Fixed point theorem of a set valued map on Cone metric space

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\[1. \text{Then in 1972, Chatterjea [23] established a fixed point} \]
\[\text{theorems on single valued maps. In [5],} \]
\[\text{it is during the year 2007 when Huang and Zhang [4]} \]
\[\text{introduced the concept of cone metric space by replacing the} \]
\[\text{range set of non negative real numbers of the metric} \]
\[\text{d by the ordered Banach space. Since then many other authors in [12]-[19]} \]
\[\text{and the references therein, have obtained the fixed point} \]
\[\text{theorems on single valued maps. In [5], the existence of a fixed} \]

\[\text{point in cone metric space for set valued mappings has been} \]
\[\text{obtained by the concept of H-Cone metric. For more recent} \]
\[\text{fixed point theorems in cone metric spaces for multivalued} \]
\[\text{mappings we refer [\{5\}-[11\}] and references therein.} \]

\section{2. Preliminaries}

Let \(E\) be a real Banach space \(P \subset E\). Then \(P\) is said to a cone if it satisfies the following conditions:

1. \(P\) is a nonempty closed subset and \(P \neq \emptyset\).
2. \(x, y \in P\) and \(a, b \in R\) where \(a \geq 0\) and \(b \geq 0\) then \(ax + by \in P\).
3. \(\text{If} \ x \in P \ \text{and} \ -x \in P, \ \text{then} \ x = \theta\).

Cone induces a Partial order relation We can define a partial order relation \(\preceq\) on \(E\) with respect to the cone \(P\) in the following way: \(x \preceq y\) if and only if \(y - x \in P\). Also \(x \ll y\) if and only if \(y \preceq x \in \text{IntP}\) and \(x \ll y\) implies \(x \preceq y\) but \(x \neq y\). If \(\text{IntP} \neq \emptyset\) then the cone is a solid cone.

\begin{definition}{1.2} \cite{4} Let \(X\) be a non empty set and \(d : X \times X \to E\) satisfying
\begin{enumerate}
\item \(\theta \preceq d(x, y)\) and \(d(x, y) = \theta\) if and only if \(x = y\).
\item \(d(x, y) = d(y, x)\).
\item \(d(x, y) \preceq d(x, z) + d(z, y)\), \(\forall x, y \in X\).
\end{enumerate}
\end{definition}
Then \( d \) is called the cone metric and the pair \((X, d)\) is called the cone metric space.

**Example 1** [5]: Let \( E = \mathbb{R}^2 \) and \( P = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\} \), \( X = \mathbb{R}^2 \) and \( d((x, y), (x', y')) = \sqrt{(x-x')^2 + (y-y')^2} \), \( \forall x, y, x', y' \in \mathbb{R}^2 \) and \( \alpha \geq 0 \). Then \((X, d)\) is a cone metric space and \( P \) is a normal cone with normal constant 1.

There are two different kinds of cones: Normal (with a normal constant ) and Non-Normal cones. Let \( E \) be a real Banach space, \( P \subset E \) be a cone and \( \rho \) be the partial ordering defined by \( P \). Then \( P \) is said to be normal if there exist positive real number \( H \) such that, for all \( x, y \in E \), \( \theta \leq x \leq y \Rightarrow \|x\| \leq H \|y\| \). Or, equivalently if \( x_n \leq y_n \leq z_n \) and \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x \), then \( \lim_{n \to \infty} z_n = n \). The least of all such constant \( H \) is known as normal constant.

**Definition 2.2.** [4]: Let \((X, d)\) be a cone metric space. Let \( \{x_n\} \) be a sequence in \( X \) and \( x \in X \). If for every \( \varepsilon \in E \) with \( \varepsilon > \theta \) there is \( N \) such that for all \( n > N \), \( d(x_n, x) < \varepsilon \). Then \( \{x_n\} \) is said to be convergent and \( x \) is the limit of \( \{x_n\} \). We denote this by, \( \lim_{n \to \infty} x_n = x \) as \( n \to \infty \).

**Definition 2.3.** : Let \((X, d)\) be a cone metric space. Let \( \{x_n\} \) be a sequence in \( X \). If for every \( \varepsilon \in E \) with \( \varepsilon > \theta \) there is a positive integer \( N \) such that for all \( n > N \), \( d(x_n, x) \leq \varepsilon \). Then \( \{x_n\} \) is said to be a Cauchy sequence in \( X \).

**Definition 2.4.** : If every Cauchy sequence \( \{x_n\} \subset M \) is convergent in \( x \in M \), then \((X, d)\) is called a complete cone metric space.

**Lemma 2.5.** [21] Let \( E \) be a Banach space.
(i) If \( a, b, c \in E \) and \( a \leq b \leq c \), then \( a \leq c \).
(ii) If \( \theta \leq a \leq c \) for each \( c > \theta \), then \( a = \theta \).
(iii) If \( E \) is a real Banach space with cone \( P \) and if \( a \leq \lambda a \) where \( a \in P \) and \( \lambda \in (0, 1) \), then \( a = \theta \).

