Edge geodetic parameters of snake graphs
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Abstract
In this paper, we investigate the different edge geodetic parameters of triangular snake graph, double triangular snake graph, alternate triangular snake graph, double alternate triangular snake graph, quadrilateral snake graph, double quadrilateral snake graph, alternate quadrilateral snake graph, double alternate quadrilateral snake graph.

Keywords
Edge geodetic set, Snake Graphs, Split edge geodetic Set.

AMS Subject Classification
05CS12.

1 Introduction
An edge geodetic set of $G$ is a set $S \subseteq V(G)$ such that every edge of $G$ is contained in a geodesic joining some pair of vertices in $S$. The edge geodetic number $g_1(G)$ of $G$ is the minimum order of its edge geodetic sets. This concept was introduced in [5]. The concept of split edge geodetic number ($g_{ss}$) was introduced in [7]. A. P. Santhakumaran et al. [6] introduced the concept of restrained edge geodetic number ($eg_r$). In [8] Venkanagouda M Goudar and Shobha introduced total edge geodetic number ($g_{t}$). The concept of strong split geodetic number ($g_{ss}$) was introduced in [1]. Further the concept of nonsplit geodetic number ($g_{ns}$) was introduced in [10]. Let $P_n : v_1, v_2, ..., v_n$ be the path of length $n−1$.

In this paper, we investigated the edge geodetic number, split edge geodetic number, strong split geodetic number of different snake graphs in terms of blocks, regions, vertex covering number. For more details on this theory, we suggest the reader to refer [2,3,4,9].

2. Edge geodetic parameters of Snake graphs

Definition 2.1. The triangular snake $T_n$ is obtained from a path $P_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $u_i$.

Theorem 2.2. Let $G = T_n$ be the triangular snake with $(n \geq 3)$ then $g_1(G) = n + 1$.

Proof. Let $G = T_n$. Let $|V(G)| = 2n − 1$ and $|E(G)| = 3(n − 1)$. Let $S = \{v_1, v_n\} \cup Q$ where $\{v_1, v_n\}$ are the end vertices of path $P_n$ and $Q = \cup u_i$ are the new vertices joined to $v_i$ and $v_{i+1}$. Since each edge of $G$ lies on a geodesic joining any two vertices of $S$ then $S$ is an edge geodetic set of $G$. Hence

$$g_1(G) = |S| = n + 1.$$

Corollary 2.3. For the triangular snake $G = T_n$ $(n \geq 4)$, $eg_r(G) = n + 1$.

Corollary 2.4. Let $T_n$ $(n \geq 3)$ be the triangular snake then $g_{ns}(T_n) = n + 1$.

Corollary 2.5. Let $G = T_n$ be the triangular snake with $(n \geq 3)$, $g_{t}(G) = n + 1$. 

References
Theorem 2.6. The strong split geodetic number for a triangular snake graph $T_n$ is

$$g_{ss}(G) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even}, \\ \frac{3n}{2} - 1 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Let $V(T_n) = \{v_1, v_2, \ldots, v_{2n}\}$. Then $|V(G)| = 3n - 1$ and $|E(G)| = 3(n - 1)$. Let $R = \{r_1, r_2, \ldots, r_{n-1}\}$ be the region set of $G$ where each region consists of $C_3 = \{v_i, u_i, v_{i+1}\}$ and $|R| = r$. Clearly $S$ is an edge geodetic set of $G$ and $V - S$ is connected. Let $S' = S \cup \{v_k\}$ where $\{v_k\}, 3 \leq k \leq n - 2$ is any one internal vertex of $P_n$. Then $V - S'$ is disconnected. Therefore

$$g_{ss}(G) = |S'|,$$

$$= |S| + n + 1,$$

$$= \frac{3n}{2}.$$

Case 2 For $n$ is odd. Consider the geodetic set $S = \{v_1, v_2, \ldots, v_{2n}\}$ and $|S| = V(G)$. Let $S' = S \cup \{v_3, v_5, \ldots, v_{2n-1}\}$. Clearly $V - S'$ is totally disconnected. Therefore

$$g_{ss}(G) = |S'|,$$

$$= |S| + n + 1,$$

$$= \frac{3n - 1}{2}.$$

\[\square\]

