Energy decay of solutions for viscoelastic wave equations with a dynamic boundary and delay term

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Abstract
In this paper, we establish a general decay result by using Nakao's technique for a system of multi-dimensional viscoelastic wave equations with dynamic boundary conditions related to the Kelvin Voigt damping and delay term acting on the boundary.

Keywords
Global existence, energy decay, blow up of solutions, nonlinear damping.

AMS Subject Classification
35L60, 35K55, 26A33, 35B44, 35B33.

1. Introduction
In this article, we investigate the following wave equation with dynamic boundary conditions and delay term:

\[
\begin{align*}
    u_{tt} - \Delta u - \int_0^t g(t-s)\Delta u(s) ds - \delta \Delta u_t &= |u|^{p-1}u, \\
    in & \quad \Omega \times (0, +\infty), \\
    u &= 0, & on & \quad \Gamma_0 \times (0, +\infty), \\
    u_t &= -a\left(\frac{\partial u}{\partial \nu}\right)(x, t) + \delta \frac{\partial u}{\partial \nu}(x, t) + \mu_1(t)u_t(x, t) + \mu_2(t)u(x, t - \tau), & on & \quad \Gamma_1 \times (0, +\infty), \\
    u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), & x & \in \Omega, \\
    u_t(x, t - \tau) &= f_0(x, t - \tau), & on & \quad \Gamma_1 \times (0, +\infty),
\end{align*}
\]

where \( u = u(x, t), \ t \geq 0, x \in \Omega \) and \( \Delta \) denotes the Laplacian operator with respect to the \( x \) variable. \( \Omega \) is a regular and bounded domain of \( \mathbb{R}^N \), \( (N \geq 1) \), \( \partial \Omega = \Gamma_1 \cup \Gamma_0, \Gamma_1 \cap \Gamma_0 = \emptyset \) and \( \frac{\partial}{\partial \nu} \) denotes the unit outer normal derivative. \( \mu_1 \) and \( \mu_2 \) are functions depend on \( t \). Moreover, \( \tau > 0 \) represents the delay and \( u_0, u_1, f_0 \) are given functions belonging to suitable spaces that will be specified later. This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term \( \Delta u_t \), indicates that the stress is proportional not only to the strain, but also to the strain rate (See [5]).

This type of problem without delay (i.e \( \mu_i = 0 \)), has been considered by many authors during the past decades and many results have been obtained (see [4], [6], [7], [13]) and the references therein.

Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass (see [4], [1], [6] for more details), which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (1.1) appear when we assume that is an exterior domain of \( \mathbb{R}^3 \) in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). This type of dynamic boundary conditions is known as acoustic boundary conditions. Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen ([7], [8], [9]) in which the author has
made contributions to this field. And In [7] the author has introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length \( L \) when the end \( x = 0 \) is rigidly fixed while the other end \( x = L \) is free to move with an attached load. This yields to a system of two second order equations of the form:

\[
\begin{align*}
    u_{tt} - u_{xx} + u_{xxx} &= 0, & x \in (0, L), & t > 0, \\
    u_{tt}(L, t) &= -\left[u_t + u_{tt}\right](L, t), & t > 0, \\
    u_t(x, 0) &= u_1(x), u_t(L, 0) = \mu, & x \in (0, L), \\
    u(L, 0) &= \eta, & u_t(L, 0) = \mu.
\end{align*}
\]

By rewriting problem (1.2) within the framework of the abstract theories of the so-called B-evolution theory, the existence of a unique solution in the strong sense has been shown and an exponential decay result was also proved in [8] for a problem related to (1.2), which describes the weakly damped vibrations of an extensible beam (See [8] for more details). Subsequently, Zang and Hu [30], considered the problem

\[ u_{tt} - p(u_t)u_t - q(u_t)x = 0, \quad x \in (0, 1), \quad t > 0, \]

\[ p(u_t)u_t + q(u_t)(1 + t) + ku_t(1 + t) = 0, \quad u(0, t) = 0, \quad t \geq 0. \]

By using the Nakao’s inequality and under appropriate conditions on \( p \) and \( q \), they established both exponential and polynomial decay rates for the energy depending on the form of the terms \( p \) and \( q \). It is clear that in the absence of the delay term and for \( \mu_1 = 0 \), problem (1.2) is the one dimensional model of (1.1). Similarly, and always in the absence of the delay term, Pellerico and Sola-Morales [22] considered the one dimensional problem of (1.1) as an alternative model for the classical spring-mass damper system and by using the dominant eigenvalues method, they proved that their system has the classical second order differential equation

\[ m u''(t) + d u'(t) + k u(t), \]

as a limit, where the parameters \( m, d \) and \( k \) are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence. (See also [21], [22]) for related results.

