Hamiltonian property of intersection graph of zero divisors of the ring $\mathbb{Z}_n$
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Abstract
The intersection graph $G'_Z(\mathbb{Z}_n)$ of zero-divisors of the ring $\mathbb{Z}_n$, the ring of integers modulo $n$ is a simple undirected graph with the vertex set is $Z(\mathbb{Z}_n)^* = Z(\mathbb{Z}_n) \setminus \{0\}$, the set of all nonzero zero-divisors of the ring $\mathbb{Z}_n$ and for any two distinct vertices are adjacent if and only if their corresponding principal ideals have a nonzero intersection. We determine some results concerning the necessary and sufficient condition for the graph $G'_Z(\mathbb{Z}_n)$ is Hamiltonian. Also, we investigate for all values of $n$ for which the graph $G'_Z(\mathbb{Z}_n)$ is Hamiltonian and as an example we show that how the results give as easy proof of the existence of a Hamilton cycle.

Keywords
Finite commutative ring, Zero-divisors, Principal ideals, Intersection graph, Hamilton Cycle.

AMS Subject Classification
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1. Introduction
In 1736, Leonard Euler [10] starts the journey of Graph theory with the famous problem Konigsberg bridge problem. A graph is said to be Hamiltonian, if it posses a Hamilton cycle. The Hamilton cycle problem is a NP-complete problem which place a central role in graph theory and which has various applications in computational theory, see [3, 11]. This Hamilton problem traces its origin to the 1850’s, named for Sir William Rowan Hamilton. Generally, the Hamilton problem is considered to be determining the conditions under which a graph contains a spanning cycle. Many authors have studied the Hamilton cycles for several types of graphs and in which those are refer [6, 13, 14].

The intersection graph is a graph that represents the pattern of intersection of a family of sets. Let $F = \{A_j : j \in J\}$ be a family of nonempty sets, then the intersection graph $G(F)$ defined on $F$ as, for two distinct vertices $A_i$ and $A_j$ are adjacent whenever $A_i \cap A_j \neq \emptyset$. Intersection graph was first introduced by Bosak in 1965 for semigroup see [7], defined as vertices are the sub semigroups of that semigroup and in which two distinct vertices are adjacent if they have non trivial intersection. Many researchers worked on these intersection graphs by considering the members of $F$ have different algebraic structures and in which those see [8, 9, 17]. In [16], the intersection graph $G'_Z(R)$ of zero-divisors of a finite commutative ring $R$ is a simple undirected graph whose vertices are the nonzero zero-divisors of $R$ and in which two distinct vertices $x$ and $y$ are adjacent if and only if their corresponding principal ideals have nonzero intersection. i.e., $x$ is adjacent to $y$ if and only if $(x) \cap (y) \neq \{0\}$, $\forall x, y \in V(G'_Z(R))$. In this paper, we illustrate some results that shows the necessary and sufficient condition for the intersection graph $G'_Z(\mathbb{Z}_n)$ is Hamiltonian. Also, we investigate the problem of existence of Hamilton cycles in the intersection graph $G'_Z(\mathbb{Z}_n)$ for all characterizations of $n$.

2. Definitions and Notations
In this section, we consider the ring theoretic definitions and notations from [1, 4]. The set of all elements in the ring of integers modulo $n$, $\mathbb{Z}_n$ can be partitioned into the disjoint union of zero-divisors and regular elements of $\mathbb{Z}_n$ and...
which are denoted by \( Z(n) \) and \( \text{Reg}(n) = Z(n) \setminus Z(n) \) respectively. The set of all nonzero zero-divisors in \( Z_n \) is denoted by \( Z(n)^{\ast} = Z(n) \setminus \{0\} \). For an element \( x \) in \( Z_n \), the principal ideal generated by \( x \) is \( (x) = \{xr : r \in Z_n\} \). For further definitions of ring theory, the reader may refer [12].