Theorem 2.7. For the triangular snake $G = T_n$ with $n \geq 6$, $g_{1s}(G) = r + 2$ where $r$ is the number of regions in $G$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_{2n}\}$ where $v_i \in V(P_n)$ and $\{u_j, j \leq n - 1\}$ are the new vertices joined to $v_i$ and $v_{i+1}$. Then $|V(G)| = 2n - 1$ and $|E(G)| = 3(n - 1)$. Let $R = \{r_1, r_2, r_3, \ldots, r_{n-1}\}$ be the region set of $G$ where each region consists of $C_3 = \{v_i, u_i, v_{i+1}\}$ and $|R| = r$. Clearly $S$ is an edge geodetic set of $G$ and $V - S$ is connected. Let $S' = S \cup \{v_k\}$ where $\{v_k\}, 3 \leq k \leq n - 2$ is any one internal vertex of $P_n$. Then $V - S'$ is disconnected. Therefore

$$g_{1s}(G) = |S'|,$$

$$= |S| + n + 1,$$

$$= \frac{3n - 1}{2}.$$
Theorem 2.13. For the double triangular snake $DT_n$ $(n \geq 3)$,

$$g_{ss}(DT_n) = \begin{cases} \frac{5n-4}{2(n-1)} & \text{if } n \equiv 0 \text{(mod 2)}, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. Let $G = DT_n$. By Definition 2.8, let $\{u_i, w_i/1 \leq i \leq n-1\}$ be the new vertices added to $v_1$ and $v_{i+1}$ in upward and downward direction. Let $\{B_1, B_2, \ldots, B_{n-1}\}$ be the blocks of $DT_n$. Now, the geodetic set of $G$ must have vertices of degree $2$ from each block and hence $S = \{u_i, w_i\}$ is the geodetic set. Hence, to attain the minimum strong split geodetic set of $DT_n$, we construct a vertex set $X \subset V(DT_n)$ as follows:

$$X = \begin{cases} \{v_1, v_2, \ldots, v_{n-1}, u_i, w_i\} & \text{if } n \equiv 0 \text{(mod 2)}, \\ \{v_2, v_4, \ldots, v_{n-1}, u_i, w_i\} & \text{otherwise}. \end{cases}$$

where $1 \leq i \leq n-1$. Then

$$|X| = \begin{cases} \frac{5n-4}{2(n-1)} & \text{if } n \equiv 0 \text{(mod 2)}, \\ \frac{2}{2} & \text{otherwise}. \end{cases}$$

Since each vertex in $V(DT_n)$ is either in $X$ or is adjacent to a vertex in $X$, it follows that $X$ is the minimum strong split geodetic set as $<V-X>$ is totally disconnected. Thus,

$$g_{ss}(DT_n) = \begin{cases} \frac{5n-4}{2(n-1)} & \text{if } n \equiv 0 \text{(mod 2)}, \\ 2 & \text{otherwise}. \end{cases}$$

Theorem 2.14. For the double triangular snake $G = DT_n$ $(n \geq 6)$, $g_{ss}(G) = 2n+1$.

Proof. By Definition 2.8, let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}$ such that $|V(G)| = 2n-1$ and $|E(G)| = 5(n-1)$. Let $S = S_1 \cup S_2 \cup S_3$ where

$$S_1 = \{v_1, v_n\},$$

$$S_2 = \cup\{w_i\},$$

$$S_3 = \cup\{u_i\}$$

and $1 \leq i \leq n-1$. Then $S$ is an edge geodetic set. But $<V-S>$ is connected, consider $S' = S \cup \{v_j\}, 3 \leq j \leq n-2$ is any one internal vertex of $P_n$. Clearly $<V-S'>$ is disconnected. Therefore

$$g_{ss}(G) = |S'| = 2n+1.$$
Here $S$ is the edge geodetic set with minimum cardinality containing $\frac{n^2}{2}$ vertices. Then $< V - S >$ is disconnected. Therefore
\[
g_{1s}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

**Case 2** Suppose $n$ is odd. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, v_n\}$. Clearly $S$ is the edge geodetic set with minimum cardinality containing $\lfloor \frac{n}{2} \rfloor + 1$ vertices. But $< V - S >$ is disconnected. Therefore $S$ is split edge geodetic set. Hence
\[
g_{1s}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

**Definition 2.21.** The double alternate triangular snake $DAT_n$ consists of two alternate triangular snake which have a common path.

**Theorem 2.22.** Let $G = DAT_n (n \geq 4)$ be the double alternate triangular snake, then
\[
g_1(G) = \left\{ \begin{array}{ll}
 b + 3 & \text{if } n \equiv 0(\text{mod}2), \\
 b + 2 & \text{if } n \equiv 1(\text{mod}2),
\end{array} \right.
\]

where $b$ is the number of blocks.

