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Existence and stability for fractional order integral equations with multiple time delay in Fréchet spaces

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Abstract

In this paper, we present some results concerning the existence of solutions for a system of integral equations of Riemann-Liouville fractional order with multiple time delay in Fréchet spaces, we use an extension of the Burton-Kirk fixed point theorem. Also we investigate the stability of solutions of this system.

Keywords: Functional integral equation, left-sided mixed Riemann-Liouville integral of fractional order, solution, stability, multiple time delay, Fréchet space, fixed point.

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1 Introduction

Integral equations occur in mechanics and many related fields of engineering and mathematical physics and others. They also form one of useful mathematical tools in many branches of pure analysis such as functional analysis [21, 27, 29]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [7], Baleanu et al. [12], Kilbas et al. [22], Lakshmikantham et al. [23], Podlubny [26]. Recently some interesting results on the attractivity of the solutions of some classes of integral equations have been obtained by Abbas et al. [1, 2, 3, 5, 6, 8], Banaś et al. [13, 14, 15], Darwish et al. [16], Dhage [17, 18, 19], Pachpatte [24, 25] and the references therein.

In [10], Avramescu and Vladimirescu presented an existence result of asymptotically stable solutions for the integral equation

$$x(t) = q(t) + \int_0^t K(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds; \text{ if } t \in R_+. \tag{1.1}$$

They used two fixed point theorems in Fréchet spaces, the Banach's contraction principle and the fixed point theorem of Burton-Kirk. In [11], the same authors studied the existence and the stability of solutions of the integral equation

$$x(t) = f(t,x(t)) + \int_0^{\nu(t)} u(t,s,x(\mu(s)))ds; \text{ if } t \in R_+, \tag{1.2}$$

by using the Schauder-Tychonoff fixed point theorem (see, e.g., [29]) in some Fréchet spaces. Recently, in [4], Abbas and Benchohra investigated the existence and uniqueness of solutions for the following fractional order integral equations for the system

$$u(x,y) = \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) + I_\theta^r f(x,y,u(x,y)); \text{ if } (x,y) \in J_1, \tag{1.3}$$

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$$u(x,y)=\Phi(x,y);\ \text{if } (x,y)\in \tilde{J}_1:=[-\xi,a]\times[-\mu,b]\backslash(0,a]\times(0,b], \tag{1.4}$$

where $J_1=[0,a]\times[0,b],\ a,b>0,\ \theta=(0,0),\ \xi_i,\mu_i\geq 0;\ i=1\dots,m,\ \xi=\max_{i=1\dots,m}\{\xi_i\},\ \mu=\max_{i=1\dots,m}\{\mu_i\},\ I_\theta^r$ is the left-sided mixed Riemann-Liouville integral of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ f:J_1\times R^n\to R^n,\ g_i: J_1\to R;\ i=1\dots m$ are given continuous functions, and $\Phi:\tilde{J}_1\to R^n$ is a given continuous function such that $\Phi(x,0)=\sum_{i=1}^m g_i(x,0)\Phi(x-\xi_i,-\mu_i)$ and $\Phi(0,y)=\sum_{i=1}^m g_i(0,y)\Phi(-\xi_i,y-\mu_i)$.

Motivated by those papers, this work deals with the existence and the stability of solutions of a class of functional integral equations of Riemann-Liouville fractional order with multiple time delay. We establish some sufficient conditions for the existence and the stability of solutions of the following fractional order integral equations for the system

$$u(t,x)=\sum_{i=1}^m g_i(t,x)u(t-\tau_i,x-\xi_i)+f(t,x,I_\theta^r u(t,x),u(t,x));\ (t,x)\in J, \tag{1.5}$$

$$u(t,x)=\Phi(t,x);\ \text{if } (t,x)\in\tilde{J}:=[-\tau,\infty)\times[-\xi,b]\backslash(0,\infty)\times(0,b], \tag{1.6}$$

