln N).$$

## 8 Numerical Illustration

The $\varepsilon$–uniform convergence of the numerical method proposed in this paper is illustrated through an example presented in this section. The following system of singularly perturbed boundary value problem for a linear parabolic second order differential equations of convection-diffusion type is considered for numerical illustration.

**Example**

$$\begin{array}{c} \frac{\partial u_1}{\partial t}(x,t) - \varepsilon_1\frac{\partial^2 u_1}{\partial x^2}(x,t) + (6 + x^2)\frac{\partial u_1}{\partial x}(x,t) + (5 + x + t)u_1(x,t) - xu_2(x,t) = e^t + x^2, \\ \frac{\partial u_2}{\partial t}(x,t) - \varepsilon_2\frac{\partial^2 u_2}{\partial x^2}(x,t) + (10 + e^x)\frac{\partial u_2}{\partial x}(x,t) - u_1(x,t) + (6 + \sin x)u_2(x,t) = 5 + e^{-x^2}, \\ \text{for } (x,t) \in (0,1) \times [0,T], with \vec{u}(0,t) = \vec{0}, \vec{u}(x,0) = \vec{0}, \vec{u}(1,t) = \vec{0}. \end{array} \quad (8.17)$$

Fixing a fine Shishkin mesh with 128 points horizontally, the problem is solved by the method suggested above. The order of convergence and the error constant are calculated for $t$ and the results are presented in Table 1. A graph of the numerical solution is presented in the Figure 1.
A fine uniform mesh on $t$ with 32 points is considered. The order of convergence and the error constant are calculated for $x$ and the results are presented in Table 2. A graph of the numerical solution is presented in the Figure 2.
Based on the algorithm found in [6], it is to be noted that Table 1 and Table 2 give the parameter-uniform order of convergence and the error constant.

Table 1:
Values of $D^N, p^N, p^*$ and $C^N_{p^*}$ for $\varepsilon_1 = \eta/4, \varepsilon_2 = \eta$ and $\alpha = 0.9$

| $\eta$ | Number of mesh points $N$ | | | | |
|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 |
| $2^{-9}$ | 0.968E-01 | 0.479E-01 | 0.248E-01 | 0.112E-01 | 0.429E-02 |
| $2^{-13}$ | 0.969E-01 | 0.480E-01 | 0.248E-01 | 0.111E-01 | 0.430E-02 |
| $2^{-17}$ | 0.969E-01 | 0.480E-01 | 0.248E-01 | 0.111E-01 | 0.430E-02 |
| $2^{-21}$ | 0.969E-01 | 0.480E-01 | 0.248E-01 | 0.111E-01 | 0.430E-02 |
| $2^{-25}$ | 0.969E-01 | 0.480E-01 | 0.248E-01 | 0.111E-01 | 0.430E-02 |
| $D^N$ | 0.968E-01 | 0.480E-01 | 0.248E-01 | 0.112E-01 | 0.430E-02 |
| $p^N$ | 0.101E+01 | 0.952E+00 | 0.115E+01 | 0.138E+01 | |
| $C^N_p$ | 0.145E+01 | 0.139E+01 | 0.139E+01 | 0.121E+01 | 0.904E+00 |
| t-order of convergence= $0.9523812E+00$ | | | | | |
| The error constant= $0.1452491E+01$ | | | | | |

Table 2:
Values of $D^N, p^N, p^*$ and $C^N_{p^*}$ for $\varepsilon_1 = \eta/4, \varepsilon_2 = \eta$ and $\alpha = 0.9$

| $\eta$ | Number of mesh points $N$ | | | |
|---|---|---|---|---|
| | 128 | 256 | 512 | 1024 |
| $2^{-13}$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $2^{-17}$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $2^{-21}$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $2^{-25}$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $2^{-29}$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $D^N$ | 0.401E+00 | 0.312E+00 | 0.222E+00 | 0.146E+00 |
| $p^N$ | 0.362E+00 | 0.489E+00 | 0.608E+00 | |
| $C^N_p$ | 0.105E+01 | 0.105E+01 | 0.958E+00 | 0.808E+00 |
| x- order of convergence= $0.362E+00$ | | | | |
| The error constant= $0.105E+01$ | | | | |

The Figure 1 displays the numerical solution for the problem (8.17), computed for $M = 8, N = 128, \varepsilon_1 = 2^{-27}$ and $\varepsilon_2 = 2^{-25}$. The solution $\vec{u}(x,t)$ have boundary layers at $(1,t)$.
The Figure 2 displays the numerical solution for the problem (8.17), computed for $M = 16, N = 32, \varepsilon_1 = 2^{-15}$ and $\varepsilon_2 = 2^{-13}$. The solution $\vec{u}(x,t)$ have boundary layers at $(1,t)$.

Figure 1:

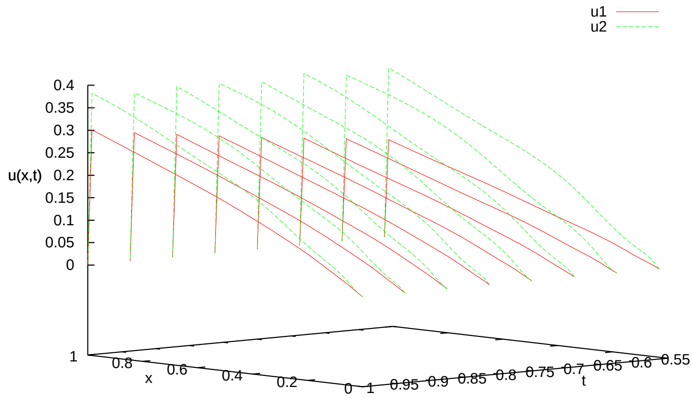
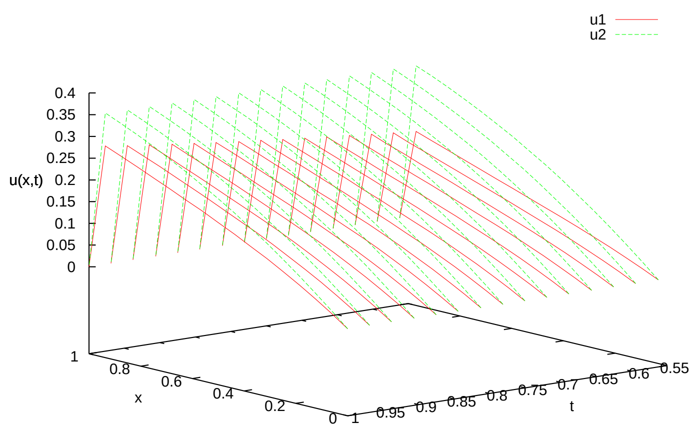



Figure 2:




## 9 Acknowledgment

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