Malaya Journal of Matematik, Vol. 6, No. 3, 563-568, 2018

https://doi.org/10.26637/MJM0603/0016

A type of Lusin's theorem for regular monotone auto continuous measure space

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Abstract
In this paper we contemplate on the study on monotone (uniformly) auto continuous regular set multi functions. Lusin's theorem is generalized to uniformly auto continuous measure spaces. Egroffs theorem for fuzzy measures is proved on this theoretical setting.

Keywords
Uniformly auto continuous, Monotone measure Regular, Lusin's theorem.

AMS Subject Classification
47H05, 26E50.

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Article History: Received **16** March **2018**; Accepted **18** June **2018**

Contents

1. Introduction

In classical real analysis, Lusin's theorem governs the continuity and the approximation of measurable functions on metric spaces in non-additive measure theory this well known theorem was generalized by Wu and Ha [13] under the conditions of continuity and auto continuity. This was furthered by Jiang et. al [4, 5] and Li and Yasuda [8], Li and Yasuda [8] proved Lusin's theorem on finite fuzzy measure spaces by using weak null–additivity. Alina Cristiana Gavriluj [1] using monotone uniformly auto continuous functions has proved the Lusin type theorems. We have proved Lusin's theorem of uniformly auto continuous functions.

The paper is organized as follows. Section 2 Preliminaries and we have exposed the interconnection between uniform auto continuity and pseudo metric generating property. In Section 3 condition (E) of monotone measures is elaborated. Section 4 is devoted to the issue of regularity of monotone measures.

2. Preliminaries

We suppose that (X,P) is a metric space and O and C are the classes of all open and closed sets in (X,P) respectively. Let B is the Borel σ-algebra on X. It is the smallest σ algebra containing O. Let F denote the class of all finite real valued measurable functions on the Borel measurable space (X,B) unless stated otherwise all the subsets mentioned are supposed to belong to B, and all the function mentioned are supposed to belong to F.

Definition 2.1. *A non-negative set function $\mu : B \to [0,+\infty]$ is said to be finite if $\mu(X) < \infty$.*

Definition 2.2. *A non-negative set function $\mu : B \to [0,\infty]$ is continuous from below if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \uparrow A$and there exists n_0 with $\mu(A_{n0}) < \infty$.*

Definition 2.3. *A non-negative set function $\mu : B \to [0,\infty]$ is continuous from above if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \downarrow A$ and there exists n_0 with $\mu(A_n) < \infty$.*

Definition 2.4. *A non–negative set function $\mu : B \to [0,\infty]$ is continuous if μ is continuous from below and above.*

Definition 2.5 ([3]). *A non–negative set function $\mu : B \to [0,\infty]$ is order continuous if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \downarrow \emptyset$.*

Definition 2.6 ([3]). *A non–decreasing set function $\mu : B \to [0,\infty]$ is exhaustive if $\lim_{n\to\infty} \mu(E_n) = 0$ for any infinite disjoint sequence $\{E_n\}_{n\in N}$.*

Definition 2.7 ([6]). *A non–decreasing set function $\mu : B \to [0,\infty]$ is strongly order continuous if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \downarrow A$ and $\mu(A) = 0$.*

Definition 2.8 ([12]). *A non–decreasing set function $\mu : B \to [0,\infty]$ is null additive if $\mu(E\cup F) = \mu(F)$ whenever $E,\, F \in B$ and $\mu(F) = 0$*

Definition 2.9 ([12]). *A non-decreasing set function $\mu : B \to [0,\infty]$ is weakly null additive if $\mu(E\cup F) = 0$ whenever $\mu(E) = \mu(F) = 0$.*