In [15], for every positive integer \( n > 1 \) can be written as \( n = p_1^{a_1}p_2^{a_2}...p_m^{a_m} \), \( p_1 < p_2 < ... < p_m \) are primes, \( a_i \) is a positive integer for every \( i = 1, 2, ..., m \) and \( m \geq 1 \). The subset \( D \) of \( \mathbb{Z}_n \) be the set of all non trivial proper divisors of \( n \), i.e., \( D = \{d : d|n \text{ and } 1 < d < n\} \). The divisor function \( d(n) \) is the cardinality of the set of all divisors of \( n \), i.e., \( |D| = d(n) + 2 \) and \( d(n) = (a_1 + 1)(a_2 + 1)...(a_m + 1) \). For any positive integer \( m \) is called the least common multiple of \( a \) and \( b \), if \( m \) is a common multiple of \( a \) and \( b \), and also \( m|m_0 \) for any common multiple \( m_0 \) of \( a \) and \( b \). We write \( m = \text{lcm}(a,b) \). For the integers \( a, b \) and \( n > 0 \), if \( n \) divides the difference of \( a \) and \( b \), we denote that \( a \) is congruent to \( b \) modulo \( n \) and defined as \( a \equiv b({\text{mod}} \ n) \). Otherwise, we denote that \( a \) is incongruent to \( b \) modulo \( n \) and defined as \( a \not\equiv b({\text{mod}} \ n) \). For any positive integer \( n \) is called a square free integer, if a positive integer \( d \) with \( d^2|n \) implies that \( d = 1 \). In particular, \( n > 1 \) is a square free integer if and only if \( n = p_1p_2...p_m \), \( p_1 < p_2 < ... < p_m \) are primes. For further definitions of number theory, see [2]. We consider the graph theoretic definitions and notations from [5, 18]. For the graph \( G \), the two distinct vertices \( x \) and \( y \) are adjacent, write \( x \sim y \). The graph \( G \) is called complete if there exist an edge between every pair of two distinct vertices. A complete graph with \( n \) vertices is denoted by \( K_n \). A vertex induced subgraph is a subgraph that can be obtained by deleting a set of vertices. i.e., for the graph \( G \) the subgraph induced with the vertex set \( T \) is denoted by \( <T> <T> = G - T \), where \( T = V(G) \). A walk in a graph is an alternating sequence of vertices and edges, which begins and ends with a vertex. A trail is a walk in which all edges are distinct, and also a path is a trail in which all vertices are distinct. The graph \( G \) is said to be connected whenever there exist a path between every pair of two distinct vertices, otherwise disconnected. A cycle is a 2-regular connected subgraph of a graph, i.e., a closed path said to be a cycle. A collection of disjoint cycles that includes all the vertices of the graph \( G \) is said to be cycle factor of \( G \). We denote cycle factor as the union of cycles, i.e., \( C_1 \cup C_2 \cup ... \cup C_n \) where all cycle are disjoint and each vertex of \( G \) belongs to some cycle \( C_i \), \( \forall 1 \leq i \leq n \). If \( t = 1 \), then \( C_1 \) is called Hamilton cycle of \( G \), i.e., the cycle which visits each vertex of the graph exactly once is called Hamilton cycle of the graph. Also a graph is said to be Hamiltonian if it has a Hamilton cycle. The following definition and results are taken from [16].

**Definition 2.1.** The intersection graph \( G'_Z(R) \) of a finite commutative ring \( R \) with unity is a simple undirected graph whose vertices are all the nonzero zero-divisors of \( R \) and in which two distinct vertices are joined by an edge if and only if their corresponding principal ideals having nonzero intersection, i.e., \( x \) is adjacent to \( y \) if and only if \( x \cap y \neq 0 \), \( \forall x, y \in V(G'_Z(R)) \).

**Theorem 2.2.** For the ring \( Z_n \), order of the intersection graph \( G'_Z(Z_n) \) is \( n - \varphi(n) - 1 \).

**Theorem 2.3.** Let \( x \) and \( y \) be any two distinct nonzero zero-divisors of a finite commutative ring \( Z_n \). Then the least common multiple of \( x \) and \( y \) is congruent to zero modulo \( n \) if and only if \( x \) is not adjacent to \( y \) in \( G'_Z(Z_n) \).

**Theorem 2.4.** If \( n = p \), \( p \) is prime, then the graph intersection graph \( G'_Z(Z_n) \) does not exist.

**Theorem 2.5.** The graph \( G'_Z(Z_n) \) is complete if and only if \( n = p^m \), \( p \) is prime and \( m > 1 \).

**Theorem 2.6.** If \( n \) can be written as a product of two distinct primes \( p_1 \) and \( p_2 \), then the graph \( G'_Z(Z_n) \) is disconnected with two components, which are complete with \( p_2 - 1 \) and \( p_1 - 1 \) vertices respectively.