where $J:=R_+\times[0,b],\ b>0,\ R_+=[0,\infty),\ \theta=(0,0),\ r=(r_1,r_2),\ r_1,r_2\in(0,\infty),\ \tau_i,\xi_i\geq 0;\ i=1\dots,m,\ \tau=\max_{i=1\dots,m}\{\tau_i\},\ \xi=\max_{i=1\dots,m}\{\xi_i\},\ f:J\times R\times R\to R,\ g_i:J\to R_+;\ i=1\dots m,\ \Phi:\tilde{J}\to R$ are given continuous functions such that

$$\Phi(t,0)=\sum_{i=1}^m g_i(t,0)\Phi(t-\tau_i,-\xi_i)+f(t,0,0,\Phi(t,0));\ t\in[0,\infty),$$

and

$$\Phi(0,x)=\sum_{i=1}^m g_i(0,x)\Phi(-\tau_i,x-\xi_i)+f(0,x,0,\Phi(0,x));\ x\in[0,b].$$

Our investigations are conducted in Fréchet spaces with an application of the fixed point theorem of Burton-Kirk for the existence of solutions of our problem, and we prove that all solutions are globally asymptotically stable. Also, we present an example illustrating the applicability of the imposed conditions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,a]\times[0,b])$, for $a,b>0$, we denote the space of Lebesgue-integrable functions $u:[0,a]\times[0,b]\to R$ with the norm

$$\|u\|_1=\int_0^a\int_0^b|u(t,x)|dxdt.$$

As usual, $\mathcal{C}:=C([-\tau,\infty)\times[-\xi,b])$ is the space of all continuous functions from $[-\tau,\infty)\times[-\xi,b]$ into R.

Definition 2.1. *([28]) Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ \theta=(0,0)$ and $u\in L^1([0,a]\times[0,b]);\ a,b>0$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}u(s,y)dyds,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt;\ \zeta>0$.

In particular, for almost all $(t,x)\in[0,a]\times[0,b]$,

$$(I_\theta^\theta u)(t,x)=u(t,x),\ \text{and } (I_\theta^\sigma u)(t,x)=\int_0^t\int_0^x u(s,y)dyds,$$

where $\sigma=(1,1)$.

For instance, $I_\theta^r u$ exists almost everywhere for all $r_1,r_2>0$, when $u\in L^1([0,a]\times[0,b])$. Moreover

$$(I_\theta^r u)(t,0)=0;\ \text{for a.a. } t\in[0,a],$$

and

$$(I_\theta^r u)(0,x) = 0, \text{ for a.a. } x \in [0,b].$$

Example 2.1. *Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$. Then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \text{ for a.a. } (t,x) \in [0,a] \times [0,b].$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*:=\{1,2,\ldots\}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \ldots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}^*$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}^*$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [20].

For each $p \in \mathbb{N}^*$ we consider following set, $C_p = C([-\tau,p] \times [-\xi,b])$, and we define in $\mathcal{C}$ the semi-norms by

$$\|u\|_p = \sup_{(t,x)\in[-\tau,p]\times[-\xi,b]} \|u(t,x)\|.$$

Then $\mathcal{C}$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Definition 2.2. *Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}^*$ there exists $k_n \in [0,1)$ such that*

$$\|N(u) - N(v)\|_n \le k_n \|u - v\|_n \quad \text{for all } u, v \in X.$$

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Theorem 2.1. *[9] Let $(X, \|.\|_n)$ be a Fréchet space and let $A, B : X \to X$ be two operators such that*

(a) A is a compact operator;

(b) B is a contraction operator with respect to a family of seminorms $\{\|.\|_n\}$;

(c) the set $\left\{x \in X : x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right),\ \lambda \in (0,1)\right\}$ is bounded.

Then the operator equation $A(u) + B(u) = u$ has a solution in X.

Let $\emptyset \ne \Omega \subset \mathcal{C}$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{2.1}$$

Now we introduce the concept of attractivity of solutions for our equations.

Definition 2.3. *([6, 7]) Solutions of equation (2.1) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space $\mathcal{C}$ such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equation (2.1) belonging to $B(u_0,\eta) \cap \Omega$, we have that, for each $x \in [0,b]$,*

$$\lim_{t\to\infty} \Big(v(t,x) - w(t,x)\Big) = 0. \tag{2.2}$$

When the limit (2.2) is uniform with respect to $B(u_0,\eta) \cap \Omega$, solutions of equation (2.1) are said to be uniformly locally attractive (or equivalently that solutions of (2.1) are locally asymptotically stable).

