Coincidence points for a pair of ordered \( F \)-contraction mappings in ordered partial metric spaces

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Abstract

The concept of ordered \( F \)-contraction in an ordered metric space was introduced by Durmaz et al. [9] and became a very important result in the existing metric fixed point theory. In this paper, we prove a fixed point theorem for a pair of compatible \( F \)-contraction maps in an ordered complete partial metric spaces. In particular, the main results generalize a fixed point theorem due to Durmaz et. al. [9] to partial metric spaces. An illustrative example is provided to support the theorem.

Keywords

Ordered partial metric spaces, \( F \)-contraction mappings, coincidence points.

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1. Introduction

In 2012, Wardowski introduced a new type of contraction known as \( F \)-contraction which generalizes the Banach Contraction Principle. In the results Wardowski defined an \( F \)-contraction map as follows:

Definition 1.1. [15] Let \((M,d)\) be a metric space, a mapping \( T : M \rightarrow M \) is said to be an \( F \)-contraction on \( M \) if there exists \( \tau > 0 \) such that, for all \( x,y \in M \),

\[
d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)).
\]

(1.1)

and \( F : \mathbb{R}_+ \rightarrow \mathbb{R} \), a mapping satisfying the following conditions:

\( F1: \) \( F \) is strictly increasing, that is for all \( x,y \in \mathbb{R}_+ \) such that \( x < y \Rightarrow F(x) < F(y) \).

\( F2: \) For each sequence \( \{\alpha_n\}_{n \geq 1} \) of positive numbers \( \lim_{n \to \infty} \alpha_n = 0 \), if and only if \( \lim_{n \to \infty} F(\alpha_n) = -\infty \).

\( F3: \) There exists \( k \in (0,1) \) such that \( \lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0 \).

We denote by \( \Delta_F \) the set of all functions satisfying the conditions \( F1 \) – \( F3 \).

Moreover, Wardowski proved that every \( F \)-contraction mapping on a complete metric space has a unique fixed point. Futhermore, several contractions in the literature can be deduced by varying suitable elements of \( \Delta_F \).

The following example shows an \( F \)-contraction in metric spaces.

Example 1.2. [15] Let \( F : \mathbb{R}_+ \rightarrow \mathbb{R} \) be defined by \( F(\alpha) = \ln(\alpha) \). It is clear that \( F \) satisfies \((F1)-(F3)\) for any \( k \in (0,1) \). Each mapping \( T : M \rightarrow M \) satisfying \( d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \) is an \( F \)-contraction such that \( d(Tx,Ty) \leq e^{-\tau} d(x,y) \) for all \( x,y \in M \). Obviously, for all \( x,y \in M \) such that \( Tx = Ty \), the inequality \( d(Tx,Ty) \leq e^{-\tau} d(x,y) \) holds and \( T \) is a Banach contraction. One can find more examples in [15].
Recently, several researchers have shown interest in mappings satisfying the $F$-contraction condition. There exist numerous literatures on and around the notion of $F$-contractions; see ([3–5, 9, 13]).

In 1992, Matthews [11], introduced the notion of partial metric spaces and proved an analogue of Banach Contraction Principle on partial metric spaces. Matthews [11] provided the following definition:

**Definition 1.3.** [11] Let $X$ be a non-empty set. A partial metric space is a pair $(X, p)$, where $p$ is a function $p : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all $x, y, z \in X$ the following axioms hold:

$(P1)$ $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$; 

$(P2)$ $p(x, x) \leq p(x, y)$; 

$(P3)$ $p(x, y) = p(y, x)$; and 

$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Clearly, by $(P1)$–$(P3)$, if $p(x, y) = 0$, then $x = y$. But, the converse is in general not true.

The most common example of partial metric spaces is a pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. More examples of partial metric spaces may be found in [7].

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ whose basis is the collection of all open $p$-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$, and $\varepsilon$ is a positive real number.

**Definition 1.4.** [2, 11] Let $(X, p)$ be a partial metric space. Then:

(i) a sequence $\{x_n\}$ in $(X, p)$ is said to be convergent to $x \in X$ if and only if $p(x, x_n) = \lim_{n \to \infty} p(x, x_n)$.

(ii) a sequence $\{x_n\}$ in $(X, p)$ is a Cauchy sequence if and only if $\lim_{n, m \to \infty} p(x_n, x_m)$ exists and is finite.

(iii) a partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges with respect to the topology $\tau_p$ to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$.

(iv) a mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)$.

**2. Preliminaries**

In this section, we recall some definitions and basic results of ordered partial metric spaces which will be used throughout the paper.

Following lemma was proved by Bukatin et al. [7] and will be useful in this paper.