**Remark 2.6.** [21]: If \( c \gg \theta \), \( \theta \ll a_0 \) and \( a_0 \to \theta \), then there exist \( N \), such that for all \( n > N \), we have \( a_n \ll c \).

A set \( A \subset M \) is closed if for any sequence \( \{x_n\} \subset A \) convergent in \( x \), then \( x \in A \).

We denote \( N(M) \) as the collection of all nonempty subsets of \( M \) and \( C(M) \) as collection of all nonempty closed subsets of \( M \).

**Definition 2.7.** An element \( x \in M \) is said to be a fixed point of a set-valued mapping \( T : M \to N(M) \) if \( x \in Tx \). Denote \( \text{Fix}(T) = \{x \in M : x \in Tx\} \).

The following is the definition of H-cone metric as given by Wardowski in [6] came in the year 2011.

**Definition 2.8.** Let \((M, d)\) be a cone metric space and \( \mathcal{A} \) be the collection of all nonempty subsets of \( M \). A map \( \mathcal{H} : \mathcal{A} \times \mathcal{A} \to E \) is called an H-cone metric with respect to \( d \) if for any \( A, B, C \in \mathcal{A} \) the following conditions hold:
1. \( \mathcal{H}(A, B) = 0 \Rightarrow A = B \).
2. \( \mathcal{H}(A, B) = \mathcal{H}(B, A) \).
3. \( \forall \varepsilon \in E \) with \( \theta \ll \varepsilon \), \( \forall x \in A_1 \), \( \exists \) at least one \( y \in A_2 \), such that \( d(x, y) \leq \mathcal{H}(A_1, A_2) + \varepsilon \).
4. anyone of the following holds there exist
   (a) \( \forall \varepsilon \in E \) with \( \theta \ll \varepsilon \), \( \exists \) at least one \( x \in A_1 \), such that \( \mathcal{H}(A_1, A_2) \leq d(x, y) + \varepsilon \). \( \forall y \in A_2 \).
   (b) \( \forall \varepsilon \in E \) and \( \theta \ll \varepsilon \), \( \exists \) at least one \( x \in A_2 \), such that \( \mathcal{H}(A_1, A_2) \leq d(x, y) + \varepsilon \). \( \forall y \in A_1 \).

For examples we refer [6] to the readers. The author in [6] have proved that if \((M, d)\) is a cone metric space and \( \mathcal{H} : \mathcal{A} \times \mathcal{A} \to E \) is H-cone metric with respect to \( d \) then the pair \((\mathcal{A}, \mathcal{H})\) is a cone metric space.

In [6], the author have proved the following result.

**Theorem 2.9.** Let \((M, d)\) be a complete cone metric space with a normal cone \( P \) with a normal constant \( H \). Let \( \mathcal{A} \) be a nonempty collection of all nonempty closed subsets of \( M \) and let \( \mathcal{H} : \mathcal{A} \times \mathcal{A} \to E \) be an H-cone metric with respect to \( d \). If for a map \( T : M \to \mathcal{A} \) \( \exists \lambda \in (0, 1) \) such that \( \forall x, y \in M \), \( \mathcal{H}(Tx, Ty) \leq \lambda d(x, y) \), then \( \text{Fix} T \neq \emptyset \).

In the year 2013, H-cone metric in the sense of Arshad and Ahmad [11] was defined in the following way to make it more comparable with a standard metric.

**Definition 2.10.** [11]: Let \((M, d)\) be a cone metric space and \( \mathcal{A} \) be a collection of all nonempty subsets of \( M \). A map \( \mathcal{H} : \mathcal{A} \times \mathcal{A} \to E \) is called an H-cone metric in the sense of Arshad and Ahmad if the following conditions hold:
1. \( \theta \leq \mathcal{H}(A, B) \) for all \( A, B \in \mathcal{A} \) and \( \mathcal{H}(A, B) = \theta \) if and only if \( A = B \);
2. \( \mathcal{H}(A, B) = \mathcal{H}(B, A) \), \( \forall A, B \in \mathcal{A} \);
3. \( \mathcal{H}(A, B) \leq \mathcal{H}(A, C) + \mathcal{H}(C, B) \), \( \forall A, B, C \in \mathcal{A} \);
4. if \( A, B \in \mathcal{A} \), \( \theta \ll \varepsilon \in E \) with \( \mathcal{H}(A, B) \ll \varepsilon \), then for each \( a \in A \) there exists \( b \in B \) such that \( d(a, b) \ll \varepsilon \).

Using this H-cone metric the following result [11], Th.3 was proved

**Theorem 2.11.** [11] Let \((M, d)\) be a complete cone metric space. Let \( \mathcal{A} \) be a nonempty collection of all nonempty closed subsets of \( M \) and let \( \mathcal{H} : \mathcal{A} \times \mathcal{A} \to E \) be an H-cone metric induced by \( d \). If for a map \( T : M \to \mathcal{A} \) \( \exists \lambda \in (0, 1) \) such that \( \forall x, y \in M \), \( \mathcal{H}(Tx, Ty) \leq \lambda d(x, y) \), then \( \text{Fix} T \neq \emptyset \).