Definition 2.4. *([6, 7]) The solution $v = v(t,x)$ of equation (2.1) is said to be globally attractive if (2.2) holds for each solution $w = w(t,x)$ of (2.1). If condition (2.2) is satisfied uniformly with respect to the set Ω, solutions of equation (2.1) are said to be globally asymptotically stable (or uniformly globally attractive).*

3 Existence and Stability Results

Let us start by defining what we mean by a solution of the problem (1.5)-(1.6).

Definition 3.1. *A function $u \in \mathcal{C}$ is said to be a solution of (1.5)-(1.6) if u satisfies equation (1.5) on J and condition (1.6) on $\tilde{J}$.*

Now, we are concerned with the existence and the stability of solutions for the problem (1.5)-(1.6). Set

$$B_p = \max_{i=1\dots m}\left\{\sup_{(t,x)\in[0,p]\times[0,b]} g_i(t,x)\right\};\ p\in\mathbb{N}^*,$$

and

$$B^* = \max_{i=1\dots m}\left\{\sup_{(t,x)\in J} g_i(t,x)\right\}.$$

Theorem 3.1. *Assume that the following hypothesis holds:*

(H) *The function f is continuous and there exist functions $P, Q : J \to R_+$ such that*

$$|f(t,x,u,v)| \le \frac{P(t,x)|u| + Q(t,x)|v|}{1+|u|+|v|},\ \textit{for } (t,x)\in J \textit{ and } u,v\in R.$$

Moreover, assume that

$$\lim_{t\to\infty} P(t,x) = \lim_{t\to\infty} Q(t,x) = 0;\ for\ x\in[0,b].$$

If $mB_p < 1$; $p \in \mathbb{N}^*$, then the problem (1.5)-(1.6) has at least one solution in the space $\mathcal{C}$. Moreover, if the functions g_i; $i = 1\dots m$ are bounded on J, and $mB^* < 1$, then solutions of (1.5)-(1.6) are globally asymptotically stable.

Proof. Let us define the operators $A, B : \mathcal{C} \to \mathcal{C}$ by

$$(Au)(t,x) = \begin{cases} 0; & (t,x)\in\tilde{J},\\ f(t,x,I_\theta^r u(t,x), u(t,x)); & (t,x)\in J, \end{cases} \tag{3.1}$$

$$(Bu)(t,x) = \begin{cases} \Phi(t,x); & (t,x)\in\tilde{J},\\ \sum_{i=1}^m g_i(t,x)u(t-\tau_i, x-\xi_i); & (t,x)\in J. \end{cases} \tag{3.2}$$

The problem of finding the solutions of (1.5)-(1.6) is reduced to finding the solutions of the operator equation $A(u) + B(u) = u$. We shall show that the operators A and B satisfied all the conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: *A is compact.*

To this aim, we must prove that A is continuous and it transforms every bounded set into a relatively compact set. Recall that $M \subset \mathcal{C}$ is bounded if and only if

$$\forall p\in\mathbb{N}^*,\ \exists \ell_p > 0 : \forall u\in M,\ \|u\|_p \le \ell_p,$$

and $M = \{u(t,x);\ (t,x)) \in [-\tau,\infty)\times[-\xi,b]\} \subset \mathcal{C}$ is relatively compact if and only if for any $p \in \mathbb{N}^*$, the family $\{u(t,x)|_{(t,x)]\in[-\tau,p]\times[-\xi,b]}\}$ is equicontinuous and uniformly bounded on $[-\tau,p]\times[-\xi,b]$. The proof will be given in several claims.