**Lemma 2.1.** [7] Let $(X, p)$ be a partial metric space. Then the mapping $p^* : X \times X \to [0, \infty)$ given by

$$p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$ defines a metric on $X$.

Bukatin et al. [7] also proved the following lemma:

**Lemma 2.2.** [7] Let $(X, p)$ be a partial metric space. Then:

(i) a sequence $\{x_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, p^*)$.

(ii) a partial metric space $(X, p)$ is complete if and only if the metric space $(X, p^*)$ is complete.

Paesano and Vetro [13] provided the following definitions regarding partially ordered set, ordered partial metric space and regularity:

**Definition 2.3.** [13] Let $(X, \preceq)$ be a partially ordered set. Let $A$ and $B$ be two non-empty subset of $X$. Two relations between $A$ and $B$ are denoted and defined as follows:

(r1) $A \prec B$ if for each $a \in A$ there exists $b \in B$ such that $a \preceq b$.

(r2) $A \preceq B$ if for each $a \in A$ and $b \in B$, we have $a \preceq b$.

**Definition 2.4.** [13] If $(X, p)$ be a partial metric space and $(X, \preceq)$ is partially ordered set, then $(X, p, \preceq)$ is called an ordered partial metric. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds. Further a self map $T : X \to X$ is called non-decreasing if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x, y \in X$ and an ordered partial metric space $(X, p, \preceq)$ is regular if the following holds:

For every non-decreasing sequence $\{x_n\}$ in $X$ converging to some $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

First results on fixed point problems in partially ordered metric spaces were obtained by Ran and Reurings [14] and followed by Nieto and Rodriguez [12]. Abbas et al. [11] used the notion of the $F$-contraction to establish order-theoretic common fixed point results. Recently, Durmaz et al. [9] introduced the concept of ordered $F$-contraction in an ordered metric space using the results of Ran and Reurings [14] and proved the following fixed point theorem.

**Theorem 2.5.** [9] Let $(X, d, \preceq)$ be an ordered complete metric space and $T : X \to X$ be an ordered $F$-contraction. Let $T$ be a non-decreasing map and there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $T$ is continuous or $X$ is regular then $T$ has a fixed point.

Durmaz et al. [9] generalized their results by fixing $f = I : X \to X$ in Theorem 2 given by Abbas et al. [1]. Moreover, they provided a condition that every pair of elements of $X$
Theorem 2.9. Let $T: X \rightarrow X$ be an ordered complete metric space and $F: X \rightarrow X$ be an ordered $F$-contraction mapping. Let $T$ be a non-decreasing map and there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. If $T$ is continuous or $X$ is regular then $T$ has a fixed point.

The purpose of this paper is to extend Theorem 2.12 to an ordered partial metric space in order to obtain a fixed point theorem for an ordered $F$-contraction map.

3. Main Results

In this section, we deal with the existence and uniqueness of fixed point of a $F$-contraction map in an ordered partial metric space. First we will provide the extension of Definition 2.11 in an ordered partial metric space which is as follows:

Definition 3.1. Let $(X, \preceq, p)$ be an ordered partial metric space and $T : X \rightarrow X$ be a mapping. Also let $Y = \{(x, y) : x \preceq y, p(Tx, Ty) > 0\}$. We say that $T$ is an ordered $F$-contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that for all $(x, y) \in Y$, we have

$$\tau + F(p(Tx, Ty)) \leq F(p(x, y)).$$

(3.1)

Next, we prove a fixed point theorem for a pair of compatible ordered $F$-contraction mappings.

Theorem 3.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric space $X$ such that $(X, p)$ is a complete partial metric space. Suppose $T$ and $g$ are continuous self $F$-contraction mappings on $X$, $T(X) \subseteq g(X)$, $T$ is a monotone $g$-non decreasing map and

$$\tau + F(p(Tx, Ty)) \leq F(M(x, y))$$

(3.2)

where

$$M(x, y) = \max \left\{ p(gx, gy), p(gx, Tx), p(gy, Ty), \frac{1}{\tau} [p(gx, Ty) + p(gy, Tx)] \right\}$$

for all $x, y \in X$ for which $gx$ and $gy$ are comparable and $\tau > 0$. If there exists $x_0 \in X$ such that $gx_0 \leq Tx_0$ and $T$ and $g$ are compatible, then $T$ and $g$ have a coincident point.

