Observations on $x^2 + y^2 + 2(x + y) + 2 = 10z^2$

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Abstract
The quadratic equation with three unknowns given by $x^2 + y^2 + 2(x + y) + 2 = 10z^2$ is analysed for its non-zero distinct integer solutions. Given a solution, formula for generating sequence of solutions is obtained.

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Second degree equation, three unknowns, lattice points.

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1. Introduction
It is well-known that there are various choices of quadratic equations with three unknowns to obtain lattice points satisfying them [1,18]. Particularly in [2-17], different types of problems are presented. This paper deals with a different quadratic equation with three unknowns given by $x^2 + y^2 + 2(x + y) + 2 = 10z^2$ to obtain a sequence of integral solutions. Further, a general formula for generating sequence of solutions based on the given solution is illustrated.

2. Method of analysis
The quadratic diophantine equation with three unknowns under consideration is

$$x^2 + y^2 + 2(x + y) + 2 = 10z^2 \quad (2.1)$$

Assuming

$$x = u + v \ , \ y = u - v \ , \ u \neq v \neq 0 \quad (2.2)$$

in (2.1), it gives

$$(u + 1)^2 + v^2 = 5z^2 \quad (2.3)$$

Solving (2.3) through various approaches and employing (2.2), different sets of integer solutions to (2.1) are obtained. The above process is illustrated below:

Method: 1

Write 5 as

$$5 = (2 + i)(2 - i) \quad (2.4)$$

Let

$$z = a^2 + b^2 \ , \ a, b \neq 0 \quad (2.5)$$

Using (2.4) and (2.5) in (2.3) and applying factorization, consider

$$u + iv = (2 + i)(a + ib)^2$$

Equating the real and imaginary parts, one gets

$$u = 2(a^2 - b^2 - ab) - 1$$
$$v = a^2 - b^2 + 4ab$$

In view of (2.2), the values of x and y are given by

$$\begin{cases} x = 3a^2 - 3b^2 + 2ab - 1 \\ y = a^2 - b^2 - 6ab - 1 \end{cases} \quad (2.6)$$

Thus, (2.5) and (2.6) represent the non-zero distinct integer solutions to (2.1).
Note: 1

Observe that (2.5) is also written as

\[ 5 = (1 + 2i)(1 - 2i) \tag{2.7} \]

Following the analysis as presented above, the corresponding values of \( x \) and \( y \) are given by

\[
\begin{align*}
    x &= 3a^2 - 3b^2 - 2ab - 1 \\
    y &= -a^2 + b^2 - 6ab - 1
\end{align*}
\]

The above values of \( x \) and \( y \) along with (2.5) represent the integer solutions to (2.1).

Method: 2

One may write (2.3) as

\[ (u + 1)^2 + v^2 = 5z^2 + 1 \tag{2.8} \]

Write 1 as

\[ 1 = \frac{(p^2 - q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2}, \quad p > q > 0 \tag{2.9} \]

Substituting (2.4), (2.5) and (2.9) in (2.8) and using factorization, define

\[
    u + 1 + iv = \frac{(p^2 - q^2 + i2pq)(2 + i)(a + ib)^2}{(p^2 + q^2)}
\]

from which, one obtains

\[
    u + 1 = \frac{1}{(p^2 + q^2)}[2(p^2 - q^2)(a^2 - b^2 - ab) - 2pq(a^2 - b^2 + 4ab)]
\]

\[
    v = \frac{1}{(p^2 + q^2)}[(p^2 - q^2)(a^2 - b^2 + 4ab) + 4pq(a^2 - b^2 - ab)]
\]

Employing (2.2), note that

\[
    \begin{cases}
        x = \frac{1}{(p^2 + q^2)}[(p^2 - q^2)(3a^2 - 3b^2 + 2ab) + 2pq(a^2 - b^2 - 6ab)] - 1 \\
        y = \frac{1}{(p^2 + q^2)}[(p^2 - q^2)(a^2 - b^2 - 6ab) - 2pq(3a^2 - 3b^2 + 2ab)] - 1
    \end{cases} \tag{2.10}
\]

Hence, (2.10) and (2.5) represent integer solutions to (2.1) for suitable choices of \( a \) and \( b \).

Note: 2

In (2.8), one may consider (2.7) for (2.5) and proceeding similarly, another choice for \( x \) and \( y \) is found.

3. Formula for generating sequence of solutions

Let \((x_0, y_0, z_0)\) satisfy (2.1). The solution may be in real integers or in Gaussian integers or in irrational numbers. Let \((x_1, y_1, z_1)\) be the second solution of (2.1), where

\[ x_1 = x_0 + 2h, y_1 = y_0 + 2h, z_1 = h - z_0 \tag{3.1} \]

in which \( h \) is an unknown to be determined.

Substitution of (3.1) in (2.1) gives

\[ h = 2x_0 + 2y_0 + 10z_0 + 4 \tag{3.2} \]

Using (3.2) in (3.1), the second solution \((x_1, y_1, z_1)\) of (2.1) is expressed in the matrix form as

\[
    (x_1, y_1, z_1)^{\text{t}} = M(x_0, y_0, z_0)^{\text{t}}
\]

where \( t \) is the transpose and

\[
    M = \begin{pmatrix}
        5 & 4 & 20 & 8 \\
        4 & 5 & 20 & 8 \\
        2 & 2 & 9 & 4 \\
        0 & 0 & 0 & 1
    \end{pmatrix}
\]

Following the above procedure, the general solution \((x_{n+1}, y_{n+1}, z_{n+1})\) of (2.1) is written in the matrix form as

\[
    \begin{pmatrix}
        x_{n+1} \\
        y_{n+1} \\
        z_{n+1}
    \end{pmatrix} =
    \begin{pmatrix}
        X_n & Y_n & 10X_n & Y_n - 1 \\
        0 & 0 & 0 & 1
    \end{pmatrix} \begin{pmatrix}
        x_0 \\
        y_0 \\
        z_0
    \end{pmatrix}
\]

\[
    n = 0, 1, 2, 3, \ldots \quad \text{where } (X_n, Y_n) \text{ is the general solution of the Pellian equation } Y^2 = 20X^2 + 1.
\]

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