Claim 1: *A is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $\mathcal{C}$. Then, for each $(t,x) \in [-\tau,\infty)\times[-\xi,b]$, we have

$$|(Au_n)(t,x) - (Au)(t,x)| \le |f(t,x,I_\theta^r u_n(t,x), u_n(t,x)) - f(t,x,I_\theta^r u(t,x), u(t,x))|. \tag{3.3}$$

If $(t,x) \in [-\tau,p]\times[-\xi,b]$; $p \in \mathbb{N}^*$, then, since $u_n \to u$ as $n \to \infty$, then (3.3) gives

$$\|A(u_n) - A(u)\|_p \to 0 \quad \text{as } n\to\infty.$$

Claim 2: *A maps bounded sets into bonded sets in C.*

Let M be a bounded set in $\mathcal{C}$, then, for each $p \in \mathbb{N}^*$, there exists $\ell_p > 0$, such that for all $u \in \mathcal{C}$ we have $\|u\|_p \leq \ell_p$. Then, for arbitrarily fixed $(t,x) \in [-\tau, p] \times [-\xi, b]$ we have

$$\begin{aligned}|(Au)(t,x)| &\leq |f(t,x,I_\theta^r u(t,x),u(t,x))| \\ &\leq (P(t,x)|I_\theta^r u(t,x)| + Q(t,x)|u(t,x))|) \\ &\times (1 + |I_\theta^r u(t,x)| + |u(t,x))|)^{-1} \\ &\leq P(t,x) + Q(t,x) \\ &\leq P_p + Q_p,\end{aligned}$$

where

$$P_p = \sup_{(t,x)\in[0,p]\times[0,b]} P(t,x) \quad and \quad Q_p = \sup_{(t,x)\in[0,p]\times[0,b]} Q(t,x).$$

Thus

$$\|A(u)\|_p \leq P_p^* + Q_p^* := \ell_p'. \tag{3.4}$$

Claim 3: *A maps bounded sets into equicontinuous sets in C.*

Let $(t_1,x_1),(t_2,x_2) \in [0,p] \times [0,b]$, $t_1 < t_2,\ x_1 < x_2$ and let $u \in M$, thus we have

$$|(Au)(t_2,x_2) - (Au)(t_1,x_1)| \leq$$

$$|f(t_2,x_2,I_\theta^r u(t_2,x_2),u(t_2,x_2)) - f(t_1,x_1,I_\theta^r u(t_1,x_1),u(t_1,x_1))|.$$

From continuity of f, I_θ^r, u and as $t_1 \to t_2,\ x_1 \to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2$, $x_1 \leq 0 \leq x_2$ is obvious. As a consequence of claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that A is continuous and compact.

Step 2: *B is a contraction.*

Consider $v, w \in \mathcal{C}$. Then, for any $p \in \mathbb{N}$ and each $(t,x) \in [-\tau, p] \times [-\xi, b]$, we have

$$\begin{aligned}|(Bv)(t,x) - (Bw)(t,x)| \quad &\leq \quad \sum_{i=1}^m g_i(t,x)|v(t-\tau_i, x-\xi_i) - w(t-\tau_i, x-\xi_i)| \\ &\leq \quad mB_p\|v-w\|_p,\end{aligned}$$

then

$$\|(B(v) - B(w)\|_p \leq mB_p\|v-w\|_p.$$

Since $mB_p < 1;\ p \in \mathbb{N}^*$, then; the operator B is a contraction.

Step 3: *the set* $\mathcal{E} := \left\{u \in \mathcal{C} : u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right),\ \lambda \in (0,1)\right\}$ *is bounded.*

Let $u \in \mathcal{C}$, such that $u = \lambda A(u) + \lambda B\left(\frac{u}{\lambda}\right)$ for some $\lambda \in (0,1)$. Then, for any $p \in \mathbb{N}^*$ and each $(t,x) \in [0,p] \times [0,b]$, we have

$$\begin{aligned}|u(t,x)| \quad &\leq \quad \lambda|(Au)(t,x)| + \lambda\left|B\left(\frac{u(t,x)}{\lambda}\right)\right| \\ &\leq \quad mB_p|u(t,x)| + Q(t,x) + P(t,x) \\ &\leq \quad mB_p\|u\|_p + P_p + Q_p,\end{aligned}$$

then,

$$\|u\|_p \leq \frac{P_p + Q_p}{1 - mB_p}.$$

On the other hand, for each $(t,x) \in [-\tau, p] \times [-\xi, b] \backslash (0,p] \times (0,b]$, we get

$$|u(t,x)| \leq |\Phi(t,x)| \leq \sup_{(t,x)\in[-\tau,p]\times[-\xi,b]\backslash(0,p]\times(0,b]} |\Phi(t,x)| := \Phi_p.$$