Definition 2.10 ([11]). *A non-decreasing set function $\mu : B \to [0,\infty]$ is called auto continuous from above (resp from below) if for every $\varepsilon > 0$ and every $A \in B$ there exists $\delta = \delta(A,\varepsilon) > 0$ such that $m(A) - \varepsilon \leqslant m(A\cup B) + \varepsilon$(resp. $m(A) - \varepsilon \leqslant m(\frac{AB}{B}) \leqslant m(A) + \varepsilon$). Whenever $B \in B$, $A\cap B = \emptyset$ (resp $B \subset A$) and $m(B) < \delta$ holds.*

Definition 2.11 ([10]). *A monotone measure on B is an extended real valued set function $\mu : B \to [0,\infty]$ satisfying*

1. $\mu(\varphi) = 0$
2. *$\mu(A) \leqslant \mu(B)$ whenever $A \subset B$ and $A, B \in B$*

when μ is a monotone measure the triple $(X, B\mu)$ is called a monotone measure space. In this paper we always assume that μ is a monotone measure on B.

Definition 2.12. *μ is called countably weakly null additive if $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$ whenever $\{A_n\}_{n\in N} \subset B$ and $\mu(A) = 0$, $n = 1,2,\ldots$.*

Definition 2.13 ([1]). *μ is called null continuous if $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$ for every increasing sequence $\{A_n\}_{n\in N} \subset B$ such that $\mu(A) = 0$, $n = 1,2,\ldots$.*

Definition 2.14 ([9]). *μ is countably weakly null additive if and only if μ is both weakly null additive and null continuous.*

Theorem 2.15 ([1]). *If μ is strongly order continuous and weakly null additive then it is null continuous.*

Theorem 2.16 ([9]). *If μ is strongly order continuous and weakly null additive then it is countably weakly null additive.*

We define the Hausdorff pseudo metric h on $P_f(x)$ as $h(M,N) = \max\{e(M,N), e(N,M)\}$ for every $M,N \in P_f(X)$ where
$e(M,N) = \sup_{X\in M} d(X,N)$, e is called the excess of M over N.

Definition 2.17. *If $\mu : C \to P_f(x)$ is a set multifunction by $|\mu|$ we mean the real extended value set function defined by $|\mu|(A) = |\mu(A)|$ for every $A \in C$ where $P_f(x)$ is the farmly of closed non-void sets of X.*

Definition 2.18. *A set multi function $\mu : C \to P_f(x)$ is said to be uniformly auto continuous if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ so that for every $A, B \in C$ with $|\mu(B)| < \delta$ we have $h(\mu(A\cup B),\, \mu(A)) < \varepsilon$.*

Remark 2.19. *If $\mu : C \to P_f(x)$ is uniformly auto continuous then it Has the pseudo metric generating property.*

3. Condition (E) of monotone measures

In [7] Li Introduced the concept of condition (E) of a set function and proved that it is a necessary and sufficient condition for Egroffs theorem. In this Section We present some properties for the condition (E) (Or Egroffs condition) of a monotone measure. They play important roles in proving regularity and Lusin's theorem.

Definition 3.1 ([7]). *A set Function $\mu : B \to [0,\infty]$ is said to fulfill condition (E) if for every double sequence $\{E_n^{(m)} \mid n, M \in N\} < B$ satisfying the conditions:*
For any fixed $m = 1,2,\ldots E_n^{(m)} \to E^m (n\to\infty)$ and $\mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0$.
There exits increasing sequences $\{n_i\}_{i\in N}$ and $\{m_i\}_{i\in N}$ of natural numbers such that $\lim_{n\to\infty} \mu\left(\bigcup_{r=k}^{\infty} E_{n_r}^{(m_r)}\right) = 0$.