The following example has been shown in [14], Eg 1.10 which indicates that Definition 2.10 is different from Definition 2.8.

**Example 2.12.** Let \( X = \{a, b, c\} \) and \( d : X \times X \to [0, +\infty) \) be defined by \( d(a, b) = d(b, a) = \frac{1}{2} \), \( d(a, c) = d(c, a) = d(b, c) = d(c, b) = 1 \), \( d(a, a) = d(b, b) = d(c, c) = 0 \). Let \( A = \{a, b, c\} \).

\( \mathcal{H} : A \times A \to [0, +\infty) \) as \( \mathcal{H}(\{a\}, \{b\}) = \mathcal{H}(\{b\}, \{a\}) = 1 \).
We define an H-cone metric \( d \) be a normal cone and let \( \mathcal{H} \) be two cone multivalued mappings and suppose that there is \( H \in \mathbb{R} \) such that \( \forall x, y \in X \) at least one of the following is holds:

1. \( \mathcal{H}(Tx, Sy) \leq d(x, y) \);
2. \( \mathcal{H}(Tx, Sy) \leq d(x, u) \) for each fixed \( u \in Tx \);
3. \( \mathcal{H}(Tx, Sy) \leq d(y, v) \) for each fixed \( v \in Sy \);
4. \( \mathcal{H}(Tx, Sy) \leq \lambda \frac{d(x,y) + d(x, y)}{2} \) for each fixed \( x \in Sy \) and \( u \in Tx \).

Then \( T \) and \( S \) have a common fixed point.

The following example is given by Wardowski [[6], Ex. 3.3] which satisfies Defn. 2.8.

**Example 2.14.** Let \( M = [0, 1] \), \( E = \mathbb{R}^2 \) be a Banach space with the standard norm, \( P = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \land y \geq 0 \} \) be a normal cone and let \( d : M \times M \to E \) be of the form \( d(x, y) = (x - y, \frac{1}{2}|x - y|) \). Let \( \mathcal{S} \) be a family of subsets of \( M \) of the form \( \mathcal{S} = \{[0, x] : x \in M \} \cup \{\{x \} : x \in M \} \). We define an H-cone metric \( \mathcal{H} : \mathcal{S} \times \mathcal{S} \to \mathbb{R} \) with respect to \( d \) by the formulae

\[
\mathcal{H}((a, b), (c, d)) = \max\{d(a, c), d(b, d)\} \\
\mathcal{H}(a, b) = \max\{d(a, b), d(b)\}
\]

In [5], the author have given the following example that satisfies defn 2.8.

**Example 2.15.** Let \( M = \{(0, 1), (1, 0), (0, 0)\} \), \( E = \mathbb{R}^2 \) be a Banach space with the standard norm, \( P = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \land y \geq 0 \} \) be a normal cone and let \( d : M \times M \to E \) be defined by

\[
d((0, 0), (1, 0)) = d((0, 1), (0, 0)) = (1, \frac{1}{2}) \\
d((0, 0), (0, 1)) = d((1, 0), (0, 0)) = (1, \frac{1}{2}) \\
d((1, 0), (0, 0)) = d((0, 1), (0, 0)) = (1, \frac{1}{2})
\]

Then the pair \( (M, d) \) is a complete cone metric space.
If we take $T_1 = T_2$, in the above theorem we get the following result due to Wardowski ([6], Th. 3.1).

**Corollary 3.2.** Let $(M, d)$ be a complete cone metric space with a normal cone $P$ with a normal constant $\lambda$. Let $\mathcal{A}$ be a nonempty collection of all nonempty closed subsets of $M$ and $\mathcal{H}: \mathcal{A} \times \mathcal{A} \to \mathcal{E}$ be an $H$-cone metric with respect to $d$. If for a map $T : M \to \mathcal{A}$ there exists $x_0 \in [0, 1]$ such that $\mathcal{H}(T(x), T(y)) \leq \lambda(d(y, x), \forall y \in M).$ then Fix$T \neq \emptyset$.

**Definition [5]:** Suppose $D(x, Tx) = \{d(x, z) : z \in T(x)\}$ and $S(x, Tx) = \{u \in D(x, Tx) : ||u|| = \inf \{||v|| : v \in D(x, Tx)\}\}$.

**Theorem 3.3.** Let $(M, d)$ be a complete cone metric space. Let $\mathcal{A}$ be a nonempty collection of all nonempty closed subsets of $M$ and $T : M \to \mathcal{A}$ be the set valued map. Consider an $H$-cone metric with respect to $d$ $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to \mathcal{E}$ satisfying Defn. 2.8.

Then if $T$ satisfies the contraction condition $\mathcal{H}(T(x), T(y)) \leq \lambda(S(x, Tx) + S(y, Ty)), \forall x, y \in M. \lambda \in [0, 1)$ then $T$ has a fixed point.