It is widely known that delay effects, which arise in many practical problems are source of some instabilities. In this way Datko and al [20] showed that a small delay in a boundary control turns to be a well-behaved hyperbolic system into a wide one which in turn, becomes a source of instability. Nicola and al [12] studied the following system of a wave equation with a linear boundary term:

\[
\begin{align*}
    u_{tt} - \Delta u(x, t) &= 0, & \text{in} & \quad \Omega \times (0, +\infty), \\
    u(x, t) &= 0, & \text{on} & \quad \Gamma_0 \times (0, +\infty), \\
    \frac{\partial u}{\partial n} &= \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau), & \text{on} & \quad \Gamma_1 \times \mathbb{R}_+, \\
    u(x, 0) &= u_0(x), & x \in \Omega, \\
    u_t(x, 0) &= f_0(x, t - \tau), & x \in \Omega, & t \in (0, \tau),
\end{align*}
\]

and proved that the energy is exponentially stable, where \( \nu \) is the unit outward normal to \( \partial \Omega \), under the condition

\[ \mu_2 < \mu_1. \]

On the contrary, if (1.4) doesn’t hold, there is a sequence of delays for which the corresponding solution of (1.3) will be instable.

The problem (1.3) with time varying delay term has been studied by Nicola and al. We refer the readers to ([11], [12]).

Recently, inspired by the works of Al and Nicola [12], Stéphane Gerber and Said El Houari [15] considered the following problem in bounded domain:

\[
\begin{align*}
    u_{tt} - \Delta u - \Delta u_t &= 0, & \text{in} & \quad \Omega \times (0, +\infty), \\
    u &= 0, & \text{on} & \quad \Gamma_0 \times (0, +\infty), \\
    u_{tt} &= -\alpha \frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u}{\partial \nu}(x, t) + \mu_1 u_t(x, t) \\
    &+ \mu_2 u_t(x, t - \tau), & \text{on} & \quad \Gamma_1 \times (0, +\infty), \\
    u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & x \in \Omega, \\
    u_t(x, t - \tau(t)) &= f_0(x, t - \tau), & \text{on} & \quad \Gamma_1 \times (0, +\infty),
\end{align*}
\]

and obtained several results concerning global existence and exponential decay rates for various sign of \( \mu_1, \mu_2 \).

Motivated by the previous works, in the present paper we investigate problem (1.1) in which we generalize the results obtained in [31] by supposing new conditions with which the global existence and stability results are assured. The stable set is used to prove the existence result and Nakao’s technique to establish energy decay rates.

The content of this paper is organized as follows: In Section 2, we provide assumptions that will be used later. In Section 3, we state and prove the global existence result. In Section 4, the stability result given in Theorem 4.1 will be proved.

## 2. Preliminary Results

In this section, we present some material in the proof of our main result. We assume

\[
(A_1) \quad \mu_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad \text{is a nonincreasing function of class} \quad C^1(\mathbb{R}_+) \quad \text{satisfying}
\]

\[
\begin{align*}
    \left| \frac{\mu'_1(t)}{\mu_1(t)} \right| &\leq M, \\
    \left| \frac{\mu'_2(t)}{\mu_2(t)} \right| &\leq M
\end{align*}
\]

\[ M > 0 \]
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(A2) $\mu_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta \mu_1(t),$$

(2.2)

$$|\mu_2'(t)| \leq \bar{M} \mu_1(t),$$

(2.3)

for some $0 < \beta < 1$ and $\bar{M} > 0$. For the relaxation function $g$ we assume

(A3) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded $C^1$ function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = I < 1.$$

(A4) There exists a nonincreasing differentiable function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$g'(t) \leq -\eta(t)g(t).$$

(A5) We suppose therefore

$$2 \leq p \leq \frac{2n-2}{n-2} \quad \text{if} \quad n \geq 3, \quad p > 2, \quad \text{if} \quad n = 1, 2.$$

Now we choose $\tilde{\tau}$ such that

$$\tau \beta < \tilde{\tau} < \tau(2 - \beta).$$

(2.5)

Lemma 2.1. (Sobolev-Poincaré’s inequality). Let $2 \leq m \leq \frac{2n}{n-2}$. The inequality

$$\|u\|_{m} \leq c_5 \|\nabla u\|_{2} \quad \text{for} \quad u \in H^1_0(\Omega),$$

holds with some positive constant $c_5$.