**Theorem 2.7.** The graph \( G'_Z(Z_n) \) is connected, not complete if and only if either of the following conditions is hold

(i) \( n \) can be written as a product of more than two primes.

(ii) \( n \) can be written as a product of at least two prime powers.

3. Hamilton cycle in the intersection graph of zero divisors of the ring \( Z_n \)

In this section, we show that the Intersection graph \( G'_Z(Z_n) \) of zero-divisors of a finite commutative ring \( Z_n \) is Hamiltonian for those characterizations of \( n \) and not Hamiltonian for those characterizations of \( n \), for all \( n \in N \). Let \( p \) be a prime. Then, in view of Theorem 2.4, the graph \( G'_Z(Z_p) \) does not exist.

**Theorem 3.1.** The intersection graph \( G'_Z(Z_{p^m}) \) is Hamiltonian if and only if \( m > 2 \).

**Proof.** Necessity. Suppose the graph \( G'_Z(Z_{p^m}) \) is Hamiltonian. But by the Theorem 2.5 the graph \( G'_Z(Z_{p^m}) \) is complete. So, there exist a Hamilton cycle \( C = (p, 2p, 3p, ..., p^m - p) \) whose length is \( p^{m-1} - 1 > 2 \). This shows \( m > 2 \).

Sufficiency. Let \( m > 2 \). Then, by Theorem 2.5, the graph is complete with at least 3 vertices. Hence the graph \( G'_Z(Z_{p^m}) \) is Hamiltonian. \(\square\)

**Example 3.2.** For \( m = 3 \), the intersection graph \( G'_Z(Z_3) = G'_Z(Z_9) \) having a Hamilton cycle \( C = (2, 4, 6) \) and is shown in Fig. 1.

**Figure 1.** The graph \( G'_Z(Z_8) \).

**Remark 3.3.** The intersection graph \( G'_Z(Z_{p^m}) \) is Hamiltonian if and only if \( m = 2 \) for all \( p \), except \( p \in \{2, 3\} \), since the graphs \( G'_Z(Z_4) \) and \( G'_Z(Z_8) \) are not Hamiltonian.

**Example 3.4.** For \( m = 2 \), the Hamilton cycle \( C \) in the intersection graph \( G'_Z(Z_5) = G'_Z(Z_{25}) \) is \( C = (5, 10, 15, 20) \) and is shown in Fig. 2.
Notation 3.5. Let $D$ be the set of all proper divisors of $n$, i.e., $D = \{d : d|n\}$ and $G_i$ be the set of all non unit divisors of $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m}$, for all $i = 1,2,...,m$, i.e., $G_i = \{d : d$ is a non unit divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m}$ \}, $\forall 1 \leq i \leq m$. Also $H_j$, for all $j = 0,1,2,...,m$ are

$$H_0 = \bigcap_{1 \leq i \leq m} G_i, H_j = \bigcup_{0 < k < j - 1} H_k, \forall j = 1,2,...,m.$$  

This shows that the cardinality of each $H_j$, for all $j = 0,1,2,...,m$, are $|H_0| = \alpha_1 \alpha_2 ... \alpha_m - 1$, $|H_1| = \alpha_1 (\alpha_2 + 1) (\alpha_3 + 1) (\alpha_m + 1) - \alpha_1 \alpha_2 ... \alpha_m$, $|H_k| = \alpha_k (\alpha_{k+1} + 1) (\alpha_{k+2} + 1) ... (\alpha_m + 1)$, $\forall k = 2,3,...,m$.

Theorem 3.6. For all $i = 1,2,...,m$ and $j = 0,1,...,m$, then the intersection graphs induced with the vertex sets $G_i$ and $H_j$ are complete induced subgraphs of the graph $G_Z(Z_n)$. In particular, $D$ is the disjoint union of $H_j$, for all $j = 0,1,...,m$.

Proof. From the notation of $G_i$, $\{d\}$ contains an element $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m} \neq 0$, for every $d \in G_i$, for all $i = 1,2,...,m$. By the Definition 2.1 there exist an edge between every pair of two distinct vertices in the intersection graph induced with the vertex set $G_i$, $\forall 1 \leq i \leq m$.