**Proof:** Suppose that $\varepsilon_n \in E$ and $\varepsilon_n \gg \theta$, such that $\varepsilon_n \to \theta$, as $n \to \infty$. Let $x_0 \in M$ be arbitrary and fixed. Then $T(x_0) \in \mathcal{A}$. Suppose $d(x_1, x_0) = \inf \{d(x_0, z) : z \in T(x_0)\}$. Then $d(x_0, x_1) = d(x_0, x_1)$. Let $x_2 \in T(x_1)$, such that $d(x_1, x_2) = \inf \{d(x_1, z) : z \in T(x_1)\}$. Then we have $S(x_1, x_2) = d(x_1, x_2)$. Inductively we have for $x_{n+1} \in T(x_n), S(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Therefore, $d(x_n, x_{n+1}) \leq \mathcal{H}(T_{n}, T_{n+1}) + \varepsilon_n$. Then $\lambda \{S(x_{n-1}, T_{n-1}) + S(x_{n}, T_{n+1})\} + \varepsilon_n$. Hence we have $\lambda \{S(x_{n-1}, T_{n-1}) + S(x_{n}, T_{n+1})\} + \varepsilon_n$. So we have, $\leq \lambda \{S(x_{n-2}, T_{n-2}) + S(x_{n-1}, T_{n+1})\} + \varepsilon_n$. Therefore, $\lambda \{S(x_{n-3}, T_{n-3}) + S(x_{n-2}, T_{n+1})\} + \varepsilon_n$. 

\[d(x_0, x_1) \leq \sum_{n=0}^{\infty} \lambda \frac{\varepsilon_n}{\varepsilon_n} \leq \varepsilon_n \leq \varepsilon_n, \text{ for all } n > N.\]

Thus if $d(x_n, x_{n+1}) \leq \varepsilon_n$, for all $n > N$. Therefore, $d(x_{2n}, x_{2m}) \leq c, \text{ for all } n > N$. which gives that $\{x_{2n}\}$ is a cauchy sequence. Since $(M, d)$ is complete $\{x_{2n}\}$ is convergent in $M$. Let $x_{2n} \to x_0$. Now $x_{2n} \in T_2(x_{2n-1}) \exists x \in T_2(x_0)$ such that $d(x_{2n}, x) \leq \mathcal{H}(T_2(x_{2n-1}), T_2(x_0)) + \varepsilon_n$. $d(x_{2n}, x) \leq \lambda d(x_{2n}, x_0) + \varepsilon_n$. Taking limit $n \to \infty$ we get, $d(x_0, x) \leq \varepsilon_n$. Therefore, $x_0 = x$. But $x \in T_1(x_0)$. So, we have $x_0 \in T_1(x_0)$. That is $x_0$ is a fixed point of $T$. Similarly, $x_{2n} \in T_1(x_{2n-2})$ then $x_{2n} \in T_2(x_2n-2)$ such that, $d(x_{2n-1}, x_0) \leq \mathcal{H}(T_2(x_{2n-1}), T_2(x_0)) + \varepsilon_n$. $d(x_{2n-1}, x_0) \leq \lambda d(x_{2n}, x_0) + \varepsilon_n$. Taking limit $n \to \infty$ we get, $d(x_0, x_0) \leq \varepsilon_n$. Therefore, $x_0 = x$. But $x \in T_0(x_0)$. So we have $x_0 \in T_0(x_0)$. That is $x_0$ is a fixed point of $T_2$. 

\[\lambda \sum_{n=0}^{\infty} \lambda \frac{\varepsilon_n}{\varepsilon_n} \leq \varepsilon_n \leq \varepsilon_n, \text{ for all } n > N.\]

Therefore, $d(x_{2n}, x_{2m}) \leq c, \text{ for all } n > N$. For $m \geq n$, we have, $d(x_{2n}, x_{2m}) \leq \sum_{j=n}^{m-1} d(x_{2n}, x_{2m}) + \varepsilon_n$. $d(x_{2n}, x_{2m}) \leq \lambda d(x_{2n}, x_{2m}) + \varepsilon_n$. Taking limit $n \to \infty$ we get, $d(x_{2n}, x_{2m}) \leq \frac{\varepsilon_n}{\varepsilon_n}$. Therefore, $x_0 = x$. But $x \in T_4(x_0)$. So we have $x_0 \in T_4(x_0)$. That is $x_0$ is a fixed point of $T_2$. 

\[\lambda \sum_{n=0}^{\infty} \lambda \frac{\varepsilon_n}{\varepsilon_n} \leq \varepsilon_n \leq \varepsilon_n, \text{ for all } n > N.\]
So, $S(y, Ty) = d\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Also, $H\left(Tx, Ty\right) = \left(\max\{0, 0 - \frac{1}{2}(y - \frac{1}{2})^2\}\right)\max\{0, 0 - \frac{1}{2}(y - \frac{1}{2})^2\} = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Hence, we have,

$$\alpha \left[S(x, Tx + S(y, Ty))\right] - H\left(Tx, Ty\right) = \left(\alpha(x + \frac{3}{8}), \frac{\alpha(x + \frac{3}{8})}{2}\right) - \left(\frac{1}{2}, \frac{1}{2}\right),$$

that is,

$$\alpha \left[S(x, Tx + S(y, Ty))\right] - H\left(Tx, Ty\right) = \left(\alpha(x + \frac{3}{8}) - \frac{1}{2}, \frac{\alpha(x + \frac{3}{8})}{2} - \frac{1}{2}\right).$$

We now find what could be the minimum value of $\alpha(x + \frac{3}{8}) - \frac{1}{2}$ for $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[\frac{1}{2}, 1\right]$.