Lemma 2.2. [2] For any $g \in C^1$ and $\phi \in H^1(0, T)$, we have

$$\int_0^t \int_\Omega g(t-s)\phi \phi_x dsdx = -\frac{d}{dt} \left( (go\phi)(t) + \int_0^t g(s)ds \|\phi\|_{2}^2 \right)$$

$$- g(t)\|\phi\|_{2}^2 + (g'\phi_{\phi})(t),$$

where

$$(go\phi)(t) = \int_0^t (g(t-s))(\phi(s) - \phi(t))^2 dsdx.$$

Lemma 2.3. [2] For $u \in H^1_0(\Omega)$, we have

$$\int_\Omega \left( \int_0^t g(t-s)(u(t) - u(s))ds \right)^2 dx \leq (1-l)c_5^2$$

$$\times (go\nabla u)(t),$$

(2.6)

where $c_5^2$ is the Poincaré’s constant and $l$ is given in (A3) and

$$(go\nabla u)(t) = \int_0^t (g(t-s))\int_\Omega |\nabla u(s) - \nabla u(t)|^2 dsdx.$$

Lemma 2.4. [32] Let $\phi$ be a nonincreasing and nonnegative function on $[0, T]$, $T > 1$, such that

$$\phi(t)^{1+r} \leq \omega_0(\phi(t) - \phi(t+1)), \quad \text{on} \quad [0, T],$$

where $\omega_0 > 1$ and $r \geq 0$. Then we have, for all $t \in [0, T]$

(i) if $r = 0$, then

$$\phi(t) \leq \phi(0)e^{-\omega_0[t-1]^+},$$

(ii) if $r > 0$, then

$$\phi(t) \leq \left( \phi(0)^{-r} + \omega_0^{-1}r[t-1]^{+} \right)^{1/r}.$$
Lemma 3.2. Let \((u,z)\) be the solution of (3.1) then, the energy equation satisfies

\[
E'(t) \leq \frac{1}{2}(g'\phi u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2
- \mu_1(t) \left(1 - \frac{c(t)}{4\tau} - \frac{\beta}{2}\right)\|u(t)\|_2^2
- \mu_1(t) \left(\frac{c(t)}{4\tau} - \frac{\beta}{2}\right)\|z(x,1,t)\|_2^2
- \delta\|\nabla u(t)\|_2^2 \leq 0.
\]

(3.3)

Proof. By multiplying the first and second equation in (3.1) by \(u(t)\), integrating the first equation over \(\Omega\) and the second equation over \(\Gamma_1\), using Green's formula, we get

\[
\frac{d}{dt} \left[ \frac{1}{2}\|u(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{2,\Gamma_1}^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 \right]
- \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} - \delta\|u(t)\|_2^2 + \mu_1(t)\|u(t)\|_{2,\Gamma_1}^2
- \int_{\Omega} g(t-s)\nabla u(s)\nabla u(t) d\Gamma + \int_{\Gamma_1} \mu_2(z)(1,1)u(t) d\gamma = 0.
\]

(3.4)

As in [??] we multiply the third equation in (3.1) by \(\zeta(t)\) and integrate over \(\Gamma_1 \times (0,1)\) to obtain

\[
\zeta(t)\tau\int_{\Gamma_1} \int_0^1 z\zeta(\gamma,1,t) d\gamma d\tau
+ \zeta(t)\int_{\Gamma_1} \int_0^1 \zeta z(x,1,t) d\gamma d\tau = 0,
\]

(3.5)

this yields

\[
\frac{\zeta(t)\tau}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau
+ \frac{\zeta(t)}{2} \int_{\Gamma_1} \int_0^1 \zeta z(x,1,t) d\gamma d\tau = 0,
\]

(3.6)

then

\[
\frac{\tau}{2} \frac{d}{dt} \left(\zeta(t)\int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau\right)
- \zeta'(t)\int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau
+ \frac{\zeta(t)}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau
- \frac{\zeta'(t)}{2}\int_{\Gamma_1} \int_0^1 z^2(x,0,t) d\gamma = 0,
\]