**Proof.** Let $V(G) = \{v_i, u_j, w_j\}$ for $1 \leq i \leq n, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ where $v_i$ are the vertices of $P_n$ and $u_j, w_j$ are the vertices obtained from a path $P_n : v_1, v_2, \ldots, v_n$ by joining $v_i$ and $v_i+1$ alternatively. Then
\[
V(G) = \left\{ \begin{array}{ll}
 2n & \text{if } n \text{ is even}, \\
 2n - 1 & \text{if } n \text{ is odd}.
\end{array} \right.
\]

Let $B = \{B_1, B_2\}$ be the blocks of $G$, where $B_1 = \{b_1, b_2, \ldots, b_{\frac{n}{2}}\}$ and $B_2 = \{b_1', b_2', \ldots, b'_{\frac{n}{2}}\}$ such that $b_i = \{v_i, u_i, w_i, v_{i+1}\}$, $b_i' = \{v_{i+1}, v_{i+2}\}$ and $|B| = b$.

Let us consider the following cases:

**Case 1** when $n$ is even. Let $S = \{u_1, u_2, \ldots, u_{\frac{n}{2}}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, w_{\frac{n}{2}}, \ldots, w_n\}$ be the geodetic set of $G$. But $S$ is not edge geodetic set. Let $S' = S \cup \{v_1, v_n\}$. Clearly $S'$ is an edge geodetic set. Hence $g_1(G) = b + 3$.

**Case 2** when $n$ is odd. Let $S = \{u_1, u_2, \ldots, u_{\frac{n-1}{2}}, v_{\frac{n-1}{2}}, v_{\frac{n-1}{2}+1}, w_{\frac{n-1}{2}}, \ldots, w_n\}$ be the geodetic set of $G$. Let $S' = S \cup \{v_1, v_n\}$. Clearly $S'$ is an edge geodetic set. Hence $g_1(G) = b + 2$.

**Corollary 2.23.** For the double alternate triangular snake $G = DAT_n (n \geq 4)$,
\[
e_{g_1}(G) = \left\{ \begin{array}{ll}
 b + 3 & \text{if } n \equiv 0(\text{mod}2), \\
 b + 2 & \text{if } n \equiv 1(\text{mod}2).
\end{array} \right.
\]

where $b$ is the number of blocks.

**Corollary 2.24.** For the double alternate triangular snake $G = DAT_n (n \geq 5)$,
\[
g_{1s}(G) = \left\{ \begin{array}{ll}
 \frac{3n}{2} & \text{if } n \text{ is even,} \\
 \frac{3n - 1}{2} & \text{if } n \text{ is odd.}
\end{array} \right.
\]

where $b$ is the number of blocks.

**Theorem 2.25.** The strong split geodetic number of double alternate triangular snake $G = DAT_n (n \geq 5)$ is,
\[
g_{ss}(G) = \left\{ \begin{array}{ll}
 \frac{3n}{2} & \text{if } n \text{ is even,} \\
 \frac{3n - 1}{2} & \text{if } n \text{ is odd.}
\end{array} \right.
\]

**Proof.** Let $G = DAT_n$. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n}{2}}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, w_{\frac{n}{2}}, \ldots, w_n\}$ be the vertex set of $G$. We discuss the following cases:

**Case 1** Suppose $n$ is even. Consider $S = \{S_1, S_2, S_3\}$ where $S_1 = \cup u_i/1 \leq i \leq \frac{n}{2}$ and $S_2 = \cup w_j/1 \leq j \leq \frac{n}{2}$. Now, $< V - S >$ contains the set of vertices $\{v_i\}$ for $1 \leq i \leq n$ such that $deg(v_i) \neq 0$. Then $S$ is not strong split geodetic set. Let $S' = \{v_k/1 \leq k \leq n - 1\}$ which are all non adjacent vertices. Clearly $< V - S' >$ is totally disconnected. Therefore $g_{ss}(G) = |S'| = \frac{3n}{2}$.

**Case 2** Suppose $n$ is odd. Let $S = \{u_1, u_2, \ldots, u_{\frac{n}{2}}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, w_{\frac{n}{2}}, \ldots, w_n\}$ be the minimum geodetic set of $G$. Let $S' = S \cup \{v_k\}$ for $1 \leq k \leq n - 1$. Clearly induced subgraph $< V - S' >$ has isolated vertices. Therefore $g_{ss}(G) = |S'| = \frac{3n - 1}{2}$.