This shows that the intersection graph induced with the vertex set $G_i$ and $H_j$ is complete by Notations 3.5. To show that $D$ is the disjoint union of $H_j$ for all $0 < j < m$, first we prove that the set $D$ can be written as the union of $G_i$ for all $i = 1,2,...,m$.

We know that the set $D = \{d : d$ is a proper divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_m^{\alpha_m}\}$ and $G_i = \{d : d$ is a non unit divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m}\}$, $\forall 1 \leq i \leq m$, i.e.,

$$G_i = \{d : d$ is a non unit divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m}\},$$

$G_2 = \{d : d$ is a non unit divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_i^{\alpha_i}...p_m^{\alpha_m}\},$

$G_m = \{d : d$ is a non unit divisor of $p_1^{\alpha_1}p_2^{\alpha_2}...p_m^{\alpha_m}\}$. 

By the above construction of $G_i$‘s, we have

$$D = \bigcup_{1 \leq i \leq m} G_i.$$ 

From the notations of $H_0,H_1,H_2,...,H_m$, we have

$$H_1 = G_1 - H_0, H_2 = G_2 - (H_0 \cup H_1),...,H_j = G_j - \bigcup_{0 \leq k < j - 1} H_k,$$

$$...H_m = G_m - \bigcup_{0 \leq k < m - 1} H_k.$$ 

Hence $D$ can be written as the disjoint union of $H_j$, $\forall j = 0,1,...,m$. 

\[\square\]

\[\text{Example 3.7. If } n = 12, 12 = 2^2 \cdot 3. \text{ Then } D = \{2,3,4,6\}, G_1 = \{2,3,6\}, G_2 = \{2,4\}, H_0 = \{2\}, H_1 = \{3,6\} \text{ and } H_2 = \{4\}. \text{ Therefore, } D \text{ is the disjoint union of } H_0, H_1, \text{ and } H_2. \text{ The subgraphs induced with vertex sets } G_1, G_2, H_0, H_1, \text{ and the graph } G_Z(Z_{12}) \text{ are shown in the following Figs. [3-4].} \]

\[\text{Figure 3. The subgraphs induced with vertex sets } G_1, G_2, H_0, H_1 \text{ and } H_2 \text{ respectively.} \]

\[\text{Figure 4. The graph } G_Z(Z_{12}). \]

Definition 3.8. For every proper divisor $d$ of $n$, $D_d$ be the set of all elements in the ring $Z_n$ such that whose principal ideal is equal to principal ideal of $d$. i.e., $D_d = \{x \in Z_n : (x) = (d)\}$, $\forall d \in D$, where $D$ be the set of all proper divisors of $n$.

Lemma 3.9. The intersection graph induced with the vertex set $D_d$ for all $d \in D$ is a complete induced subgraph of the graph $G_Z(Z_n)$.

Proof. By the Definition 3.8, for $d \in D$ we have $(x) = (d)$ for every $x \in D_d$. Clearly, $d \neq 0$, since $d$ is a proper divisor of $n$. Hence, there exists an edge between every pair of two distinct vertices in the intersection graph with the vertex set $D_d$ for all $d \in D$ from the Definition 2.1. Hence the proof follows.

\[\square\]

Theorem 3.10. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_m^{\alpha_m}, m > 2$. Then the intersection graph $G_Z(Z_n)$ is Hamiltonian.

Proof. The set $D$ can be written as $D = H_0 \cup H_1 \cup H_2 \cup ... \cup H_m$, where $H_i \cap H_j = \emptyset$ for distinct $i$ and $j$ varying from 0 to $m$ from Theorem 3.6. We shall prove that the graph $G_Z(Z_n)$ is Hamiltonian. For this we construct a path $P$, which contains vertices are the set of all elements in $D$ as follows.

In $H_0$ there is an edge between $x_{01} = p_1^{\alpha_1 - 2}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$ and $x_{02} = p_1^{\alpha_1 - 2}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$, and $x_{03} = p_1^{\alpha_1 - 2}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$, $\forall 0 \leq k < j - 1$. Similarly in $H_1$ there exist an edge between $x_{11} = p_1^{\alpha_1 - 1}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$ and $x_{12} = p_1^{\alpha_1 - 1}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$, $\forall 0 \leq k < m - 1$.