(3.7)

consequently

\[
\frac{\tau}{2} \frac{d}{dt} \left(\zeta(t)\int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau\right)
- \zeta'(t)\int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau
+ \frac{\zeta(t)}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma,1,t) d\gamma d\tau
+ \frac{\zeta'(t)}{2}\int_{\Gamma_1} \int_0^1 z^2(x,0,t) d\gamma,
\]

(3.8)

finally from (3.4) and (3.7), we get

\[
E(t) + \left(\mu_1(t) - \frac{\zeta(t)}{2\tau}\right)\|u(t)\|_{2,\Gamma_1}^2
+ \mu_2(t)\int_{\Gamma_1} \zeta(\gamma,1,t)u(t,\gamma,t) d\gamma
- \frac{\zeta(t)}{2}\int_{\Gamma_1} \int_0^1 z^2(\gamma,k,t) d\gamma d\tau + \frac{\zeta(t)}{2\tau}\int_{\Gamma_1} \zeta^2(\gamma,1,t) d\gamma
- \frac{1}{2}\int_0^t (g'\phi u)(s) ds + \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2 ds = 0.
\]

(3.9)

Due to Young's inequality, we have

\[
\int_{\Gamma_1} z(\gamma,1,t) u(t,\gamma,t) d\gamma \leq \frac{1}{2}\|u(t)\|_{2,\Gamma_1}^2
+ \frac{1}{2}\int_{\Gamma_1} \zeta^2(\gamma,1,t) d\gamma.
\]

(3.10)

Noting that \(\zeta'(t) \leq 0\). Inserting (3.10) into (3.9) and deriving it, we get the desired result. \(\square\)

Now we are in position to state the local existence result to problem (3.1), which can be established by combining arguments of ([19],[20],[22]).

Theorem 3.3. Let \(u_0 \in H^1_{\Gamma_0}(\Omega)\), \(u_1 \in L^2(\Omega)\) and \(f_0 \in L^2(\Gamma_1 \times (0,1))\) be given. Suppose that (A1) — (A3) hold. Then the problem (3.1) admits a unique weak solution \((u,z)\) satisfying

\[u \in L^\infty((0,T);H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T);H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T);L^2(\Gamma_1)),\]

\[u_{tt} \in L^\infty((0,T);L^2(\Omega)) \cap L^\infty((0,T);L^2(\Gamma_1)).\]

Now we will prove that the solution obtained above is global and bounded in time. For this purpose let us define

\[
I(t) = \left(1 - \int_0^t g(s) ds\right)\|\nabla u\|_2^2 + (g\phi u)(t)
- \|u\|_{p+1}^{p+1} + \zeta(t)\int_{\Gamma_1} \int_0^1 z^2(\gamma,k,s) d\gamma d\tau,
\]

(3.11)

\[
J(t) = \frac{1}{2}(g\phi u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s) ds\right)\|\nabla u\|_2^2
- \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\zeta(t)}{2}\int_{\Gamma_1} \int_0^1 z^2(\gamma,k,s) d\gamma d\tau,
\]

(3.12)

and

\[
E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_{2,\Gamma_1}^2.
\]

(3.13)

Lemma 3.4. Suppose that assumptions (A3) — (A4) are fulfilled. Let \((u,z)\) be the solution of the problem (12). Assume further that \(I(0) > 0\) and

\[
\alpha = \left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-2}{2}} < 1,
\]

(3.14)

then \(I(t) > 0\) for all \(t \geq 0\).
Proof. Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that for all $t \in [0, T^*)$, we have

$$I(t) \geq 0.$$  \hfill (3.15)

From (3.11) and (3.12) we obtain

$$J(t) \geq \frac{p-1}{2(p+1)} \left[ (1 - \int_0^t g(s)ds) \| \nabla u \|_2^2 + g_0 \nabla u(t) \right] + \frac{1}{p+1} I(t)$$
$$+ \frac{(p-1)}{2(p+1)} \left[ \zeta(t) \int_0^1 \int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma \right]$$
$$\geq \frac{p-1}{2(p+1)} \left[ (1 - \int_0^t g(s)ds) \| \nabla u \|_2^2 + (g_0 \nabla u(t)) \right]$$
$$+ \frac{(p-1)}{2(p+1)} \left[ \zeta(t) \int_0^1 \int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma \right]$$
$$\geq \frac{p-1}{2(p+1)} l\| \nabla u \|_2^2.$$  \hfill (3.16)