**Theorem 2.26.** Let $G = DAT_n (n \geq 6)$ be the double alternate triangular snake, then
\[
g_{1s}(G) = \left\{ \begin{array}{ll}
 n + 3 & \text{if } n \text{ is even,} \\
 n + 2 & \text{if } n \text{ is odd.}
\end{array} \right.
\]

**Proof.** Let $G$ be the double alternate triangular snake with $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n}{2}}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, w_{\frac{n}{2}}, \ldots, w_n\}$. We discuss the following cases:

**Case 1** For $n$ is even, $|V(G)| = 2n$. Let $S = \{v_1, v_n\} \cup S_1 \cup S_2$ where $S_1 = \cup u_i/1 \leq i \leq \frac{n}{2}$ and $S_2 = \cup w_j/1 \leq j \leq \frac{n}{2}$. Thus $I[S] = V(G)$. Clearly $S$ is an edge geodetic set. But $< V - S >$ is connected. Consider $S' = S \cup \{v_k\}, 3 \leq k \leq n - 2$ where $v_k$ is any one internal vertex so that $< V - S' >$ is disconnected. Thus $S'$ is the split edge geodetic set. Therefore
\[
g_{1s}(G) = \left| S' \right| = S + 1 = 2 + \frac{n}{2} + \frac{n}{2} + 1 = n + 3.
\]

**Case 2** For $n$ is odd, $|V(G)| = 2n - 1$. Let $S = \{v_1, v_n\} \cup \{u_j\} \cup \{w_j\}$. Thus $I[S] = V(G)$. Clearly $S$ is an edge geodetic set. But $< V - S >$ is connected. Consider $S' = S \cup \{v_k\}$ for
3 \leq k \leq n - 2 \text{ where } \{v_k\} \text{ is the only internal vertex so that } < V - S' > \text{ is disconnected. Therefore}
\begin{align*}
g_{1s}(G) &= |S'| \\
&= S + 1 \\
&= n + 2.
\end{align*}

Example: For the Double alternate triangular snake graph \(DAT_7\) given in Figure 2. The empty color vertices is its split edge geodetic set.

![Figure 2. G](image)

\(S = \{v_1, u_1, u_2, u_3, w_1, w_2, w_3, v_7\}\) is the edge geodetic set so that \(g_1(DAT_7) = 8\) and \(S' = \{v_1, u_1, u_2, u_3, v_3, v_1, w_1, w_2, w_3, v_7\}\) is the split edge geodetic set. Therefore \(g_{1s}(DAT_7) = 9\).

**Definition 2.27.** A quadrilateral snake \(Q_n\) is obtained from a path \(P_n\) by joining \(v_i\) and \(v_{i+1}\) to new vertices \(u_i\) and \(w_i\) respectively and then joining \(u_i\) and \(w_i\) for \(1 \leq i \leq n - 1\) that is every edge of a path is replaced by a cycle \(C_4\).

**Theorem 2.28.** For the quadrilateral snake \(G = Q_n, (n \geq 2)\), \(g_1(G) = d - 2\) where \(d\) is the diameter of \(G\).

Proof. In order to obtain \(Q_n\) replace every edge of \(P_n\) by a cycle \(C_4\). Let \(|V(G)| = 3n - 2\) and \(|E(G)| = 4(n - 1)\). Let \(B = \{B_1, B_2, \ldots, B_{n-1}\}\) be the blocks of \(G\). Let \(\{u_i, w_i\}\) be the vertices of the block \(B_i\) for \(1 \leq i \leq n - 1\). Let \(S = \{u_1, u_2, \ldots, u_{n-2}, w_{n-1}\}\) be the geodetic set of \(G\). Clearly all the edges lie on any geodesic joining a pair of vertices of \(S\) and hence \(S\) is also an edge geodetic set of \(G\). Since \(d(u_i, w_{n-1}) = \text{diam}(G) = n + 1\) we have \(g_1(G) = |S| = d - 2\).

**Corollary 2.29.** For the quadrilateral snake \(G = Q_n, (n \geq 3)\), \(e_g(G) = d - 2\) where \(d\) is the diameter of \(G\).

**Corollary 2.30.** For the quadrilateral snake \(G = Q_n, (n \geq 2)\), \(g_{ns}(G) = d - 2\) where \(d\) is the diameter of \(G\).

**Theorem 2.31.** Let \(G = Q_n, (n \geq 3)\) be the quadrilateral snake then \(g_{ss}(G) = b + n - 2\) where \(b\) is the number of blocks.