Observe that $x + \frac{3}{8}$ is maximum and $\alpha(x + \frac{3}{8})$ is minimum. If $\frac{1}{2}(y - \frac{1}{2})^2$ is minimum, $\frac{1}{2}(y - \frac{1}{2})^2$ is maximum.

But, $\alpha\left(x + \frac{3}{8}\right)$ is maximum if $y$ is maximum i.e., $y = 1$, so $\frac{1}{2}(y - \frac{1}{2})^2 = \frac{1}{8}$.

and $\alpha(x + \frac{3}{8})$ is minimum if $x$ is minimum i.e., $x = 0$, so $\alpha(x + \frac{3}{8}) = \frac{3}{8}$.

So, we have,

$$\frac{1}{2}(y - \frac{1}{2})^2 = \frac{1}{8},$$

$$\alpha(x + \frac{3}{8}) = \frac{3}{8}.$$

Thus, $\alpha \geq \frac{1}{2}$. Taking $\alpha = \frac{1}{2}$, we get,

$$H\left(Tx, Ty\right) \leq \frac{1}{2} S\left[x, Tx + S(y, Ty)\right], \forall x, y \in M, \lambda \in \left[0, \frac{1}{2}\right].$$

Then $H = H\left(Tx, Ty\right)$ has a fixed point.

Proof : Suppose that $\varepsilon_n \in E$ and $\varepsilon_n \to \theta$, such that $\varepsilon_n \to \theta$ as $n \to \infty$. Let $x_0 \in M$ be arbitrary and fixed.

Then $T(x_0) \in \mathcal{A}$. Let $x_1 \in T(x_0)$, then $S(x_1, T(x_0)) = \theta$.

Let $x_2 \in T(x_1)$, such that $\|d(x_0, x_2)|| = \inf \{\|d(x_0, z)|| \in \mathbb{R} : z \in T(x_1)\}$.

Then we have $S(x_0, T(x_1)) = d(x_0, x_2)$.

Hence, we have, $d(x_1, x_2) = H\left(Tx_0, T_1\right) + \varepsilon_1$.

$\leq \lambda S(x_0, T(x_1)) + S(x_1, T(x_0)) + \varepsilon_1$.

$\leq \lambda d(x_0, x_2) + \varepsilon_1$.

$\leq \lambda d(x_0, x_2) + d(x_1, x_2) + \varepsilon_1$.

Then we have $S(x_1, T_2) = d(x_1, x_2)$.

Hence, we have, $d(x_2, x_3) = H\left(Tx_1, T_2\right) + \varepsilon_1$.

$\leq \lambda S(x_1, T_2) + S(x_2, T_2) + \varepsilon_2$.

$\leq \lambda d(x_1, x_2) + \varepsilon_2$.

$\leq \lambda d(x_1, x_2) + d(x_2, x_3) + \varepsilon_2$.

$\leq \lambda d(x_1, x_2) + d(x_2, x_3) + \frac{1}{2} \varepsilon_2$.

Inductively we have for $x_{n+1} \in T(x_n)$,

$$d(x_n, x_{n+1}) \leq \frac{\lambda}{\lambda - \frac{1}{2}} d(x_n, x_{n-1}) + \frac{\varepsilon_n}{\lambda - \frac{1}{2}}.$$

So we have,

$$\lim_{n \to \infty} \left[\left(\frac{\lambda}{\lambda - \frac{1}{2}} d(x_n, x_{n-1}) + \frac{\varepsilon_n}{\lambda - \frac{1}{2}}\right)\right] = \theta, \quad \lambda > \frac{1}{2}.$$

We now find what could be the minimum value of $\alpha(x + \frac{3}{8}) - \frac{1}{2}$ for $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[\frac{1}{2}, 1\right]$.

Therefore, we have,

$$d(x_n, x_{n+1}) \leq \alpha^n \left[\|d(x_0, x_1)|| + \sum_{j=1}^{n-1} \frac{\lambda^{n-j}}{(\lambda - \frac{1}{2})^{n-j}} \varepsilon_j\right].$$

Where $\alpha = \frac{1}{\lambda - \frac{1}{2}} < 1$.