Theorem 3.2 ([7]). *If μ is a finite continuous monotone measure then it fulfills the condition (E)*

Theorem 3.3 ([7]). *If μ fulfills condition (E) then it is strongly order continuous and hence it is order continuous and exhaustive.*

Theorem 3.4 ([9]). *If μ is weakly null additive and fulfills condition (E) then it is null continuous and countably weakly null additive.*

Theorem 3.5 ([9]). *The following conditions are equivalent:*

1. *μ fulfills the Condition (E).*
2. *For any $\varepsilon > 0$ and a double sequence $\{\left\{E_n^{(m)} \mid n, m \in N\right\}$ $\subset B$ satisfying the conditions for any fixed $m = 1,2,\ldots$, $E_n^{(m)} \to E^{(m)} (n\to\infty)$ and $\mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0$. There exists an increasing sequence $\{n_m\} M \in N$ of natural numbers such that $\mu\left(\bigcup_{m=1}^{\infty} E_{n_m}^{(m)}\right) < \varepsilon$.*

Theorem 3.6 ([9]). *The following conditions are equivalent*

1. *μ is weakly null additive and fulfills condition (E)*

2. *For any $\varepsilon > 0$ and double sequence $\left\{E_n^{(m)} | n, m \in N\right\}$ $\subset B$ satisfying the conditions for any fixed $m = 1, 2, \ldots$, $E_n^{(m)} \to E^{(m)} (n \to \infty)$ and $\mu\left(E^{(m)}\right) = 0$ there exists an increasing sequence $\{n_m\}_{M \in N}$ of natural numbers such that $\mu\left(\bigcup_{m=1}^{\infty} E_{n_m}^{(m)}\right) < \varepsilon$.*
3. *For any fixed K∈N and any double sequence $\left\{E_n^{(m)} | n, m \in N\right\} \subset B$ satisfying conditions for any fixed m = 1,2,... $E_n^{(m)} \to E^{(m)} (n \to \infty)$ and $\mu\left(E^{(m)}\right) = 0$ there exists an increasing sequence $\{n_m(K)\} m \in N$ of natural numbers such that $\mu\left(\bigcup_{m=1}^{\infty} E_{n_{m(k)}}^{(m)}\right) < \frac{1}{K}$. and for any fixed $m = 1, 2, \ldots$, $n_{m(k)} \leqslant n_{m(K+1)}$.*

Note: In the above statement (3) for the double subsequence $\left\{E_{n_{m(K)}}^{(m)} \mid m, K \in N\right\}$ of $\left\{E_n^{(m)} | n, m \in N\right\}$, We have $\bigcup_{m=1}^{\infty} E_{n_{m(k)}}^{(m)}$ $\supseteq \bigcup_{m=1}^{\infty} E_{n_{m(k+1)}}^{(m)}$ K = 1,2,... and $\lim_{K \to \infty} \mu\left(\bigcup_{m=1}^{\infty} E_{n_{m(k)}}^{(m)}\right) = 0$.

4. Regularity of monotone measures on metric spaces

It is known that every probability measure P on a metric space is regular. Now we prove that this property is also enjoyed by monotone measures with condition (E).

Definition 4.1. *A set function μ is called regular if for every $A \in B$ and $\varepsilon > 0$ there exists a closed set F_ε and an open set G_ε of X such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon - F_\varepsilon) < \varepsilon$.*

Theorem 4.2. *If μ fulfils condition (E) and auto continuous then μ is regular.*

Proof. Let A be the class of all sets $A \in B$ such that for any $\varepsilon > 0$ there exist a closed set F_ε and an open set G_ε satisfying $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon - F_\varepsilon) < \varepsilon$.

To prove the theorem it is sufficient to show that $B \subset A$.

It is easy to verify that A is an algebra we shall now prove that A is closed under the formation of pairwise disjoint countably unions. Let $\left\{A^{(m)}\right\}_{M \in N} < A$ be the sequence of pairwise disjoint sets and $\varepsilon > 0$ be given.

By Remark 2.19, since μ is auto continuous it possesses pseudo metric generating properly.