Proof. Let $x_0$ be such that $gx_0 \leq Tx_0$ since $T(X) \subseteq g(X)$, we can choose $x_1 \in X$ so that $gx_1 = Tx_0$. Since $T(x_1) \in g(X)$, there exists $x_2 \in X$ such that $gx_2 = Tx_1$. By induction, we can construct a sequence $\{x_n\}$ in $X$ such that $gx_{n+1} = Tx_n$ for every $n \geq 0$. Since $T$ is a monotone $g$-non decreasing mapping, $gx_0 \leq Tx_0$ implies $Tx_0 \leq Tx_1$. Similarly, since $gx_1 \leq gx_2$ we obtain $Tx_1 \leq Tx_2$ and $gx_2 \leq gx_3$. Continuing with this process we obtain

$$Tx_0 \leq Tx_1 \leq Tx_2 \leq \cdots \leq Tx_n \leq Tx_{n+1} \leq \cdots$$

Suppose that $p(Tx_n, Tx_{n+1}) > 0$ for all $n = 0, 1, 2, \ldots$. If not then $Tx_{n+1} = Tx_n$ for some $n$, $Tx_{n+1} = gx_{n+1}$ that is $T$ and...
have a coincident point \( x_{n+1} \) and so this will end the proof. Consider
\[
\tau + F(p(gx_{n+1}, gx_{n+2})) = \tau + F(p(Tx_n, Tx_{n+1})) \leq F(M(x_n, x_{n+1})).
\]
where \( M(x_n, x_{n+1}) = \max \{ p(gx_n, gx_{n+1}), p(gx_n, Tx_n), p(gx_{n+1}, Tx_n), \frac{p(gx_n, Tx_{n+1}) + p(gx_{n+1}, Tx_n)}{2} \} \]
\[
= \max \{ p(Tx_{n-1}, Tx_n), p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}), \frac{p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Tx_n)}{2} \}
\]
\[
= \max \{ p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}) \}. \]

Suppose
\[
\max \{ p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}) \} = p(Tx_n, Tx_{n+1})
\]
then
\[
\tau + F(p(gx_{n+1}, gx_{n+2})) = \tau + F(p(Tx_n, Tx_{n+1})) \leq F(p(Tx_n, Tx_{n+1})),
\]
which is a contradiction. Hence
\[
\max \{ p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}) \} = p(Tx_{n-1}, Tx_n).
\]
Then, for all \( n \in \mathbb{N} \), we can write
\[
F(p(Tx_n, Tx_{n+1})) \leq F(p(Tx_{n-1}, Tx_n)) - \tau \leq ... \leq F(p(Tx_0, Tx_1)) - n\tau. \tag{3.3}
\]
From (3.3), we obtain \( \lim_{n \to \infty} F(p(Tx_n, Tx_{n+1})) = -\infty \). Since \( F \in \Delta_F \) then by (F2) we have,
\[
\lim_{n \to \infty} p(Tx_n, Tx_{n+1}) = 0. \tag{3.4}
\]
By (F3) there exists \( k \in (0, 1) \) such that
\[
\lim_{n \to \infty} (p(Tx_n, Tx_{n+1}))^k F(p(Tx_n, Tx_{n+1})) = 0. \tag{3.5}
\]
Following (3.3), for all \( n \in \mathbb{N} \) we obtain
\[
(p(Tx_n, Tx_{n+1}))^k (F(p(Tx_n, Tx_{n+1})) - F(p(Tx_0, Tx_1))) \leq - (p(Tx_n, Tx_{n+1}))^k n\tau \leq 0. \tag{3.6}
\]
Taking into account (3.4), (3.5) and letting \( n \to \infty \) in (3.6) we get
\[
\lim_{n \to \infty} (p(Tx_n, Tx_{n+1}))^k = 0. \tag{3.7}
\]
Since (3.7) holds, there exists \( n_1 \in \mathbb{N} \) such that
\[
n(p(Tx_n, Tx_{n+1}))^k \leq 1,
\]
for all \( n \geq n_1 \). This implies that
\[
(p(Tx_n, Tx_{n+1}))^k \leq \frac{1}{n_1}, \text{ for all } n \geq n_1. \tag{3.8}
\]
Next, we will show that \( \{Tx_n\} \) is a Cauchy sequence. Consider \( n, m \in \mathbb{N} \) such that \( m > n \geq n_1 \), then by (3.8) and axiom (P3) of Definition 1.3 we have
\[
p(Tx_n, Tx_m) \leq p(Tx_n, Tx_{n+1}) + ... + p(Tx_{m-1}, Tx_m)
\]
\[
- \sum_{j=n+1}^{m-1} p(Tx_j, Tx_j)
\]
\[
\leq p(Tx_n, Tx_{n+1}) + p(Tx_{n+1}, Tx_{n+2}) + ... + p(Tx_{m-1}, Tx_m)
\]
\[
= \sum_{i=n}^{m-1} p(Tx_i, Tx_{i+1})
\]
\[
\leq \sum_{i=n}^{\infty} \frac{1}{i}.
\]
The convergence of the series \( \sum_{i=n}^{\infty} \frac{1}{i} \) implies that
\[
\lim_{n \to \infty} p(Tx_n, Tx_m) = 0.
\]
By Lemma 2.1 we get that, for any \( n, m \in \mathbb{N} \),
\[
p'(T_x, Tx_m) \leq 2 p(Tx_n, Tx_m) \to 0
\]
as \( n \to \infty \). This implies that, \( \{Tx_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence with respect to \( p' \) and hence converges by Lemma 2.2. Thus there exists \( u \in X \) such that, \( \lim Tx_n = u \). By the continuity of \( T \), we have \( \lim Tx_n = Tu \). Since \( gx_{n+1} = Tx_n \to u \) and the pair \( (T, g) \) is compatible, we have
\[
\lim_{n \to \infty} p(g(Tx_n), T(gx_n)) = 0. \tag{3.9}
\]
By axiom (P3) of Definition 1.3 we have
\[
p(Tu, gu) \leq p(Tu, T(gx_n)) + p(T(gx_n), g(Tx_n)) \\
+ p(g(Tx_n), gu) - p(T(gx_n), T(gx_n)) \\
- p(g(Tx_n), g(Tx_n)).
\] (3.10)