**Proof.** Let \(G = Q_n\). By Definition 2.27, \(V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}\). Let \(B = \{B_1, B_2, \ldots, B_{n-1}\}\) be the blocks of \(G\) where each block contains 4 vertices such that 3 vertices are of degree 2 and one vertex is of maximum degree 4 which is a common vertex for adjacent blocks and \(|B| = b\). Let \(S = \{u_1, u_2, \ldots, u_{n-2}, w_{n-1}\}\) be the geodetic set. But induced subgraph \(< V - S >\) is connected. Let \(S' = S \cup \{v_i / 2 \leq k \leq n - 1\}\) where \(\Delta(v_k) = 4\). Clearly induced subgraph \(< V - S' >\) is an independent set. Hence \(g_{ss}(G) = |S'| = b + n - 2\).

**Theorem 2.32.** For the quadrilateral snake \(G = Q_n, (n \geq 4)\), \(g_{1s}(G) = n\).

Proof. Let \(G = Q_n\). Let \(V(G) = V_1 \cup V_2 \cup V_3\) where \(V_1 = \{v_i/1 \leq i \leq n\}, V_2 = \{v_j/1 \leq j \leq n\}, V_3 = \{v_k/1 \leq j \leq n\}\). Then \(|V(G)| = 3n - 2\) and \(|E(G)| = 4(n - 1)\). Let \(S = \{u_1, u_2, \ldots, u_{n-1}\}\) be the edge geodetic set of \(G\). Clearly, \(< V - S >\) has two components. Therefore \(g_{1s}(G) = |S| = n - 1 + n = n\).

**Definition 2.33.** The double quadrilateral snake \(DQ_n\) is obtained by path \(P_n\) by joining \(v_i\) and \(v_{i+1}\) to new vertices \(u_i\) and \(w_i\) for \(i = 1, 2, \ldots, n - 1\) in upward direction and \(u_i, w_i\) for \(i = 1, 2, \ldots, n - 1\) in downward direction. Let \(V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}\).

**Theorem 2.34.** For the double quadrilateral snake \(DQ_n, (n \geq 3)\), \(g_1(DQ_n) = \frac{n}{2} + n - 1\) where \(m\) is the number of edges in \(DQ_n\).

Proof. Let \(G = DQ_n\). By Definition 2.33, \(|V(DQ_n)| = 5n - 4\) and \(|E(DQ_n)| = 7(n - 1)\). Let \(B_1, B_2, \ldots, B_{n-1}\) be the blocks of \(DQ_n\). Now, consider \(S = \{u_i, w_j\}\) such that in each block \(d(u_i, w_j) = 3\) be the geodetic set of \(G\). Since every edge of \(G\) lies on a geodesic joining \(u_i\) and \(v_i\), then \(S\) is also an edge geodetic set of \(G\). Therefore, \(g_1(G) = \frac{n}{2} + n - 1\).

**Corollary 2.35.** For the double quadrilateral snake \(DQ_n, (n \geq 3)\), \(e_g(DQ_n) = \frac{n}{2} + n - 1\) where \(m\) is the number of edges in \(DQ_n\).

**Corollary 2.36.** Let \(G = DQ_n, (n \geq 3)\) be the double quadrilateral snake, then \(g_{ns}(G) = \frac{n}{2} + n - 1\) where \(m\) is the number of edges in \(DQ_n\).

**Theorem 2.37.** Let \(G = DQ_n\) be the double quadrilateral snake \((n \geq 4)\), \(g_{ss}(DQ_n) = 3n - 3\).

Proof. Let \(G\) be a double quadrilateral snake. Let \(V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}\). Consider \(S = \{u_1, u_2, \ldots, u_{n-2}, w_{n-1}, u_1, u_2, \ldots, u_{n-2}, w_{n-1}\}\) be the geodetic set of \(G\). Let \(S' = S \cup \{v_2, v_3, \ldots, v_{n-1}\}\) such that \(|S'| = V(G)\). Since \(< V - S' >\) contains isolated vertices, then \(g_{ss}(G) = |S'| = 3n - 4\).
Theorem 2.38. For the double quadrilateral snake $G = DQ_n$ $(n \geq 4)$, $g_{1s}(G) = 2n - 1$. 