From the definition of A and $A^{(m)} \in A$ we know that for every $m = 1, 2, \ldots$, there exists a sequence $\left\{G_n^{(m)}\right\}_{n=1}^{\infty}$ is decreasing and $\left\{F_n^{(m)}\right\}_{m=1}^{\infty}$ is increasing. Therefore for any fixed $m = 1, 2, \ldots$, $\left\{G_n^{(m)} - F_n^{(m)}\right\}_{n=1}^{\infty}$ is a decreeing sequence of sets with respect to n an as $n \to \infty$

$$G_n^{(m)} - F_n^{(m)} \to \bigcap_{n=1}^{\infty} G_n^{(m)} - F_n^{(m)}$$

Denote $D_m = \bigcap_{n=1}^{\infty}\left(G_n^{(m)} - F_n^{(m)}\right)$ Then $G_n^{(m)} - F_n^{(m)} \to D_m$ as $n \to \infty$, noting that $\mu(D_m) \leqslant \mu\left(G_n^{(m)} - F_n^{(m)}\right) < \frac{1}{n}$; $n = 1, 2, \ldots$.

We have $\mu(D_m) = 0$; $m = 1, 2, \ldots$. Applying Theorem 3.6 to the double sequence $\left\{G_n^{(m)} - F_n^{(m)} | n, m \in N\right\}$ and the sequence $\{D_m\}_{m=1}^{\infty}$ of sets then for $\delta > 0$ mentioned above there exists a sub sequence $\left\{G_{n_m}^{(m)} - F_{n_m}^{(m)}\right\}$ of $\left\{G_n^{(m)} - F_n^{(m)}\right\}$, such that $\mu\left(\bigcup_{m=1}^{\infty} G_{n_m}^{(m)} - F_{n_m}^{(m)}\right) < \delta$.

On the other hand $\bigcup_{m=1}^{\infty} F_{n_m}^{(m)} - \bigcup_{m=1}^{K} \to \varphi(K \to \infty)$.

We observe that μ fulfills condition (E) if follows from Theorem 3.3 that it is order continuous. So we have

$$\lim_{K \to \infty} \mu\left(\bigcup_{m=1}^{\infty} F_{n_m}^{(m)} - \bigcup_{m=1}^{K} F_{n_m}^{(m)}\right) = 0$$

There exists to such that

$$\mu\left(\bigcup_{m=1}^{\infty} F_{n_m}^{(m)} - \bigcup_{m=1}^{Ko} F_{n_m}^{(m)}\right) = \delta$$

Denote $G_\varepsilon = \bigcup_{m=1}^{\infty} G_{n_m}^{(m)}$ and $F_\varepsilon = \bigcup_{m=1}^{Ko} F_{n_m}^{(m)}$ then G_ε is an open set and F_ε is a closed set and $F_\varepsilon \subset \bigcup_{m=1}^{\infty} A^{(m)} \subset G_\varepsilon$ and

$$\begin{aligned}
&\mu(G_\varepsilon - F_\varepsilon) \\
&= \mu\left(\bigcup_{m=1}^{\infty} G_{n_m}^{(m)} - \bigcup_{m=1}^{\infty} F_{n_m}^{(m)}\right) \\
&\leqslant \mu\left(\bigcup_{m=1}^{\infty} G_{n_m}^{(m)} - F_{n_m}^{(m)}\right) \cup \left(\bigcup_{m=1}^{\infty} G_{n_m}^{(m)} - \bigcup_{m=1}^{\infty} F_{n_m}^{(m)}\right) < \varepsilon \\
&\therefore \bigcup_{m=1}^{\infty} A^{(m)} \in A
\end{aligned}$$

Thus we proved that A is a σ-algebra.

We know that for any closed set $F \in$ C there exists sequence of open sets $\{G_m\}_{m=1}^{\infty}$ such that $G_m - F \to \varphi(m \to \infty)$. Therefore it follows from the order continuity of μ that $\lim m \to \infty \mu(G_m - F) = 0$. Thus $C \subset A$ since A is a closed under the formation of complements we have $O \subset A$.

Thus shows that A is a σ-algebra containing O.