Let $x_{01}$ be in $H_0$ and $x_{11} = p_1^{\alpha_1 - 1}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$ be in $H_1$. Since Icm$(x_{01},x_{11}) \neq 0$ (mod $n$), then there exist an edge between $x_{01}$ and $x_{11}$. Similarly in $H_0$ there exist an edge between $x_{11}$ and $x_{12} = p_1^{\alpha_1 - 1}p_2^{\alpha_2 - 1}...p_m^{\alpha_m - 1}$, $\forall 0 \leq k < m - 1$. Hence $G_Z(Z_n)$ is Hamiltonian.
In $H_2$, there is an edge between $x_2$ and $x_2' = \alpha_1 p_1 p_1^p \alpha_2 \ldots p_m$ in $H_2$, and $x_2$ and $x_2 = \alpha_1 p_1^p \alpha_2 \ldots p_m$ in $H_2$ whereas lcm($x_2 H_2$, $x_2$) = 0($\mod n$).

Continuing in this way, we get an edge from $x_m H_2$ = $\alpha_1 p_1^p \alpha_2 \ldots p_m$ in $H_m$ and $x_m H_2$ = $\alpha_1 p_1^p \alpha_2 \ldots p_m$ in $H_m$ by considering lcm($x_m H_2$, $x_m$) = 0($\mod n$). In the same way in $H_2$ we can find an edge between $x_m H_2$ and $x_m H_2$ = $\alpha_1 p_1^p \alpha_2 \ldots p_m$ in $H_m$. Let $x_m = \alpha_1 p_1^p \alpha_2 \ldots p_m$ in $H_m$.

Thus we get a path $P$ that joining all the elements $D$ with the initial vertex $x_0$ and the terminal vertex $x_m H_2$ as follows and also shown in Fig. 5.

$$P: x_01, x_02, \ldots, x_0 H_2, x_{11}, x_{12}, \ldots, x_{1 H_2}, x_{21}, x_{22}, \ldots, x_{2 H_2}, \ldots, x_{m H_2}, x_m H_2.$$

Similarly in the Theorem 3.10, we replace the spanning path $P_2$ of the intersection graph induced with the vertex set $D'$, $\forall d \in D$ in place of $d$ in the above path $P$.

Thus we obtain a spanning path of the graph $G'_2(Z_n)$, and finally we join the initial vertex $x_01$ of the spanning path $P_2$ corresponding to the element $x_01$ in $H_1$ and the terminal vertex $n - x_m H_2$ of the spanning path $P_{m H_2}$ corresponding to the element $x_m H_2$ in $H_m$, since lcm($x_{01}, n - x_m H_2$) = 0($\mod n$) and hence the graph $G'_2(Z_n)$ is Hamiltonian.

**Remark 3.11.** The set $D$ can be written as the disjoint union of $H_1, H_2, \ldots, H_m$ from Theorem 3.6 when $n$ is a square free integer. Since $H_0$ is empty from the notation of $H_0$.

**Theorem 3.12.** Let $n$ be $s$ a square free integer except $n \neq pq$. Then the intersection graph $G'_2(Z_n)$ is Hamiltonian.

**Proof.** Consider $n = p_1 p_2 \ldots p_m$ with $m > 2$. Then the set $D$ can be written as $D = H_1 \cup H_2 \cup \ldots \cup H_m$, $H_1 \cap H_2 = \phi$ for distinct $i$ and $j$ varying from 1 to $m$. Also $|H_1| = 2^{m-1} - 1$, $|H_2| = 2^{m-2} - 1$, $|H_3| = 2^{m-3} - 1$, $|H_m| = 1$.

Now, we construct a path $P$ with the vertex set $D$ follows.

In $H_1$ there is an edge between $x_1 = p_2$ and $x_2 = p_2 p_3$; $x_2$ and $x_3 = p_2 p_3 \ldots p_n$ in $H_2$ whereas lcm($x_2$, $x_3$) = 0($\mod n$).

Again, there exist an edge between $x_2$ and $x_2 = p_1 p_4 p_5 \ldots p_m$ whereas lcm($x_2$, $x_2$) = 0($\mod n$).

Continuing in this way, we get an edge between $x_2 H_2 = p_1 p_2 p_3 \ldots p_m$ in $H_3$ whereas lcm($x_2 H_2$, $x_3$) = 0($\mod n$).