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ where $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{u_1, u_2, \ldots, u_{n-1}\}$, $V_3 = \{w_1, w_2, \ldots, w_{n-1}\}$, $V_4 = \{w_1', w_2', \ldots, w_n'\}$, $V_5 = \{w_1, w_2, \ldots, w_{n-1}\}$. Let $S = \{u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\}$ be the edge geodetic set. Choose any two vertices of $S$ such that $d(u_i, w_j) = 3$. Then $< V - S >$ is connected so that $S$ is not split edge geodetic set. Let $S' = S \cup \{v_k\}$ where $v_k/2 \leq k \leq n - 1$ is only one internal vertex of path $P_n$. But $< V - S >$ is disconnected, therefore $g_{1s}(G) = |S'| = 2n - 2 + 1 = 2n - 1$. 

Example: For a double quadrilateral snake graph $DQ_5$ given in Figure 3. The empty color vertices is its split edge geodetic set.

![Figure 3. G](image)

$S = \{u_1, u_2, u_3, u_4, w_1', w_2', w_3', w_4'\}$ is the edge geodetic set so that $g_1(DQ_5) = 8$ and $S' = \{v_1, u_1, u_2, u_3, u_4, w_1', w_2', w_3', w_4'\}$ is the split edge geodetic set. Therefore $g_{1s}(DQ_5) = 9$.

Definition 2.39. The alternate quadrilateral snake $AQ_n$ is obtained from a path by joining $v_i$ and $v_{i+1}$ (alternatively) to new vertices $u_i$ and $w_i$ respectively and then joining $u_i$ and $w_i$.

Theorem 2.40. Let $G = AQ_n(n \geq 4)$ be an alternate quadrilateral snake, then

$$g_1(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 0 \pmod{2}, \\ \frac{n+2}{2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

where $b$ is the number of blocks.

Proof. Let $G = AQ_n$. Let $\{B_1, B_2\}$ be the number of blocks in $G$ such that $\{B_1, B_2\} = b$. We observe that $B_1 = \{b_1, b_2, \ldots, b_{\frac{n}{2}}\}$ where each block $\{b_i/1 \leq i \leq \frac{n}{2}\}$ is $C_4$ such that $b_1 = \{v_1, u_1, w_1, v_2\}$, $b_2 = \{v_3, u_2, w_2, v_4\}$, ..., $b_{\frac{n}{2}} = \{v_{n-1}, u_{\frac{n}{2}}, w_{\frac{n}{2}}, v_n\}$ and $B_2 = \{b_1', b_2', \ldots, b_{\frac{n}{2}}'\}$ where each block $\{b_i'/1 \leq i \leq \frac{n}{2}\}$ is $K_2$ such that $b_1' = \{v_1', u_1', w_1', v_2', w_2\}$, $b_2' = \{v_3, u_2, w_2, v_4\}$, ..., $b_{\frac{n}{2}}' = \{v_{n-1}, u_{\frac{n}{2}}, w_{\frac{n}{2}}, v_n\}$. We have the following cases:

Case 1 For $n$ is even.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, v_n\}$ be the geodetic set where $\{u_1, u_2, \ldots, u_{n-2}\}$ and $\{v_n\}$ are the vertices chosen from block $B_1$. Since every edge of $G$ lies on a geodesic joining any two vertices in $S$, then $g_1(G) = |S| = \left\lceil \frac{n}{2} \right\rceil$.

Case 2 For $n$ is odd.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, v_n\}$ be the minimum geodetic set where $\{u_1, u_2, \ldots, u_{n-2}\}$ and $\{v_n\}$ are the vertices chosen from block $B_1$. Clearly $S$ is an edge geodetic set of $G$. Therefore, $g_1(G) = |S| = \frac{n+2}{2}$.

Corollary 2.41. Let $G = AQ_n(n \geq 4)$ be an alternate quadrilateral snake, then

$$eg_r(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 0 \pmod{2}, \\ \frac{n+2}{2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

where $b$ is the number of blocks.

Corollary 2.42. For an alternate quadrilateral snake $G = AQ_n(n \geq 4)$,

$$g_{ns}(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 0 \pmod{2}, \\ \frac{n+2}{2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

where $b$ is the number of blocks.

Theorem 2.43. For an alternate quadrilateral snake $AQ_n(n \geq 4)$, then

$$g_{ss}(AQ_n) = \begin{cases} \alpha_0 & \text{if } n \text{ is even}, \\ \alpha_0 - 1 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Let $V(AQ_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{\frac{n}{2}}, w_1, w_2, \ldots, w_{\frac{n}{2}}\}$ and $\alpha_0$ is the vertex covering number of $AQ_n$. We have the following cases:

Case 1 For $n$ is even.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, w_{\frac{n}{2}}\}$ be the geodetic set of $G$. But $< V - S >$ is connected. Let $S' = S \cup \{v_2, v_4, \ldots, v_{n-2}\} \cup \{v_{n-1}\}$. Clearly $< V - S' >$ has isolated vertices. Hence,

$$g_{ss}(G) = |S'| = n = \alpha_0.$$

Case 2 For $n$ is odd.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, w_{\frac{n}{2}}\}$ be the minimum geodetic set of $G$. Let $S' = S \cup \{v_2, v_4, \ldots, v_{n-2}\}$. Clearly $< V - S' >$ is totally disconnected. Hence,

$$g_{ss}(G) = |S'| = n - 1 = \alpha_0 - 1.$$


Theorem 2.44. Let $G = AQ_n(n \geq 4)$ be an alternate quadrilateral snake, then
\[
g_{1s}(G) = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0(\text{mod} 2), \\ \frac{n+3}{2} & \text{if } n \equiv 1(\text{mod} 2). \end{cases}
\]

Proof. Let $G = AQ_n$ and by Definition 2.39
\[
V(G) = \begin{cases} 2n & \text{if } n \equiv 0(\text{mod} 2) \\ 2n-1 & \text{if } n \equiv 1(\text{mod} 2) \end{cases}
\]

Let $G = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{u_1, u_2, \ldots, u_2\}$, $V_3 = \{w_1, w_2, \ldots, w_2\}$. We have the following cases:

Case 1 Let $n$ be even. Let $S = \{u_1, u_2, \ldots, u_{n-2}, w_2, w_2\}$. Then $S$ is an edge geodetic set. But $S - V$ is connected. Let $S' = S \cup \{v_j\}$, where $\{v_j\}$ for $2 \leq j \leq n-1$ is any one internal vertex of $P_n$. Clearly, $S' - V$ has two components. Therefore,
\[
g_{1s}(G) = |S'| = S + \{v_j\} = \frac{n+2}{2}.
\]

Case 2 Let $n$ be odd. Let $S = \{u_1, u_2, \ldots, u_{n-3}, w_2, w_2\}$. Clearly $S$ is the minimum edge geodetic set. But $S - V$ is connected. Let $S' = S \cup \{v_j\}$, where $\{v_j\}$ for $2 \leq j \leq n-1$ is any one internal vertex of $P_n$. Clearly, $S' - V$ is disconnected, therefore
\[
g_{1s}(G) = |S'| = S + \{v_j\} = \frac{n+3}{2}.
\]

Example: For an alternate quadrilateral snake graph $AQ_7$ given in Figure 4. The empty color vertices is its split edge geodetic set. Therefore $g_{1s}(AQ_7) = 4$.

Definition 2.45. The double alternate quadrilateral snake $DAQ_n$ consists of two alternate quadrilateral snakes that have a common path.

Theorem 2.46. Let $G = DAQ_n(n \geq 4)$ be double alternate quadrilateral snake, then $g_1(G) = n$.

Proof. Let $G = DAQ_n$ be the graph obtained from joining $v_i$ and $v_{i+1}$ alternatively to new vertices $u_i, u_1$ and $w_i, w_i$ respectively. Then
\[
V(G) = \begin{cases} 3n & \text{if } n \equiv 0(\text{mod} 2) \\ 3n-2 & \text{if } n \equiv 1(\text{mod} 2) \end{cases}
\]

We have the following cases:

Case 1 Let $n$ be even. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ where $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{u_1, u_2, \ldots, u_2\}$, $V_3 = \{w_1, w_2, \ldots, w_2\}$, $V_4 = \{u'_1, u'_2, \ldots, u'_2\}$, $V_5 = \{w'_1, w'_2, \ldots, w'_2\}$ such that $u_i, u_i$ and $w_i, w_i$ are the new vertices added in upward and downward direction to $v_i$ and $v_{i+1}$ for $1 \leq i \leq n-1$. Let $S = \{u_1, u_2, \ldots, u_2, w'_1, w_1, w_2, \ldots, w'_2\}$. Choose the vertices of $S$ such that $d(u_i, w'_i) = 3$. Clearly $S$ is the minimum edge geodetic set of $G$. Therefore,
\[
g_1(G) = \frac{|S'|}{n} = \frac{n}{2} + \frac{n}{2} = \frac{n}{2}.
\]

Case 2 Let $n$ be odd. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ where $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{u_1, u_2, \ldots, u_{n-1}\}$, $V_3 = \{w_1, w_2, \ldots, w_{n-1}\}$, $V_4 = \{u'_1, u'_2, \ldots, u'_{n-1}\}$, $V_5 = \{w'_1, w'_2, \ldots, w'_{n-1}\}$. Let $S = \{u_1, u_2, \ldots, u_{n-1}, w'_1, w'_2, \ldots w'_{n-1}, v_n\}$. Choose the vertices of $S$ such that $d(u_i, w'_i) = 3$. Therefore $S$ is an edge geodetic set of $G$. Hence,
\[
g_1(G) = \frac{|S'|}{n} = \frac{n-1}{2} + \frac{n-1}{2} + 1 = \frac{n}{2}.
\]

Corollary 2.47. Let $G = DAQ_n(n \geq 4)$ be double alternate quadrilateral snake, then $eg_1(G) = n$.