$\therefore B \subset A$. □

Corollary 4.3. *Under the assumptions of Theorem 4.2 for any $E \in B$ there exists an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets and a decreasing sequence $\{G_n\}_{n=1}^{\infty}$ of open sets such for every $n = 1, 2, \ldots$,*
$F_n \subset E \subset G_n \mu(G_n - E) < \frac{1}{n}$ and $\mu(E - F_n) < \frac{1}{n}$.

By Theorem 4.2 and invoking the condition that μ is continuous from below, exhaustive and auto continuous then if fulfills the condition (E) we can obtain the following corollary immediately.

Lemma 4.4 ([8]). *μ is weakly null additive if and only if for any $\varepsilon > 0$ and any double sequence $\left\{A_n^{(K)}\right\} n \geqslant 1, K \geqslant 1 \subset B$ satisfying*
$A_n^{(K)} \to D_n (K \to \infty)$, $\mu(D_n) = 0$, $n = 1, 2, \ldots$, there exists a subsequence $\left\{A_n^{(Km)}\right\}$ of $\left\{A_n^{(K)}\right\}$, $n \geqslant 1$, $K \geqslant 1$ such that $\mu\left(\bigcup_{n=1}^{\infty} A_n^{(Kn)}\right) < \varepsilon (K_1 < K_2 < \cdots)$.

5. Lusin's theorem on monotone measure spaces

In [7] Li Proved that the condition (E) is a necessary and sufficient condition for Egroffs theorem on monotone measure spaces

Theorem 5.1 ([7]). *The following conditions are equivalent:*

1. *μ fulfils condition(E)*
2. *For $f \in F$ and $\{f_n\}_{n \in N} \subset F$ if any $\{f_n\}$ converges to f almost every where on X then for any $\varepsilon > 0$ there is a subset $X_\varepsilon \in B$ such that $\mu(X / X_\varepsilon) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on X_ε.*

Theorem 5.2 (Egroffs theorem). *let a monotone measure μ fulfils condition (E) and auto continuous. If $\{f_n\}$ converges to f almost everywhere on X then for any $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \in C$ such that $\mu(X - F_\varepsilon) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on F_ε.*

Proof. Given that $\{f_n\}$ converges to f almost everywhere on X. Then there exists an increasing sequence $\{X_m\}_{m=1}^{\infty} \subset$B such that $\mu\left(X - \bigcup_{m=1}^{\infty} X_m\right) = 0$ and f_n converges to f on X_m uniformly for any fixed $m = 1, 2, \ldots$.

Denote $H = X - \bigcup_{m=1}^{\infty} X_m$ then $\mu(H) = 0$.

μ is weakly null additive. Then for any E$\in$B there exists a sequence $\left\{F^{(k)}\right\}_{k=1}^{\infty}$ of closed sets and a sequence $\left\{G^{(k)}\right\}_{k=1}^{\infty}$ of open sets such that for every
$K = 1, 2, \ldots$, $F^{(K)} \subset E \subset G^{(K)} \mu\left(G^{(K)} - E\right) < \frac{1}{K}$ and $\mu\left(E - F^{(k)}\right) < \frac{1}{k}$.

Here the sequence $\left\{F^{(k)}\right\}_{k=1}^{\infty}$ is increasing in K and the sequence $\left\{G^{(k)}\right\}_{k=1}^{\infty}$ is decreasing in K.

$\therefore$ for every fixed X_m, $m = 1, 2, \ldots$. There exists a sequence $\left\{F_m^{(k)}\right\}_{k=1}^{\infty}$ of closed sets satisfying $F_m^{(k)} \subset X_m$ and $\mu\left(X_m - F_m^{(k)}\right) < \frac{1}{k}$ for any $K = 1, 2, \ldots$, without loss of generality we can assume that for fixed $m = 1, 2, \ldots$. $\left\{X_m - F_m^{(k)}\right\}_{k=1}^{\infty}$ is decreasing as $K \to \infty$ thus

$X_m - F_m^k \downarrow \bigcap_{k=1}^{\infty} \left(X_m - F_m^{(k)}\right)$ as K$\to \infty$. Write

$$\begin{aligned} D_m &= \left(\bigcap_{k=1}^{\infty} \left(X_m - F_m^{(k)}\right)\right) \\ &= \lim_{k \to \infty} \mu\left(X_m - F_m^{(k)}\right) \\ &= 0 \end{aligned}$$

and log the weakly null additivity of μ.