Therefore, we get a path $P$ whose vertices are the set of all elements $D$ with the initial vertex $x_01$ and the terminal vertex $x_m H_2$ as follows and also shown in Fig. 6.

$$P: x_{01}, x_{12}, \ldots, x_{(m-1) H_2} , x_{21}, x_{22}, \ldots, x_{(m-2) H_2} , \ldots, x_{(m-1)} H_2 , x_m H_2.$$
The spanning path \( P_d \), for all \( d \in D \) are \( P_2 : 2, 4, 8, 14, 16, 22, 26, 28, P_3 : 3, 9, 21, 27, P_5 : 5, 25, P_6 : 6, 12, 18, 24, P_{10} : 10, 20, P_{15} : 15 \).

Now, we replace \( P_d \) in place of \( d \) in the above path \( P \), we get a spanning path of the graph \( G'_2(Z_{30}) \) with the initial vertex \( x_{11} = 3 \) and the terminal vertex \( 30 - x_{31} = 24 \) as follows and also shown in Fig. 8.

Spanning path of the graph \( G'_2(Z_{30}) \),
\[ P : 3, 9, 21, 27, 15, 5, 25, 10, 20, 2, 4, 8, 14, 16, 22, 26, 28, 6, 12, 18, 24. \]

![Figure 8. The spanning path of \( G'_2(Z_{30}) \).](image)

Finally, we join the initial vertex 3 and the terminal vertex 24, since \( \text{lcm}(3, 24) \neq 0 \text{ (mod 30)} \). Thus we get a Hamilton cycle in the graph \( G'_2(Z_{30}) \). Also the intersection graph \( G'_2(Z_{30}) \) including its Hamilton cycle with thick lines is shown in Fig. 9.

![Figure 9. The graph \( G'_2(Z_{30}) \).](image)

Remark 3.14. Theorem 3.10 is not sufficient for \( m = 2 \), it is true for \( m > 2 \) only. Since every vertex in \( H_1 \) is not adjacent to every vertex in \( H_2 \), when \( m = 2 \).

Now, we shall study the Hamilton property of graph \( G'_2(Z_n) \) for \( m = 2 \) and \( n \neq p q \). Since the graph \( G'_2(Z_{p q}) \) is never Hamilton. Now, we prove the Hamiltonian property for \( n = p^2 q \) and \( n \neq p^2 q, p < q \) are primes separately, since when \( n = p^2 q \) and \( n \neq p^2 q \) then \( H \) consists of only one element and more than one element respectively.

Theorem 3.15. If \( n = p^2 q \) and \( n \neq p^2 q, p < q \). Then the intersection graph \( G'_2(Z_n) \) Hamiltonian.

Proof. The set \( D \) can be written as the disjoint union of \( H_0, H_1 \) and \( H_2 \) such that \( H_0 = \{ x_{01}, x_{02}, ..., x_{06} \} \) (assume), where \( k = \alpha_1 \alpha_2 - 2 > 1, H_1 = \{ p^2 q, p_1 p^2 q, p_1 p^2 q, ..., p_1 \alpha_1 - 1 \alpha_2 \} \) and \( H_2 = \{ p^2 q, p_1 p_2, p_1 p_2, ..., p_1 p_2 \}. \)

We now construct a path \( P \) with vertices are all the elements in \( D \) as follows.

Let \( x_{01} \) in \( H_0 \) and \( x_{11} = p_{0}^{\frac{3}{2}} \) in \( H_1 \). Then there exist an edge between \( x_{01} \) and \( x_{11} \), since \( \text{lcm}(x_{01}, x_{11}) \neq 0 \text{ (mod n)} \).

In \( H_1 \) there is an edge between \( x_{11} \) and \( x_{12} = p_1 p_{0}^{\frac{3}{2}} \); \( x_{12} \) and \( x_{13} = p_1 p_{0}^{\frac{3}{2}} ; x_{13}(\alpha_1 - 1) = p_1 \alpha_1 - 1 p_{0}^{\frac{3}{2}} \) and \( x_{14} = p_1 \alpha_1 - 1 p_{0}^{\frac{3}{2}} \).

Let \( x_{10} \) in \( H_1 \) and \( x_{02} \) in \( H_0 \), then \( \text{lcm}(x_{10}, x_{02}) \neq 0 \text{ (mod n)} \).

So, there exist an edge between \( x_{10} \) and \( x_{02} \).