For $m \geq n$, we have,

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} \alpha^n \left[\|d(x_0, x_1)|| + \sum_{j=1}^{n-1} \frac{\lambda^{n-j}}{(\lambda - \frac{1}{2})^{n-j}} \varepsilon_j\right].$$

Taking limit $n \to \infty$, we get, $\sum_{j=n}^{m-1} \alpha^n \left[\|d(x_0, x_1)|| + \sum_{j=1}^{n-1} \frac{\lambda^{n-j}}{(\lambda - \frac{1}{2})^{n-j}} \varepsilon_j\right] \to 0$.

Let $c \in \mathbb{N}$, then there exist a natural number $N$ such that,

$$\sum_{j=n}^{m-1} \alpha^n \left[\|d(x_0, x_1)|| + \sum_{j=1}^{n-1} \frac{\lambda^{n-j}}{(\lambda - \frac{1}{2})^{n-j}} \varepsilon_j\right] \leq \frac{\varepsilon}{c},$$

for all $n \geq N$.

Hence, we have $d(x_n, x_m) \leq \varepsilon$ for all $n > N$.

Therefore, $\{x_n\}$ is a Cauchy sequence.

Since, $(M, d)$ is complete, $\{x_n\}$ is convergent. Let us suppose that $x_n \to x^* \in M$. We claim that $x^*$ is the fixed point of $T$ i.e., $x^* \in T(x^*)$.

Now since $x_n \to x^*$ as $n \to \infty$, we get, $\|d(x_n, x^*)\| \to 0$ as $n \to \infty$, again since $x_n \in T(x_n)$, therefore $S(x^*, T(x_n)) = \theta$.

Suppose that, $x_n \in T(x^*)$, such that, $\|d(x_n, x^*)\| = \inf \{\|d(x_n, z)|| : z \in T(x^*)\}$.

Hence, $S(x_n, T(x^*)) = \theta$.

Now, for $x_n \in T(x_n)$, $\exists x_1 \in T(x^*)$, such that, $d(x_n, x_1) \leq H\left(Tx_n, T(x^*)\right) + \varepsilon_n$.

$\leq \lambda S(x_n, T(x^*)) + S(x_1, T(x_n)) + \varepsilon_n$.

$\leq \lambda d(x_n, x_1) + \varepsilon_n$.

$\leq \lambda d(x_n, x_1) + d(x_2, x_3) + \frac{\varepsilon_2}{2}$.

Taking limit $n \to \infty$, we get, $d(x^*, x_1) \leq \lambda \left[\|d(x^*, x^*)\|\right]$. Since, $\lambda < 1$.

$\|d(x^*, x^*)\| = \theta$.

Hence, $x^* \in T(x^*)$.

Example 3.6. Consider the Example 2.15. There we take the following $T(\{0, 1\}) = \{\{0, 0\}, \{0, 1\}, \{0, 1\}\}$.}

Case 1: If $x \in \{0, 0\}$ and $y \in \{0, 0\}$, then $x = 0 = 0$.}

Case 2: $x = 0 = 0$.}

Case 3: $x = 0 = 0$.}

Case 4: $x = 0 = 0$.}

Hence, $\mathcal{S}(x, Ty) = (\frac{1}{2}, \frac{1}{2})$.

Let $x = (0, 0)$, $Ty = (0, 0)$.}

Hence, $\mathcal{S}(x, Ty) = (\frac{1}{2}, \frac{1}{2})$.
\[ \lambda(S(x, Ty) + S(y, Tx)) = \lambda(2, \frac{3}{2}). \]
\[ \lambda(S(x, Ty) + S(y, Tx)) - H(Tx, Ty) = \lambda(2, \frac{3}{2}) \in P, \text{ for any } \lambda \in [0, \frac{1}{2}). \]

Hence, \( H(Tx, Ty) \leq \lambda(S(x, Ty) + S(y, Tx)) \), for any \( \lambda \in [0, \frac{1}{2}). \)

**Case 2:** If \( x \in \{0, 0\} \) and \( y \in \{0, 0\} \), then \( x = (0, 0) \)
y = (0, 1).

\[ Tx = T(0, 0) = \{0, 1\} \] and \( Ty = T(1, 0) = \{0, 1\} \).

So, \( H(Tx, Ty) = H(\{0, 1\}, \{0, 1\}) = (0, 0) \).

\[ D(x, Ty) = D(0, 0), T(1, 0) = d((0, 0), (0, 1)) = (1, \frac{2}{3}). \]

Hence, \( S(x, Ty) = (1, \frac{2}{3}) \).

\[ D(y, Tx) = D(1, 0), T(0, 0) = d((1, 0), (0, 0)) = (0, 0). \]

Hence, \( S(y, Tx) = (0, 0) \).

\[ \lambda(S(x, Ty) + S(y, Tx)) = \lambda(1, \frac{2}{3}). \]

\[ \lambda(S(y, Tx)) + \lambda(S(x, Ty)) = \lambda(1, \frac{2}{3}). \]

\[ S(y, Tx) \in P. \]

**Case 3:** If \( x \in \{0, 0\} \) and \( y \in \{0, 1\} \), then \( x = (0, 0) \)
y = (1, 0).

\[ Tx = T(0, 0) = \{0, 1\} \] and \( Ty = T(1, 0) = \{0, 0\} \).