Thus by using (3.13) and (3.14) we deduce $\forall t \in [0, T^*)$

$$l\| \nabla u \|_2^2 \leq \left( 1 - \int_0^t g(s)ds \right) \| \nabla u \|_2^2$$
$$\leq \frac{2(p+1)}{p+1} E(t) \leq \frac{2(p+1)}{p+1} E(0).$$  \hfill (3.17)

Exploiting Lemma 2.1 and formula (3.17), we obtain

$$\| u \|_{p+1} \leq \| \nabla u \|_2 \| u \|_{p+1} \leq \frac{c_{p+1}^p}{l} E(t)$$
$$\leq \frac{c_{p+1}^p}{l} \left( 2(p+1) \| \nabla u \|_2 \right) \| u \|_{p+1} \leq \alpha l \| \nabla u \|_2$$  \hfill (3.18)

Hence for all $t \in [0, T^*)$, we have

$$I(t) = \left( 1 - \int_0^t g(s)ds \right) \| \nabla u \|_2^2 + (g_0 \nabla u(t)) - \| u \|_{p+1}$$
$$+ \zeta(t) \int_0^1 \int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma > 0.$$  

Repeating this procedure and using the fact that

$$\lim_{t \to T^*} \frac{c_{p+1}^p}{l} \left( \frac{2(p+1)}{2l(p-1)} E(u(t)) \right) \leq \alpha < 1,$$

we can take $T^* = T$. This completes the proof.

\hfill \Box

4. Stability result

Theorem 4.1. Assume that (A3) – (A5) hold. Let $u_0 \in H^1_0(\Omega)$, $u_1 \in L^p(\Omega)$ and $f_0 \in L^2(\Gamma_1 \times (0,1))$ be given. Then the solution of the problem (3.1) is global and bounded in time. Furthermore, there exists $\theta > 0$, such that

$$\theta > \frac{4 - 3l - l^2}{2l},$$  \hfill (4.1)

and we have the following decay estimate:

$$E(t) \leq E(0)e^{-\sigma t}, \quad \forall t \geq 0, \quad \sigma = \ln \left( \frac{c_{12}}{c_{12} - 1} \right),$$

where $c_{12}$ is a positive constant.

Proof. First, we prove $T = \infty$. It is sufficient to show that $l\| \nabla u \|_2^2$ is bounded independently of $t$. From (24) we have

$$E(0) \geq E(t) = \frac{1}{2} \| u_t \|_2^2 + \| u_t \|_{L^2(\Gamma_1)} + J(t) \| u \|_2^2 \geq l\| \nabla u \|_2^2.$$  

Therefore $l\| \nabla u \|_2^2 \leq \rho E(0)$, where $\rho$ is a positive constant which depends only on $p$, thus we obtain the global existence result. From now and on, we focus our attention to the decay rate of the solutions to problem (3.1). In order to do so, we will derive the decay rate of the energy function for problem (3.1) by Nakao’s method, as in [32]. For this aim, we have to show that the energy function defined by (3.13) satisfies the hypotheses of Lemma 2.4. By integrating (3.3) over $[t, t+1]$, we have

$$E(t) - E(t+1) = D(t)^2,$$  \hfill (4.2)

where

$$D(t)^2 = c_1 \int_t^{t+1} \mu_1(s) \| u_t \|_{L^2(\Gamma_1)}^2 ds$$
$$+ c_2 \int_t^{t+1} \mu_2(s) \int_{\Gamma_1} z^2(\gamma,1,s) d\gamma ds$$
$$- \int_t^{t+1} \frac{1}{2} (g' \delta \nabla u)(s) ds$$
$$+ c_3 \int_t^{t+1} \| \nabla u \|_2^2 ds.$$  \hfill (4.3)