Let \( x_{02} \) in \( H_0 \) and \( x_{21} = p_0 \alpha_1 \) in \( H_2 \). There is an edge between \( x_{02} \) and \( x_{21} \), whereas \( \text{lcm}(x_{02}, x_{21}) \neq 0 \text{ (mod n)} \).

Again, in \( H_2 \), there exist an edge between \( x_{21} \) and \( x_{22} = p_1 \alpha_1 p_2 ; x_{22} \) and \( x_{23} = p_1 \alpha_1 p_2 ; x_{23}(\alpha_2 - 1) = p_1 \alpha_1 \alpha_2 - 1 p_2 \) and \( x_{24} = p_1 \alpha_1 \alpha_2 - 1 p_2 \).

If \( H_0 \) consists of more than two elements, then consider \( x_{26} \) in \( H_2 \) and \( x_{03} \) in \( H_0 \). So that there exists an edge between \( x_{26} \) and \( x_{03} \), because \( \text{lcm}(x_{26}, x_{03}) \neq 0 \text{ (mod n)} \). Also there exist an edge between \( x_{03} \) and \( x_{04} \), \( x_{04} \) and \( x_{05} \), ..., \( x_{0(k - 1)} \) and \( x_{0k} \).

Thus, we get a path \( P \) with the vertex set \( D \) having the initial vertex \( x_{01} \) and the terminal vertex \( x_{26} \), if \( H_0 \) consists of only two elements or \( x_{0k} \), if \( H_0 \) consists of more than two elements as follows and also shown in Figs. 10 and 11 respectively.

\[ P : x_{01}, x_{11}, ..., x_{1\alpha_1}, x_{02}, x_{21}, x_{22}, ..., x_{2\alpha_2}, \] if \( H_0 \) consists of only two elements or

\[ P : x_{01}, x_{11}, ..., x_{1\alpha_1}, x_{02}, x_{21}, x_{22}, ..., x_{2\alpha_2}, x_{03}, x_{04}, ..., x_{0(k - 1)}, x_{0k}, \] if \( H_0 \) consists of more than two elements.

![Figure 10. The path \( P \) if \( H_0 \) consists of only two elements.](image)

![Figure 11. The path \( P \) if \( H_0 \) consists of more than two elements.](image)
Example 3.16. Consider the intersection graph $G'_{Z}(Z_{24})$, where $24 = 2^3$. The set $D$ can be written as the disjoint union of $H_0$, $H_1$ and $H_2$, where $D = \{2, 3, 4, 6, 8, 12\}$, $H_0 = \{2, 4\}$, $H_1 = \{3, 6, 12\}$ and $H_2 = \{8\}$. Similarly in Example 3.13, we construct a path $P$ with the vertex set $D$ as follows.

Let $x_{01} = 2$ in $H_0$ and $x_{11} = 3$ in $H_1$. Then there exist an edge between $2$ and $3$, since $\text{lcm}(2, 3) \not\equiv 0(\text{mod } 24)$. In $H_1$, there is an edge between $3$ and $6$; $6$ and $12$ from Corollary 3.10. Let $12$ in $H_1$ and $4$ in $H_0$. So there is an edge between $12$ and $4$, since $\text{lcm}(12, 4) \not\equiv 0(\text{mod } 24)$. Also we can find an edge between $4$ in $H_0$ and $8$ in $H_2$, because $\text{lcm}(4, 8) \not\equiv 0(\text{mod } 24)$.

Thus we get path $P$ with vertices are set of elements in $D$ and whose initial vertex $2$ and the terminal vertex $8$ as $P: 2, 3, 6, 12, 4, 8$, also shown in the Fig. 12. The spanning path of $G'_{Z}(Z_{24})$ with thick lines is shown in Fig. 14.

Figure 12. The path $P$.

Figure 13. The spanning path of $G'_{Z}(Z_{24})$.

Finally, we join the initial vertex $2$ and the terminal vertex $16$ in the spanning path, since $\text{lcm}(2, 16) \not\equiv 0(\text{mod } 24)$. Also the intersection graph $G'_{Z}(Z_{24})$ including its Hamilton cycle with thick lines is shown in Fig. 14.

Figure 14. The graph $G'_{Z}(Z_{24})$.

Remark 3.17. The proof of Theorem 3.15 is not sufficient for $n = p^2q$. Because in this case it is not possible to draw a closed path with the vertex set $D$, since $H_0$ consists of only one element $p$.