We get $\mu(D_m) = \mu\left(\bigcap_{k=1}^{\infty} \left(X_m - F_m^{(K)}\right) \cup H\right) = 0$, $m = 1, 2, \ldots$

By Lemma 4.4 the double sequence, $\left\{\left(X_m - F_m^{(k)}\right) \cup H\right\}$ of sets and the sequence $\{D_m\}_{m=1}^{\infty}$ of sets then for any $\varepsilon > 0$ there exists a subsequence $\left\{\left(X_m - F_m^{(km)}\right) \cup H\right\}$ of $\left\{\left(X_m - F_m^{(k)}\right) \cup H\right\}$ such that
$\mu\left(\bigcup_{m=1}^{\infty} \left(X_m - F_m^{(km)}\right) \cup H\right) < \varepsilon$.

Since $X - \bigcup_{m=1}^{\infty} F_m^{(km)} \subset \bigcup_{m=1}^{\infty} \left(X_m - F_m^{(k)}\right) \cup H$.

We have $\mu\left(X - \bigcup_{m=1}^{\infty} F_m^{(km)}\right) < \varepsilon$.

From $X - \bigcup_{m=1}^{N} F_m^{(km)} \downarrow X - \bigcup_{m=1}^{\infty} F_m^{(km)}$ as N$\to \infty$ and the continuity from above of μ, We have

$$\lim_{N \to \infty} \mu\left(X - \bigcup_{m=1}^{N} F_m^{(km)}\right) = \mu\left(X - \bigcup_{m=1}^{\infty} F_m^{(km)}\right) < \varepsilon$$

$\therefore$ there exists N_0 such that

$$\mu\left(X - \bigcup_{m=1}^{N_0} F_m^{(km)}\right) < \varepsilon$$

Denote $F_\varepsilon = \bigcup_{m=1}^{N_0} F_m^{(km)}$ then F_ε is a closed set $\mu(X - F_\varepsilon) < \varepsilon$ and from $F_\varepsilon \subset \bigcup_{m=1}^{N_0} X_m$, we know that $\{f_n\}_n$ converges to f uniformly on F_ε. □

Theorem 5.3. *Lusin's Theorem*

Let a monotone measure μ fulfils condition (E) auto continuous and f be a real valued measurable function on X. Then for each $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \in C$ such that $\mu(X - F_\varepsilon) < \varepsilon$ and f / F_ε the restriction of f to F_ε is continuous on F_ε.

Proof. We prove the theorem in the following two situations.

1. Suppose that f is a simple function
   (i.e.,) $f(x) = \sum_{n=1}^{s} C_k \chi_{E_n}(x); x \in \chi$,
   where $\chi_{E_n}(x)$ is the characteristic function of the set E_n

and X= $\bigcup_{n=1}^{s} E_n$ (a disjoint finite union). For every fixed E_n, $n = 1,2,\ldots$, μ being weakly null additive then for any $E \in B$ there exists a sequence $\left\{F^{(k)}\right\}_{k=1}^{\infty}$ of closed set and a sequence $\left\{G^{(k)}\right\}_{k=1}^{\infty}$ of open sets such that for each $k = 1,2,\ldots$, $F^{(k)} \subset E \subset G^{(k)}$,

$$\mu\left(G^{(k)} - E\right) < \frac{1}{k} \text{ and } \mu\left(E - F^{(k)}\right) < \frac{1}{k}.$$