By virtue of (4.3) and Holder’s inequality, we observe that

$$\int_t^{t+1} \int_{\Gamma_1} \mu_1(s) \| u_t \|^2 d\gamma ds$$
$$+ \int_t^{t+1} \mu_2(s) \int_{\Gamma_1} |z(\gamma,1,s)|^2 d\gamma ds + \int_t^{t+1} \| \nabla u \|_2^2 ds$$
$$\leq c(\Gamma_1) D(t)^2,$$  \hfill (4.4)

where $c(\Gamma_1) = vol(\Gamma_1)$. Applying the mean value, then there exist $t_1 \in [t, t+1]$, $t_2 \in [t+\frac{1}{4}, t+1]$ and $t_3 \in [t+\frac{1}{4}, t+1]$ such that for $i = 1, 2$, we get

$$\mu_1(t_i) \| u_t(t_i) \|_{L^2(\Gamma_1)}^2 + \mu_2(t_i) \| z(\gamma,1,t_i) \|_{L^2(\Gamma_1)}^2 + \| \nabla u_t(t_i) \|_2^2$$
$$\leq c(\Gamma_1) D(t)^2.$$  \hfill (4.5)
Multiplying the first equation in (3.1) by $u$ and integrating over $\Omega \times [t_1, t_2]$, multiplying the second equation in (3.1) by $u$ and integrating over $\Gamma_1 \times [t_1, t_2]$, adding and subtracting the following term $\int_{t_1}^{t_2} \int_{\Omega} \xi(t) z^2(\gamma, k, t) dk d\gamma$, we obtain

\[
\int_{t_1}^{t_2} I(t) dt \leq \sum_{i=1}^{2} \|u(t_i)\|_2 \|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt
\]
\[+ \int_{t_1}^{t_2} \int_{\Omega} \nabla u_t \nabla u dt dx + \int_{t_1}^{t_2} (g(t) \nabla u)(t) dt
\]
\[+ \int_{t_1}^{t_2} \int_{t_0}^{t} g(t-s) \nabla u(t) \nabla u(s) - \nabla u(t) ds d\gamma dt
\]
\[+ \int_{t_1}^{t_2} \xi(t) \int_{t_0}^{t} \zeta^2(x, k, t) dk d\gamma dt
\]
\[- \int_{\Gamma_1} \int_{t_1}^{t_2} \mu_2(t) z(x, 1, t) u dt d\gamma - \int_{\Gamma_1} \int_{t_1}^{t_2} \mu_1(t) \int_{\Gamma_1} u u d\gamma dt.
\]

Then (4.6) takes the form

\[
\int_{t_1}^{t_2} I(t) dt \leq \sum_{i=1}^{2} \|u(t_i)\|_2 \|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt
\]
\[+ \int_{t_1}^{t_2} \int_{\Omega} \nabla u_t \nabla u dt dx + \int_{t_1}^{t_2} (g(t) \nabla u)(t) dt
\]
\[+ \int_{t_1}^{t_2} \int_{t_0}^{t} g(t-s) \nabla u(t) \nabla u(s) - \nabla u(t) ds d\gamma dt
\]
\[- \int_{\Gamma_1} \int_{t_1}^{t_2} \mu_2(t) z(x, 1, t) u dt d\gamma - \int_{\Gamma_1} \int_{t_1}^{t_2} \mu_1(t) \int_{\Gamma_1} u u d\gamma dt.
\]

(4.7) and Lemma 2.1, we have

\[
\|u_t(t_i)\|_2 \|u(t_i)\|_2 \leq c_c c(\Gamma) \frac{1}{2} D(t) \sup_{t_1 \leq s \leq t_2} \|\nabla u(s)\|_2
\]
\[\leq D(t) c_c c(\Gamma) \frac{1}{2} \left( \frac{2(p+1)}{p-1} \right)^\frac{1}{2} E(s)^\frac{1}{2}
\]
\[\leq D(t) c_c c(\Gamma) \frac{1}{2} \left( \frac{2(p+1)}{p-1} \right)^\frac{1}{2} E(t)^\frac{1}{2}.
\]

As in [33], by employing Young’s inequality for convolution $\|\varphi * \phi\| \leq \|\varphi\| \|\phi\|$ and noting that

\[
l\|\nabla u(t)\|_2^2 \leq \frac{1}{\theta} I(t),
\]

then we have

\[
\int_{t_1}^{t_2} g(t-s) \|\nabla u(s)\|_2^2 ds dt
\]
\[\leq \int_{t_1}^{t_2} g(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt
\]
\[\leq (1-l) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \leq \frac{1-l}{\theta} \int_{t_1}^{t_2} I(t) dt.
\]

By exploiting (4.7), we obtain

\[
\frac{1}{2} \int_{t_1}^{t_2} (g(t) \nabla u(t)) dt =
\]
\[\frac{1}{2} \int_{t_1}^{t_2} g(t-s) \|\nabla u(s) - \nabla u(t)\| ds dt
\]
\[\leq \int_{t_1}^{t_2} g(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\| dt
\]
\[\leq \int_{t_1}^{t_2} g(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\| dt \leq \frac{1-l}{2 \theta} \int_{t_1}^{t_2} I(t) dt.
\]