So, \( H(Tx, Ty) = H(\{0, 1\}, \{0, 0\}) = (1, \frac{2}{3}). \)

\[ D(x, Ty) = D(0, 0), T(1, 0) = d((0, 0), (0, 0)) = (0, 0). \]

Hence, \( S(x, Ty) = (0, 0) \).

\[ D(y, Tx) = D(1, 0), T(0, 0) = d((1, 0), (0, 0)) = (0, 0). \]

Hence, \( S(y, Tx) = (0, 0) \).

\[ \lambda(S(x, Ty) + S(y, Tx)) = \lambda(1, \frac{2}{3}). \]

\[ \lambda(S(y, Tx)) + \lambda(S(x, Ty)) = \lambda(1, \frac{2}{3}). \]

\[ S(y, Tx) \in P. \]

**Case 4:** If \( x \in \{0, 1\} \) and \( y \in \{0, 1\} \), then \( x = (1, 0) \)
y = (0, 1).

\[ Tx = T(1, 0) = \{0, 0\} \] and \( Ty = T(0, 1) = \{0, 1\} \).

So, \( H(Tx, Ty) = H(\{0, 0\}, \{0, 1\}) = (1, \frac{2}{3}). \)

\[ D(x, Ty) = D(1, 0), T(0, 1) = d((1, 0), (0, 1)) = (\frac{2}{3}, \frac{5}{3}). \]

Hence, \( S(x, Ty) = (\frac{2}{3}, \frac{5}{3}). \)

\[ D(y, Tx) = D(0, 1), T(1, 0) = d((0, 1), (0, 0)) = (0, 0). \]

Hence, \( S(y, Tx) = (0, 0) \).

\[ \lambda(S(x, Ty) + S(y, Tx)) = \lambda(\frac{2}{3}, \frac{5}{3}). \]

\[ \lambda(S(y, Tx)) + \lambda(S(x, Ty)) = \lambda(\frac{2}{3}, \frac{5}{3}). \]

\[ S(y, Tx) \in P. \]

**Case 5:** If \( x \in \{0, 1\} \) and \( y \in \{0, 1\} \), then \( x = (0, 1) \)
y = (0, 1).

\[ Tx = T(0, 0) = \{0, 1\} \] and \( Ty = T(1, 0) = \{0, 1\} \).

So, \( H(Tx, Ty) = H(\{0, 1\}, \{0, 1\}) = (0, 0). \)

\[ D(x, Ty) = D(0, 1), T(1, 0) = d((0, 1), (0, 1)) = (0, 0). \]

Hence, \( S(x, Ty) = (0, 0) \).

\[ D(y, Tx) = D(0, 1), T(0, 1) = d((0, 1), (0, 1)) = (0, 0). \]

Hence, \( S(y, Tx) = (0, 0) \).

\[ \lambda(S(x, Ty) + S(y, Tx)) = \lambda(0, 0). \]

\[ \lambda(S(y, Tx)) + \lambda(S(x, Ty)) = \lambda(0, 0). \]

\[ S(y, Tx) \in P. \]

\[ H(Tx, Ty) \leq \lambda(S(x, Ty) + S(y, Tx)), \forall x, y \in M, \text{ with } \lambda \in [0, \frac{1}{2}). \]

**Theorem 3.7.** Let \( (M, d) \) be a complete cone metric space. Let \( \mathcal{A} \) be a nonempty collection of all nonempty closed subsets of \( M \) and \( T : M \to \mathcal{A} \) be the set valued map. Consider an H-cone metric with respect to \( d \). Let \( \mathcal{A} : \mathcal{A} \times \mathcal{A} \to E \) satisfying Defn. 2.8. Then if \( T \) satisfies the contraction condition \( H(Tx, Ty) \leq \{a_1S(x, Tx) + a_2S(y, Ty) + a_3d(x, y)\}, \forall x, y \in M, \) \( a_1 \geq 0 \forall, i = 1, 2, 3 \) and \( a_1 + a_2 + a_3 < 1 \). Then \( T \) has a fixed point.

**Proof:** Suppose that \( \varepsilon_n \in E \) and \( \varepsilon_n \rhd \theta \), such that \( \varepsilon_n \to \theta \) as \( n \to \infty \).

Let \( x_0 \in M \) be arbitrary and fixed.