Applying this to the double sequence $\left\{E_n - F_n^{(k)}\right\}$, $n = 1,2,\ldots$,
$k = 1,2,\ldots$ of sets there exists a sub sequence $\left\{E_n - F_n^{(kn)}\right\}$ of $\left\{E_n - F_n^{(k)}\right\}$ such that

$$\mu\left(\bigcup_{n=1}^{s}\left(E_n - F_n^{(kn)}\right)\right) < \varepsilon.$$

We take $F_\varepsilon = \bigcup_{n=1}^{s} F_n^{(kn)}$ then f is continuous on the closed set F_ε on X and

$$\mu\left(X - F_\varepsilon\right) \leqslant \mu\left(\bigcup_{n=1}^{s} E_n - \bigcup_{n=1}^{s} F_n^{(kn)}\right)$$
$$\leqslant \mu\left(\bigcup_{n=1}^{s}\left(E_n - F_n^{(kn)}\right)\right) < \varepsilon$$

2. Let f be a real valued measurable function. Then there exists a sequence $\{\varphi_n(x)\}_{n=1}^{\infty}$ of simple functions such that $\varphi_n \to f(n \to \infty)$ on X. By the result obtained in (a) for each simple function φ_n and every $k = 1,2,\ldots$, there exists closed set $X_n^{(k)} \subset X$ such that φ_n is continuous on $X_n^{(k)}$ and $\mu\left(X - X_n^{(k)}\right) < \frac{1}{k}$ $k = 1,2,\ldots$.

   There is no loss of generality in assuming the sequence $\left\{X_n^{(k)}\right\}_{n=1}^{\infty}$ of closed sets in increasing with respect to k for any fixed n.

   $\therefore \quad X - X_n^{(k)} \downarrow \bigcap_{k=1}^{\infty}\left(X - X_n^{(k)}\right)$ as $k \to \infty$.

   And thus we have

$$\mu\left(\bigcap_{k=1}^{\infty}\left(X - X_n^{(k)}\right)\right) = \lim_{n\to\infty} \mu\left(X - X_n^{(k)}\right)$$
$$= 0,\ n = 1,2,\ldots.$$

   Now we consider the double sequence $\left\{X - X_n^{(K)} | n \geqslant 1, K \geqslant 1\right\}$ of sets. By Lemma 4.4 for every m, $m = 1,2,\ldots$, we may take a sub sequence $\left\{X - X_n^{Kn^{(m)}}\right\}_{n=1}^{\infty}$ of $\left\{X - X_n^{(K)}\right\}$, $n \geqslant 1$, $K \geqslant 1$ such that

$$\mu\left(\bigcup_{n=1}^{\infty} X - X_n^{Kn^{(m)}}\right) < \frac{1}{m} \text{ namely } \mu\left(X - \bigcap_{n=1}^{\infty} X_n^{Kn^{(m)}}\right) < \frac{1}{m}.$$

   Since the double sequence $\left\{X - X_n^{(K)}\right\}$, $n \geqslant 1$, $K \geqslant 1$ of sets is decreasing in k for fixed n, without any loss of generality we can assume that for fixed n, $n = 1,2,\ldots$, $K_n^{(1)} < K_n^{(2)} < \cdots < K_n^{(M)} \ldots$.

   Write $H_M = \bigcap_{n=1}^{\infty} X_n^{Kn^{(m)}}$, $m = 1,2,\ldots$.

   Then we obtain a sequence $\{H_m\}_{m=1}^{\infty}$ of closed sets satisfying
   $H_1 \subset H_2 \subset \cdots$

$$\mu\left(X - \bigcup_{m=1}^{\infty} H_m\right) = \lim_{n\to\infty} \mu\left(X - H_m\right) = 0.$$

   Noting that φ_n is continuous on $X_n^{Kn^{(M)}}$ and $H_m \subset X_n^{Kn^{(m)}}$, $n = 1,2,\ldots$. $\therefore$ For each H_m, φ_n is continuous on H_m for every $n = 1,2,\ldots$.