Theorem 3.18. If $n = p^2q$, $p < q$ are primes. Then the intersection graph $G'_{Z}(Z_{n})$ is Hamiltonian.

Proof. We have the set $D$ can be written as the disjoint union of $H_0$, $H_1$ and $H_2$ such that $H_0 = \{p\}$, $H_1 = \{q, pq\}$ and $H_2 = \{p^2\}$. We now construct a trail $P$ with the vertex D as follows.

Consider $p$ in $H_0$ and $q$ in $H_1$. Then there exist an edge between $p$ and $q$, because $\text{lcm}(p, q) \not\equiv 0(\text{mod } n)$. There exist an edge between $q$ and $pq$ in $H_1$. Let $qp$ in $H_1$ and $p$ in $H_0$. So there exist an edge between $qp$ and $p$ whereas $\text{lcm}(qp, p) \not\equiv 0(\text{mod } n)$. Let $p$ in $H_0$ and $p^2$ in $H_2$, then $\text{lcm}(p, p^2) \not\equiv 0(\text{mod } n)$ and thus there exist an edge between $p$ and $p^2$.

Thus, we get a trail $P$ whose vertex set $D$ with the initial vertex $p$ and the terminal vertex $p^2$ as $P: p, q, pq, p, p^2$.

We construct the spanning path of the graph $G'_{Z}(Z_{n})$ by replacing $q, qp$ and $p^2$ by $P_0, P_{qp}$ and $P_{p^2}$, also repetition of $p$ by $P_p - \{p\}$ in the above trail $P$, here $P_d$ is the spanning path of the intersection graph induced with the vertex set $D_d$, $\forall d \in D$. Finally, we join the initial vertex $p$ and the terminal vertex $n - p^2$ of the spanning path $P_{p^2}$ corresponding to the element $p^2$ in $H_2$, because $\text{lcm}(p, n - p^2) \not\equiv 0(\text{mod } n)$. Thus we a Hamilton cycle of the graph $G'_{Z}(Z_{n})$.

Example 3.19. Consider the graph $G'_{Z}(Z_{12})$, where $12 = 2^2$. Then $D$ can be written as the disjoint union of $H_0, H_1$ and $H_2$ such that $D = \{2, 3, 4, 6\}$, $H_0 = \{2\}$, $H_1 = \{3, 6\}$ and $H_2 = \{4\}$. We construct a trail $P$ with the vertex set $D$ as follows.

Let $2$ in $H_0$ and $3$ in $H_1$. Then there exist an edge between $2$ and $3$, since $\text{lcm}(2, 3) \not\equiv 0(\text{mod } 12)$. In $H_1$, there exist an edge between $3$ and $6$. Let $6$ in $H_1$ and $2$ in $H_0$. So there is an edge between $6$ and $2$, because $\text{lcm}(6, 2) \not\equiv 0(\text{mod } 12)$. Also we can find an edge between $2$ in $H_0$ and $4$ in $H_2$, whereas $\text{lcm}(2, 4) \not\equiv 0(\text{mod } 12)$.

Now, we get the trail $P$ with vertices are all the elements in $D$ whose initial vertex $2$ and terminal vertex $4$ as $P: 2, 3, 6, 2, 4$ and also shown in the Fig. 16. The spanning path $P_d$, for all
d ∈ D are \( P_2 : 2, 10, P_3 : 3, 9, P_4 : 4, 8, P_6 : 6 \). Now we replace \( P_3, P_4, P_6 \) in place of 3, 4, 6 and also \( P_2 = \{2\} \) in place of repetition of 2 in the above trail \( P \). Thus, we find a spanning path of the graph \( G'_{Z}(Z_{12}) \) with the initial vertex 2 and the terminal vertex \( 12 - 4 = 8 \) as follows. Also shown in Fig. 17.

Spanning path of the graph \( G'_{Z}(Z_{12}) : 2, 3, 9, 6, 10, 4, 8 \).

Figure 17. The spanning path of \( G'_{Z}(Z_{12}) \).

Finally, we join the initial vertex 2 and the terminal vertex 8 in the spanning path, since \( \text{lcm}(2, 8) \neq 0(\text{mod } 12) \). The intersection graph \( G'_{Z}(Z_{12}) \) including its Hamilton cycle with thick lines is shown in Fig. 18.

Figure 18. The graph \( G'_{Z}(Z_{12}) \).

References