Then \( T(x_0) \in \mathcal{A} \). Let \( x_1 \in T(x_0) \), be such that \( \|d(x_0, x_1)\| = \inf_{x \in T(x_0)} \|d(x_0, x)\|, \forall x \in T(x_0). \)

Then \( S(x_0, Tx_0) = d(x_0, x_1). \)

Let \( x_2 \in T(x_1) \), such that \( \|d(x_1, x_2)\| = \inf_{x \in T(x_1)} \|d(x_1, x)\|, \forall x \in T(x_1). \)

Then we have \( S(x_1, Tx_1) = d(x_1, x_2). \)

Inductively we have for \( x_{n+1} \in T(x_n), S(x_n, Tx_n) = d(x_n, x_{n+1}). \)

Therefore, \( d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + c \varepsilon_n \).

\[ \leq \{a_1S(x_{n-1}, Tx_{n-1}) + a_2S(x_n, Tx_n) + a_3d(x_{n-1}, x_n)\} + \varepsilon_n. \]

\[ \leq \{a_1d(x_{n-1}, x_n) + a_2d(x_n, x_{n+1}) + a_3d(x_n, x_{n+1})\} + \varepsilon_n. \]

\[ (1 - a_2) d(x_n, x_{n+1}) \leq (a_1 + a_3) d(x_{n-1}, x_n) + \varepsilon_n. \]

\[ d(x_n, x_{n+1}) \leq \left(\frac{a_1 + a_3}{1 - a_2}\right) d(x_{n-1}, x_n) + \frac{\varepsilon_n}{1 - a_2}. \]

So we have, \( \leq \left(\frac{a_1 + a_3}{1 - a_2}\right)^2 d(x_{n-2}, x_{n-1}) + \frac{a_1 + a_3}{1 - a_2} \frac{\varepsilon_{n-1}}{1 - a_2} + \frac{\varepsilon_{n-1}}{1 - a_2}. \)

\[ \leq \left(\frac{a_1 + a_3}{1 - a_2}\right)^3 d(x_{n-3}, x_{n-2}) + \left(\frac{a_1 + a_3}{1 - a_2}\right)^2 d(x_{n-2}, x_{n-1}) + \frac{a_1 + a_3}{1 - a_2} \frac{\varepsilon_{n-2}}{1 - a_2} + \frac{\varepsilon_{n-2}}{1 - a_2} + \frac{\varepsilon_{n-2}}{1 - a_2}. \]

\[ \leq \varepsilon_n + \frac{\varepsilon_n}{1 - a_2} + \frac{\varepsilon_n}{1 - a_2} + \frac{\varepsilon_n}{1 - a_2}. \]

\[ \leq \varepsilon_n + \frac{\varepsilon_n}{1 - a_2} + \frac{\varepsilon_n}{1 - a_2} + \frac{\varepsilon_n}{1 - a_2}. \]

\[ \vdots \]

\[ \varepsilon_n \to \theta \]
Therefore, we have,
\[ d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1) + \sum_{j=1}^{n} \left( \frac{(a_1+a_3)^{n-j}}{(1-a_2)^{n-j}} \right) \varepsilon_j. \]
Where \( \alpha = \frac{a_1}{1-a_2} < 1. \)

For \( m \geq n \), we have,
\[ d(x_m, x_n) \leq \sum_{j=n}^{m-1} \left[ \alpha^j d(x_0, x_1) + \sum_{r=1}^{j} \left( \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j-r}} \right) \varepsilon_r. \]

Taking limit \( n \to \infty \), we have,
\[ \sum_{j=1}^{\infty} \alpha^j d(x_0, x_1) \to \theta \quad \text{and} \quad \sum_{r=1}^{\infty} \varepsilon_r = \sum_{j=1}^{\infty} \left( \frac{(a_1+a_3)^{j-r}}{(1-a_2)^{j-r}} \right) \to \theta. \]

Let \( c \in IntP \), then there exist a natural number \( N \) such that,
\[ \sum_{j=1}^{N-1} \alpha^j d(x_0, x_1) \leq \zeta \quad \text{and} \quad \sum_{j=N}^{\infty} \alpha^j d(x_0, x_1) \leq \zeta, \]
for all \( n > N \).

Therefore, we have,
\[ d(x_n, x_N) \leq c. \]

Since \( (M, d) \) is complete, \( \{x_n\} \) is convergent. Let us suppose that \( x_n \to x^* \) in \( M \). We claim that \( x^* \) is the fixed point of \( T \) i.e., \( x^* \in T x^* \).

Suppose that \( x_1 \in T x^* \), such that,
\[ \|d(x^*, x_1)\| = \inf \{ \|d(x^*, z)\| : z \in T x^* \}. \]

Now, for \( x_n \in T x_{n-1} \), \( \exists x_1 \in T x^* \), such that,
\[ d(x_n, x_1) \leq H(T x_{n-1}, T x^*) + \varepsilon_n. \]
\[ \leq [a_1 S(x_{n-1}, T x_{n-1}) + a_2 S(x^*, T x^*) + a_3 d(x_{n-1}, x^*)] + \varepsilon_n. \]
\[ \leq [a_1 d(x_{n-1}, x_1) + a_2 d(x^*, x_1) + a_3 d(x_{n-1}, x^*)] + \varepsilon_n. \]

Taking \( n \to \infty \), we get,
\[ d(x^*, x_1) \leq a_2 d(x^*, x_1). \] Since, \( a_2 < 1 \), \( d(x^*, x_1) = \theta. \)

\[ x^* = x_1 \in T x^*. \]

\[ \text{Hence, } x^* \in T x^*. \]

References