Corollary 2.48. For double alternate quadrilateral snake $DAQ_n(n \geq 4)$, $g_{n}(DAQ_n) = n$.

Theorem 2.49. For double alternate quadrilateral snake $DAQ_n(n \geq 4)$,
\[
g_{ss}(DAQ_n) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even}, \\ \frac{3n-1}{2} & \text{if } n \text{ is odd}. \end{cases}
\]
We discuss the following cases:

**Theorem 2.50.**

Let $G = DAQ_n$. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, u_3, w_1, w_2, \ldots, w_{n-2}, u'_1, u'_2, \ldots, u'_{n-2}, w'_1, w'_2, \ldots, w'_{n-2}\}$. 

We consider the following cases:

**Case 1** Let $n$ be even.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, w_1, u'_1, u'_2, \ldots, u'_{n-2}, w'_1, w'_2, \ldots, w'_{n-2}\}$. Clearly $S$ is a geodetic set of $G$. But $\langle V - S \rangle$ is connected. Let $S' = S \cup \{v_2, v_4, \ldots, v_{n-1}\}$. Clearly $\langle V - S' \rangle$ is an independent set and hence

$$g_{ss}(G) = |S'| = \frac{3n}{2}.$$ 

**Case 2** Let $n$ be odd.

Let $S = \{u_1, u_2, \ldots, u_{n-1}, w_1, u'_1, u'_2, \ldots, u'_{n-1}, v_n\}$ be the minimum geodetic set of $G$. But $\langle V - S \rangle$ is connected. Consider $S' = S \cup \{v_2, v_4, \ldots, v_{n-1}\}$. Clearly $\langle V - S' \rangle$ is an independent set and therefore,

$$g_{ss}(G) = |S'| = \frac{3n - 1}{2}.$$ 

\[\Box\]

**Theorem 2.50.** For double alternate quadrilateral snake $DAQ_n(n \geq 4)$, $g_{ss}(G) = n + 1$.

**Proof.** Let $G = DAQ_n$. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, u_3, w_1, w_2, \ldots, w_{n-2}, u'_1, u'_2, \ldots, u'_{n-2}, w'_1, w'_2, \ldots, w'_{n-2}\}$. We consider the following cases:

**Case 1** Let $n$ be even.

Let $S = \{u_1, u_2, \ldots, u_{n-2}, w_1, u'_1, u'_2, \ldots, u'_{n-2}\}$. Choose the vertices such that $d(u_i, u'_i) = 3$. Then $S$ is an edge geodetic set. Let $S' = S \cup \{v_k\}$ where $\{v_k\}$ for $2 \leq k \leq n - 1$ is only one internal vertex of $P_n$. Then $V - S'$ is disconnected. Therefore,

$$g_{ss}(G) = |S'| = n + \frac{n}{2} + 1 = n + 1.$$ 

**Case 2** Let $n$ be odd.

Let $S = \{u_1, u_2, \ldots, u_{n-1}, w_1, u'_1, u'_2, \ldots, u'_{n-1}, v_n\}$ such that $d(u_i, u'_i) = 3$. Clearly $S$ is an edge geodetic set. But $V - S$ is connected. Hence $S$ is not split edge geodetic set. Let $S' = S \cup \{v_k\}$ where $\{v_k\}$ for $2 \leq k \leq n - 1$ is only one internal vertex of $P_n$. Since $\langle V - S' \rangle$ is disconnected, then

$$g_{ss}(G) = |S'| = \frac{n - 1}{2} + \frac{n - 1}{2} + 2 = n + 1.$$ 

\[\Box\]

**3. Conclusion**

In this paper, edge geodetic parameters for some snake graphs like triangular snake graph, double triangular snake graph, alternate triangular snake graph, double alternate triangular snake graph, quadrilateral snake graph, double quadrilateral snake graph, alternate quadrilateral snake graph, double alternate quadrilateral snake graph are determined.

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**References**