Multiplying the second equation in (3.1) by $\xi_z$ and integrating the result over $\Gamma_1 \times (0, 1)$, we get

\[
\frac{\tau}{2} \frac{d}{dt} \left( \xi(t) \int_{\Gamma_1} \int_{0}^{1} z^2(\gamma, k, t) dk d\gamma \right) =
\]
\[\frac{\tau}{2} \frac{\xi'(t)}{2} \int_{\Gamma_1} \int_{0}^{1} z^2(\gamma, k, t) dk d\gamma
\]
\[+ \frac{\tau}{2} \frac{\xi'(t)}{2} \int_{\Gamma_1} \int_{0}^{1} z^2(x, 0, t) d\gamma.
\]
Recalling that $\zeta'(t) \leq 0$, we have

\[ \int_{t_1}^{t_2} \zeta(t) \int_0^1 z^2(x,k,t)dkd\gamma dt \leq \int_{t_1}^{t_2} \zeta(s) \left\| u_t(s) \right\|_{L_1}^2 ds dv \]

\[ \leq c \left( \int_{t_1}^{t_2} dv \left( \int_{t_1}^{t_2} \mu(s) \left\| u_t(s) \right\|_{L_1}^2 ds \right) \right) \]

\[ \leq c(\Gamma_1)(t_2-t_1)D(t)^2. \]

Using Sobolev’s inequality, also we get

\[ \left| \int_{t_1}^{t_2} \mu_2(s)z(x,1,t)ud\gamma dt \right| \leq \int_{t_1}^{t_2} \mu_2(s)\left\| z(x,1,t) \right\|_2\left\| u_t \right\|_2 dt \leq c_5 \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \]

\[ \times \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} \int_{t_1}^{t_2} \mu_2(s)\left\| z(x,1,t) \right\|_2 dt \]

\[ \leq c_5 c(\Gamma_1)^{\frac{1}{2}} \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} E(t)^{\frac{1}{2}}D(t), \]

and

\[ \int_{t_1}^{t_2} \left\| u_t \right\|_2^2 dt \leq c_5^2 \int_{t_1}^{t_2} \left\| \nabla u_t \right\|_2^2 dt \leq c_5^2 c(\Gamma_1)D(t)^2, \]

\[ \int_{t_1}^{t_2} \left| \int_{\Gamma_1} \mu_1(s)u_t ud\gamma dt \right| \]

\[ \leq \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} \int_{t_1}^{t_2} \left\| \nabla u_t \right\|_2 dt \]

\[ \leq c(\Gamma_1) \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} E(t)^{\frac{1}{2}}D(t), \]

also we have

\[ \left| \int_{t_1}^{t_2} \mu_1(s)su_t ud\gamma dt \right| \leq \int_{t_1}^{t_2} \mu_1(s)\left\| u_t \right\|_{L_1} \left\| u_t \right\|_{L_1} dt \leq c_6 \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \]

\[ \times \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} \int_{t_1}^{t_2} \mu_1(s)\left\| u_t \right\|_{L_1} dt \]

\[ \leq c_6 c(\Gamma_1)B \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} E(t)^{\frac{1}{2}}D(t), \]

therefore, from (4.9) – (4.18) we deduce

\[ \int_{t_1}^{t_2} I(t) dt \leq c(\Gamma_1)^{\frac{1}{2}} \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \]

\[ \times \left[ c_4(B+2)+1 \right] E(t)^{\frac{1}{2}}D(t) \]

\[ + \left( \frac{2(1-l)}{l\theta} + \frac{1-l}{\theta} \right) \int_{t_1}^{t_2} I(t) dt \]

\[ + \left( \frac{3}{4} c(\Gamma_1) + c_5^2 c(\Gamma_1) \right) D(t)^2. \]

Then, rewriting (4.19), we get

\[ c_5 \int_{t_1}^{t_2} I(t) dt \leq c_4 D(t)^2 + c_5 E(t)^{\frac{1}{2}}D(t), \]

with

\[ c_5 = \left[ 1 - \frac{(2-2l)}{l\theta} - \frac{1-l}{2\theta} \right], c_4 = c(\Gamma_1)^{\frac{3}{4}} + c_5^2 c(\Gamma_1), \]

and

\[ c_3 = c(\Gamma_1)^{\frac{1}{2}} \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} [c_4(B+2)+1]. \]