   Since $\varphi_n \to F(n \to \infty)$ on X by Egroffs Theorem there exists an increasing sequence $\{X_m\}_{m=1}^{\infty}$ of closed sets satisfying

$$X - X_m \to X - \bigcup_{m=1}^{\infty} X_m (n \to \infty)$$

   $\mu\left(X - \bigcup_{m=1}^{\infty} X_m\right) = 0$ and $\{\varphi_n\}$ converges to f uniformly on closed set for each $m = 1,2,\ldots$.

   Considering the sequence $\{(X - H_m) \cup (X - X_m)\}_{m=1}^{\infty}$ of sets then as $m \to \infty$

$$(X - H_m) \cup (X - X_m) \to \left(X - \bigcup_{m=1}^{\infty} H_m\right) \cup \left(X - \bigcup_{m=1}^{\infty} X_m\right)$$

   By using the continuity from above and weakly null additivity of fuzzy measures we have

$$\lim_{m\to\infty} \mu\left((X - H_m) \cup (X - X_m)\right)$$
$$= \mu\left(\left(X - \bigcup_{m=1}^{\infty} H_m\right) \cup \left(X - \bigcup_{m=1}^{\infty} X_m\right)\right) = 0$$

   That is $\lim_{n\to\infty} \mu\left(X - H_m \cap X_m\right) = 0$.

   Therefore given $\varepsilon > 0$ we can take $m_o\varepsilon$ such that $\mu\left(X - H_{mo} \cap X_{mo}\right) < \varepsilon_s$. Put $F_\varepsilon = H_{mo} \cap X_{mo}$ then F_ε is a closed set and $\mu\left(X - F_\varepsilon\right) < \varepsilon$ we how show that f is continuous on F_ε. In fact $F_\varepsilon \subset H_{mo}$ and φ_n is continuous on H_{mo}. Therefore φ_n is continuous on F_ε for every $n = 1,2,\ldots$. We observe that $\{\varphi_n\}$ converges to f on F_ε uniformly then f is continuous on F_ε. $\square$

References

[1] A. Alin Cristiana Gavrilut, Lusin type Theorem for regular monotone uniformly auto continuous set multifunctions, *Fuzzy Sets and Systems,* 6(2010), 2909–2918.

[2] S. Asahina, K. Uchino and T. Murofushi, Relationship among continuity conditions and null additivity conditions in non-additive measure theons, *Fuzzy Sets and Systems,* 157(2006), 691–698.

[3] I. Dobrakov, On sub measures I, *Dissertation Math.,* 112(1974), 1–35.

[4] Q. Jiang and H. Suzuku, Fuzzy Measures on Metric spaces, *Fuzzy Sets and Systems,* 83(1996), 99–106.

[5] Q. Jiang, S. Wang and D. Zion, A further investigation for fuzzy measures on metric spaces, *Fuzzy Sets and Systems,* 105(1999), 293–297.

[6] J. Li, Order continuous of monotone set function and convergence of measurable functions sequence, *Applied Mathematics and Computation,* 135(2002), 211–218.

[7] J. Li, A further investigation for Egroffs theorem with respect to monotone set functions, *Kybernrtika,* 39(2003), 753–760.

[8] J. Li and M. Yasuda, Lusin's theorem on fuzzy measure spaces, *Fuzzy Sets and Systems,* 146(2004), 121–133.

[9] J. Li and R. Mesiar, Lusin's theorem on monotone measure spaces, *Fuzzy Sets and Systems,* 175(2001), 75–86.

[10] E. Pap, *Null Additive Set Functions,* Kluwev Dordrecht, 1995.

[11] E. Pap, The continuity of the null additive fuzzy measures, *Novi Sad Journal of Mathematics,* 27(2)(1997), 37–47.

[12] Z. Wang and G. J. Kliv, *Generalised Measure Theory,* Springer, 2009.

[13] C. Wa and M. Ha, On the Regularity of fuzzy Measures on metric fuzzy measure spaces, *Fuzzy Sets and Systems,* 66(1994), 373–379.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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