Characterization of fuzzy number fuzzy measure using fuzzy integral

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Abstract
By using the concepts of fuzzy number fuzzy measures \([2, 3]\) and fuzzy valued functions \([3]\) a theory of fuzzy integrals is investigated. In this paper we have established the fuzzy version of Generalised monotone Convergence theorem and generalised Fatou’s lemma.

Keywords
Fuzzy number, Fuzzy-valued functions, Fuzzy integral, Fuzzy number fuzzy measure.

AMS Subject Classification
26E50, 03E72.

1. Introduction

In this paper, we have introduced a concept of fuzzy number fuzzy measures, defined the fuzzy integral of a function with respect to a fuzzy number fuzzy measure and shown some properties and generalized convergence theorems. It is well-known that a fuzzy-valued function \([3, 4]\) is an extension of a function (point-valued), and the fuzzy integral of fuzzy-valued functions with respect fuzzy measures (point-valued) has been studied \([3]\); so it is natural to ask whether we can establish a theory about fuzzy integrals of fuzzy valued function with respect to fuzzy number fuzzy measures, the answer is just the paper’s purpose. In fact, it is also a continued work of \([3]\). Since what we will discuss in the following is a generalization of works in \([2, 3]\).

Throughout the paper, \(R^+\) will denote the interval \([0, \infty]\), \(X\) is an arbitrary fixed set, \(\tilde{A}\) is a fuzzy \(\sigma\)-algebra \([1]\) formed by the fuzzy-subsets of \(X\), \((X, \tilde{A})\) is a fuzzy measurable space, \(\mu: \tilde{A} \rightarrow R^+\) is a fuzzy measure in Sugeno’s sense, \(\int_A f d\mu\) is the resulting fuzzy integral \([1]\). Operation \(E\{+, \cdot, \land\}\), \(F(x)\) is the set of all \(\tilde{A}\)-measurable functions from \(x\) to \(R^+\), \(M(x)\) denotes the set of all fuzzy measures, \((R^+)\) denotes the set of interval-numbers, \(R^+\) denote the set of fuzzy numbers \([2, 3]\), \(\tilde{F}(x)\) denotes the set of all \(\tilde{A}\)-measurable interval-valued functions \([3]\). \(\bar{F}(x)\) denotes the set of all \(\tilde{A}\)-measurable fuzzy valued functions \([3]\). \(M(x)\) denotes the set of interval number fuzzy measures \([2]\), \(\bar{M}(x)\) denotes the set of fuzzy Number fuzzy Measures \([2]\), we will adopt the preliminaries in \([2–4]\). Here we omit them for brevity, for more details see \([2–4]\).

2. Definitions and Properties

Definition 2.1. Let \(\tilde{f} \in \tilde{F}(x), A \in \tilde{\sigma}, \bar{\mu} \in \bar{M}(x)\). Then the fuzzy integral of \(\tilde{f}\) and \(A\) with respect to \(\bar{\mu}\) is defined as
\[
\int_A \tilde{f} d\bar{\mu} = \left[ \int_A \tilde{f}^- d\mu^- - \int_A \tilde{f}^+ d\mu^+ \right] \text{ where } \tilde{f}(x) = \inf \tilde{f}(x) \text{ and } \tilde{f}^+(x) = \sup \tilde{f}^+(x) \bar{\mu}(x) = \inf \bar{\mu}(x) \text{ and } \mu^+(x) = \sup \mu^+(x).
\]

Definition 2.2. Let \(\tilde{f} \in \tilde{F}(x), A \in \tilde{\sigma}, \bar{\mu} \in \bar{M}(x)\). Then the fuzzy integral of \(\tilde{f}\) and \(A\) with respect to \(\mu\) is defined as
\[
\int_A \tilde{f} d\mu(r) = \sup \{ \tilde{\lambda} \in (0, 1) : r \in \int_A f d\mu \}, \text{ where } \tilde{\lambda}_x = \{ r \in (0, 1) : f(x)(r) > \lambda \} \text{ and } \mu_x \text{ is similar.}
\]

Theorem 2.3. Let \(\varepsilon \in (\tilde{F}(x), \tilde{A}, \bar{\mu} \varepsilon \bar{M}(x))\). Then \(\varepsilon \int_A f d\mu - R^+\) and the following equation holds:
\[
\left( \int_A \tilde{f} d\bar{\mu} \right)_{\lambda} = \int_A \tilde{f}_x d\mu_{\lambda} \text{ for } (0, 1].
\]  

Proof. The condition is sufficient. To prove that the condition is necessary it is enough to verify equation (2.1).

For a fixed \(\tilde{\lambda} \in (0, 1)\) let \(\lambda_n = (1 - 1/n + 1)\tilde{\lambda}\) then \(\lambda_n \uparrow \lambda\).
Theorem 2.5. Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measures have the following property:
\[ \int_B f d\bar{\mu} = \int_A f \bar{\mu} \]
Then we have \( \bar{\mu} \lambda_n \uparrow \lambda \), \( \lambda_{\lambda_n} \uparrow f^+ \), Similarly \( \mu_{\lambda_n} \uparrow \mu_{\lambda} \), \( \mu_{\lambda_n} \uparrow \mu_{\lambda} \). We have \( \int_A f d\bar{\mu} \uparrow \int_A f d\bar{\mu} \uparrow \int_A f d\bar{\mu} \). Hence
\[
\left( \int_A f d\bar{\mu} \right)_{\lambda} = \lim_{n \rightarrow \infty} \int_A f d\bar{\mu}_{\lambda_n} = \int_A f d\bar{\mu}.
\]