From the condition (4.1) and observing that is equivalent to $c_5 > 0$, thus

\[ \int_{t_1}^{t_2} I(t) dt \leq c_7 \left[ D(t)^2 + E(t)^{\frac{1}{2}}D(t) \right], \]

where $c_7 = \max(c_1,c_3)$. On the other hand, from the definition of $E(t)$ and by (3.11) and (3.13) we have

\[ \int_{t_1}^{t_2} E(t) dt \leq \frac{p-1}{2(p+1)} \]

\[ \times \int_{t_1}^{t_2} \left[ (go\nabla u)(t) + \left( 1 - \int_0^t g(s) ds \right) \left\| \nabla u_t \right\|_2^2 \right] \]

\[ + \frac{p-1}{2(p+1)} \int_{t_1}^{t_2} \int_{t_1}^{t_2} z^2(x,k,s)dkd\gamma dt \]

\[ + \frac{1}{p+1} \int_{t_1}^{t_2} I(t) dt + \int_{t_1}^{t_2} \frac{1}{2} \left\| u_t \right\|_2^2 dt \]

\[ + \int_{t_1}^{t_2} \frac{1}{2} \left\| u_t \right\|_{L_1}^2 dt \]

\[ \leq c_8 \frac{p-1}{2(p+1)} \left( \frac{1}{\theta} + \frac{1-l}{2\theta} + \frac{1}{p+1} \right) \]

\[ \times (D(t)^2 + E(t)^{\frac{1}{2}}D(t)) \]

\[ + \left[ c_6^2 c(\Gamma_1)(1+B) + c(\Gamma_1)^{\frac{3}{4}} \right] D(t)^2 \]

\[ \leq c_8 D(t)^2 + c_6 E(t)^{\frac{1}{2}}D(t) \]

\[ \leq c_{10} \left[ D(t)^2 + E(t)^{\frac{1}{2}}D(t) \right]. \]
where
\[ c_8 = c_7 \frac{p - 1}{2(p + 1)} \left[ 1 - \frac{1}{2\theta} + \frac{1}{p + 1} \right] + c_7^2 c(G_1)(1 + B) + \frac{3}{4} c(G_1), \]
\[ c_9 = c_7 \frac{p - 1}{2(p + 1)} \left[ 1 - \frac{1}{2\theta} + \frac{1}{p + 1} \right], \]
and using (4.22), we obtain
\[ c_{10} = \max(c_8, c_9). \]
Moreover, integrating (3.3) over \((t, t_2)\) and using (4.22) and the fact that \(E(t_2) \leq \int_{t_1}^{t_2} E(t)dt\), due to \(t_2 - t_1 \geq \frac{T}{2}\), we obtain
\[ E(t) = E(t_2) + \int_{t_1}^{t_2} \frac{1}{2} \left( g' o \nabla u(t) \right) dt + \int_{t_1}^{t_2} \mu_1(t) \left( 1 - \frac{\bar{c}}{2\pi} - \frac{\beta}{2} \right) \| u(t) \|_{L^2(G_1)}^2 dt \]
\[ + \int_{t_1}^{t_2} \mu_1(t) \left( \frac{\bar{c}}{2\pi} + \frac{\beta}{2} \right) \| \rho(t) \|_{L^2(G_1)}^2 dt \]
\[ + \delta \int_{t_1}^{t_2} \| \nabla u(t) \|_{L^2}^2 dt \leq \int_{t_1}^{t_2} E(t) dt, \]
hence by exploiting (4.22) we arrive at
\[ E(t) \leq c_{11} \left[ D(t)^2 + E(t)^2 D(t) \right]. \]
(4.24)
Then a simple application of Young’s inequality gives, for all \(t \geq 0\)
\[ E(t) \leq c_{12} D(t)^2, \]
(4.25)
where \(c_{11}, c_{12}\) are some positive constants. Therefore, using the formula from (4.25), we get
\[ E(t) \leq c_{12} |E(t) - E(t + 1)|. \]
Here we choose \(c_{12} > 1\). Thus by Lemma 2.4, we obtain for \(t \geq 0\)
\[ E(t) \leq E(0) e^{-\sigma t} \quad \text{with} \quad \sigma = \ln \left( \frac{c_{12}}{c_{12} - 1} \right). \]

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### References


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