Now we apply Lemma 2.10. Letting \( n \to \infty \) in (3.10) and using the fact that \( T \) and \( g \) are continuous, we obtain that \( p(Tu, gu) = 0 \) that is \( Tu = gu \) and \( u \) is a coincidence point of \( T \) and \( g \).

One can deduce the following corollary from Theorem 3.2:

**Corollary 3.3.** Let \((X, p)\) be a complete partial metric space. Let \( T, g : X \to X \) be continuous mapping satisfying
\[
\tau + F(p(Tx, Ty)) \leq F(p(gx, gy))
\]
for all \( x, y \in X \) where \( F \in \Delta_F \) and \( \tau > 0 \). If \( Tgx = gTx \) and the mappings \( T, g \) satisfy the condition \( T(X) \subseteq g(X) \) of Theorem 3.2 then the mappings have a coincidence point.

4. Example

**Example 4.1.** Let \( M = [0, 1] \) with the usual order and let \((X, p)\) be a complete partial metric space defined by \( p(x, y) = \max\{x, y\} \) for all \( x, y \in M \). Let \( T, g : M \to M \) be a pair of compatible \( F \)-contraction mappings given by \( Tx = \frac{x^3}{3x + 9} \) and \( gx = \frac{y^2}{x + 3} \).

Let \( F : \mathbb{R}^+ \to \mathbb{R} \) be defined by \( F(\alpha) = \ln(\alpha) \) for all \( \alpha \in \mathbb{R}^+ \), and also let \( \tau = \ln(3) \). We show that the condition (3.2) of Theorem 3.2 is satisfied. If \( x, y \in X \) is such that \( p(Tx, Ty) > 0 \), this implies that
\[
\tau + F(p(Tx, Ty)) = \tau + \ln \left[ \max \left\{ \frac{x^3}{3x + 9}, \frac{y^3}{3y + 9} \right\} \right].
\]
Now suppose that \( y \geq x \), Without loss of generality, we obtain that,
\[
\tau + \ln \left[ \max \left\{ \frac{x^3}{3x + 9}, \frac{y^3}{3y + 9} \right\} \right] \leq F(M(x, y)),
\]
where \( M(x, y) = \max\{gx, gy\} = p(gx, gy) \).

Therefore
\[
\ln(3) + \ln \left[ \max \left\{ \frac{x^3}{3x + 9}, \frac{y^3}{3y + 9} \right\} \right] = \ln(3) + \ln \left( \frac{y^3}{3y + 9} \right) \leq \ln \left( \frac{y^2}{y + 3} \right) = F(p(x, y)).
\]
Likewise, if \( y \leq x \) we obtain that \( \tau + F(p(Tx, Ty)) \leq F(p(gx, gy)) \).

**Remark 4.2.** As we observe in Example 4.1, if the assumption that every pair of elements has a lower bound and upper bound is not satisfied then, a fixed point of \( T \) may not be unique.

5. Conclusion

In this paper, an approach has been developed for existence and uniqueness of coincidence points for a pair of ordered \( F \)-contraction mappings in an ordered partial metric space. The results due to Durmaz et al. [9] are extended for a pair of compatible ordered \( F \)-Contraction mappings.

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References