Thus

$$\|u\|_p \leq \max\left\{\frac{P_p + Q_p}{1 - mB_p}, \Phi_p\right\} =: \ell_p^*.$$

Hence, the set $\mathcal{E}$ is bounded. As a consequence of steps 1 and 3 together with Theorem 2.1, we deduce that $A+B$ has a fixed point u in $\mathcal{C}$ which is a solution to problem (1.5)-(1.6).

Now, we show the stability of solutions of the problem (1.5)-(1.6). Let u and v be any two solutions of (1.5)-(1.6), then for each $(t,x)\in[-\tau,\infty)\times[-\xi,b]$, we have

$$\begin{aligned}
|u(t,x)-v(t,x)| &= |(Au)(t,x)-(Av)(t,x)+(Bu)(t,x)-(Bv)(t,x)| \\
&\le \sum_{i=1}^{m} g_i(t,x)|u(t-\tau_i,x-\xi_i)-v(t-\tau_i,x-\xi_i)| \\
&+ |f(t,x,I_\theta^r u(t,x),u(t,x))-f(t,x,I_\theta^r v(t,x),v(t,x)))| \\
&\le mB^*|u(t,x)-v(t,x)|+2P(t,x)+2Q(t,x).
\end{aligned}$$

Thus

$$|u(t,x)-v(t,x)|\le \frac{2(P(t,x)+Q(t,x))}{1-mB^*}. \tag{3.5}$$

By using (3.5), we deduce that

$$\lim_{t\to\infty}(u(t,x)-v(t,x))=0.$$

Consequently, the problem (1.5)-(1.6) has a least one solution and all solutions are globally asymptotically stable. □

4 Example

Consider the following system of fractional order integral equation of the form

$$u(t,x)=\frac{t^3x}{1+8t^3}u\left(t-\frac{3}{4},x-3\right)+\frac{t^4x^2}{1+12t^4}u\left(t-2,x-\frac{1}{2}\right)+\frac{1}{4}u\left(t-1,x-\frac{3}{2}\right)$$

$$+\frac{\frac{1}{1+t+x}|I_\theta^r u(t,x)|+e^{2-t+x}|u(t,x)|}{1+\frac{1}{1+t+x}|I_\theta^r u(t,x)|+e^{2-t+x}|u(t,x)|};\ (t,x)\in R_+\times[0,1], \tag{4.6}$$

$$u(t,x)=0;\ \text{if } (t,x)\in\tilde{J}:=[-2,\infty)\times[-3,1]\backslash(0,\infty)\times(0,1], \tag{4.7}$$

where $r=\left(\frac{1}{2},\frac{33}{5}\right)$. Set

$$(\tau_1,\xi_1)=\left(\frac{3}{4},3\right),\ (\tau_2,\xi_2)=\left(2,\frac{1}{2}\right),\ (\tau_3,\xi_3)=\left(1,\frac{3}{2}\right),$$

$$g_1(t,x)=\frac{t^3x}{1+8t^3},\ g_2(t,x)=\frac{t^4x^2}{1+12t^4},\ g_3(t,x)=\frac{1}{4},$$

and

$$f(t,x,u,v)=\frac{\frac{|u|}{1+t+x}+|v|e^{2-t+x}}{1+\frac{|u|}{1+t+x}+|v|e^{2-t+x}};\ (t,x)\in R_+\times[0,1].$$

We have $m=3,\ (\tau,\xi)=(2,3)$ and $B_p\le B^*=\frac{1}{4};\ p\in\mathbb{N}$.
The function f is continuous and satisfies assumption (H), with

$$P(t,x)=\frac{1}{1+t+x}\ \text{and}\ Q(t,x)=e^{2-t+x}.$$

Hence by Theorem 3.1, the problem (4.6)-(4.7) has a solution defined on $[-2,\infty)\times[-3,1]$ and all solutions are globally asymptotically stable.

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