Hence the theorem. □

Theorem 2.4. Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measures have the following property:
\[ f_1 \leq f_2 \leq \mu_{\lambda_2} \Rightarrow \int_A f_1 d\mu_1 \leq \int_A f_2 d\mu_2. \]

Proof. \( \lambda \in (0, 1] \). Let \( \lambda_n = (1 - 1/n + 1) \lambda \) then \( \lambda_n \uparrow \lambda \). It is easy to see that
\[
(\tilde{f}_n)(x) = \bigcap_{k < 1} \tilde{f}_{\lambda_k}(x) = \bigcap_{n=1}^{\infty} \tilde{f}_{\lambda_n}(x) = \lim_{n \rightarrow \infty} \tilde{f}_{\lambda_n}(x).
\]
Then we have \( (\tilde{f}_n)_{\lambda_n} \uparrow (\tilde{f}_n)_{\lambda_n} \uparrow (\tilde{f}_n)_{\lambda_n} \). By generalised monotone convergence theorem
\[
\int_A (\tilde{f}_1)_{\lambda_n} d\bar{\mu}_n \uparrow \int_A \tilde{f}_{\lambda_n} d\bar{\mu} = \int_A f_{\lambda_n} d\mu_{\lambda_n} \uparrow \int_A f_{\lambda_n} d\mu_{\lambda_n} \uparrow \int_A f_{\lambda_n} d\mu_{\lambda_n}.
\]
Hence
\[
\left( \int_A (\tilde{f}_1) d\bar{\mu}_1 \right) = \lim_{n \rightarrow \infty} \int_A (\tilde{f}_1)_{\lambda_n} d\bar{\mu}_n = \int_A f_1 d\mu_1 \leq \int_A f_2 d\mu_2.
\]

Hence the proof. □

3. Convergence theorems

In this section we canvass the convergence of sequences of fuzzy integrals.

Theorem 3.1 (Generalised Monotone Convergence theorem).
Let \( \{\tilde{f}_n (n \geq 1), \tilde{f}\} \subset \tilde{F}(x), \{\mu_n (n \geq 1), \mu\} \subset \tilde{M}(x) \).

\[ \begin{align*}
\text{(i)} \quad & \tilde{f}_n \uparrow \tilde{f} \quad \text{on} \ A, \ \lambda \tilde{\mu} \uparrow \tilde{\mu} \quad \Rightarrow \quad \int_A f_n d\mu_n \uparrow \int_A f d\mu \quad (3.1) \\
\text{(ii)} \quad & \lambda \tilde{f} \uparrow \tilde{f} \quad \text{on} \ A, \ \mu_n \uparrow \mu \quad \Rightarrow \quad \int_A f_n d\mu \uparrow \int_A f d\mu. \quad (3.2)
\end{align*} \]

Proof. To prove (i) it is sufficient to verify equation (3.1). For \( \lambda_k = (1 - 1/1 + k) \lambda \) then \( \lambda_k \uparrow \lambda \). By the proof of Theorem 2.3 we obtain
\[
\tilde{f}_k = \lim_{n \rightarrow \infty} \tilde{f}_{\lambda_n} \quad \mu_k = \lim_{n \rightarrow \infty} \mu_{\lambda_n}.
\]
Then

\[
\left( \lim_{n \to \infty} \int_A f_n d\mu_n \right)_{\lambda_k} = \int \lim\left( \int_A f_n d\mu_n \right)_{\lambda_k} = \lim\left( \int_A (f_n)_{\lambda_k} d\mu_n \right) = \int A f_{\lambda} d\mu_{\lambda}.
\]

This proves (i) and (ii) is similar.

\[\square\]

**Theorem 3.2** (Generalised Fatous lemma). Let \( \{f_n (n \geq 1), f\} \subset \mathcal{F}(x)\), \( \{\mu_n (n \geq 1), \lim\mu_k, \lim\mu_n \subset M(x)\).

Then (i) \( \int f_{\lambda} \lim f_n d\mu_n \leq \lim f_n f_{\lambda} d\mu_n \)

(ii) \( \left( \lim_{n \to \infty} \int f_n d\mu_n \right)_{\lambda} \leq \int \lim_{n \to \infty} \left( \int f_n d\mu_n \right)_{\lambda_k} \)

Proof. To prove(i), for \( \lambda \in (0, 1] \) let \( \lambda_k = (1 - 1/k)\lambda \) then \( \lambda_k \to \lambda \).

\[
f_{\lambda} = \lim_{k \to \infty} f_{n\lambda_k} \quad \mu_{\lambda} = \lim_{k \to \infty} \mu_{n\lambda_k}.
\]

Then

\[
\left( \lim_{n \to \infty} \int_A f_n d\mu_n \right)_{\lambda} = \int \lim\left( \int_A f_n d\mu_n \right)_{\lambda_k} = \int \lim\left( f_{\lambda} \right)_{\lambda} d\mu_{\lambda_k} = \int \lim\left( \int f_n d\mu_n \right)_{\lambda_k} d\mu_{\lambda_k} \leq \int \left( \lim_{n \to \infty} f_{\lambda} \right)_{\lambda_k} d\mu_{\lambda_k} = \lim\left( \int f_n d\mu_n \right)_{\lambda_k}.
\]

Hence the theorem. \[\square\]

**References**


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