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A survey on magnetic curves in 2-dimensional lightlike cone

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Abstract
The impact of magnetic fields on the moving particle trajectories by variational approach to the magnetic flow associated with the Killing magnetic field on lightlike cone $\mathbf{Q}^2 \subset E_1^3$ is examined. Different magnetic curves are found in the 2-dimensional lightlike cone using the Killing magnetic field of these curves. Some characterizations and definitions and examples of these curves with their shapes are given.

Keywords
Magnetic curve, lightlike cone, killing vector field.

AMS Subject Classification
53B30, 53B50, 53C80.

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1. Introduction

In differential gometry, there are many important consequences and properties of curves. Researchers follow labours about the curves. Working on the degenerate submanifolds of Lorentzian manifolds with degenerate metric provides us with meaningful relavance between null submanifolds and spacetime [1].

Although we know much about the submanifolds of the pseudo-Riemannian space forms, we have very few papers on submanifolds of the pseudo-Riemannian lightlike cone. A simply connected Riemannian manifold of dimension $n \geq 3$ is conformally flat if and only if it can be isometrically immersed as a hypersurface of the lightlike cone [3, 4]. In [5], Bozkurt et al. invastigated the magnetic flow associated with the Killing magnetic field in a three-dimensional oriented Riemann manifold (M^3, g). In [15], cone curves are studied in Minkowski space and also. Some applications of magnetic in same space space are given in [9]. In [16], the author gave the basic formulas on $\mathbf{Q}^2$ and $\mathbf{Q}^3$ and defined as following

$$\delta(s) = \frac{1}{2h_s^{-1}}(h^2 - 1, 2h, h^2 + 1)$$

where $\delta : I \to \mathbf{Q}^2 \subset E_1^3$. In [20], the author defined magnetic curves on a Riemannian manifold (M, g) according to trajectories of charged particles moving on M under the action of a magnetic field F^*. In [9], the authors investigated magnetic curves corresponding to the Killing magnetic field W in the 3$-$dimensional Minkowski space. In [5], the authors examined the trajectories of these magnetic fields and gave some characterizations and examples of these curves.

2. Preliminaries

Let E_1^3 be the pseudo-Euclidean space with the

$$g^*(X^*, Y^*) = \langle X^*, Y^* \rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

for all $X^* = (x_1, x_2, x_3)$, $Y^* = (y_1, y_2, y_3) \in E_1^3$. E_1^3 is a flat pseudo-Riemannian manifold of signature $(2, 1)$.

Let M be a submanifold of E_1^3. If the pseudo$-$Riemannian metric g^* of E_1^3 induces a pseudo-Riemannian metric $\widetilde{g}$ (respectively, a Riemannian metric, a degenerate quadratic form) on M, then M is called a timelike (respectively, spacelike, degenerate) submanifold of E_1^3.

The lightlike cone is defined by

$$\mathbf{Q}^2 = \left\{ \delta \in E_1^3 : g^*(\delta,\delta) = 0 \right\}.$$

Let E_1^3 be 3-dimensional Minkowski space and $\mathbf{Q}^2$ be the lightlike cone in E_1^3. A vector $W \neq 0$ in E_1^3 is called spacelike, timelike or lightlike, if $\langle W,W\rangle > 0$, $\langle W,W\rangle < 0$ or $\langle W,W\rangle = 0$, respectively. A frame field $\{\delta,\alpha,y\}$ on E_1^3 is called an asymptotic orthonormal frame field, if

$$\begin{aligned} \langle \delta,\delta\rangle &= \langle y,y\rangle = \langle \delta,\alpha\rangle = \langle y,\alpha\rangle = 0,\\ \langle \delta,y\rangle &= \langle \alpha,\alpha\rangle = 1. \end{aligned}$$

Let's curve $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a regular curve in $\mathbf{Q}^2$ for $t \in I$. From here, we suppose that the curve is regular.

Using $\delta'(s) = \alpha(s)$, from an asymptotic orthonormal frame along the curve $\delta(s)$ and the cone Frenet formulas of $\delta(s)$ can be written as follows:

$$\begin{aligned} \delta'(s) &= \alpha(s)\\ \alpha'(s) &= \kappa(s)\delta(s) - y(s) \\ y'(s) &= -\kappa(s)\alpha(s), \end{aligned} \tag{2.1}$$

where the function $\kappa(s)$ is called cone curvature function of the curve $\delta(s)$, [15].

Let $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ with arc length parameter s. Then $\delta = \delta(s) = (\delta_1,\delta_2,\delta_3)$ can be given as

$$\delta(s) = \frac{h_s^{-1}}{2}(h^2-1, 2h, h^2+1), \tag{2.2}$$

for some non constant function $h(s)$ and $h_s = h'$, [16].

The Lorentzian cross-product $\times : E_1^3 \times E_1^3 \to E_1^3$ is given as follows:

$$v^* \times w^* = \begin{bmatrix} i & j & -k\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3 \end{bmatrix},$$

where $v^* = (v_1,v_2,v_3)$, $w^* = (w_1,w_2,w_3) \in E_1^3$. Here i,j,k have classic sense. This product is skew-symmetric and $v^* \times w^*$ is ortogonal on both v^* and w^* as in E^3.

The Lorentz force ψ of a magnetic field F^* on $\mathbf{Q}^2$ is defined to be a skew-symetric operator given by

$$g^*(\psi(X^*),Y^*) = F^*(X^*,Y^*),$$

for all $X^*,Y^* \in \mathbf{Q}^2$.

The $\alpha-$magnetic trajectories of F^* are δ on $\mathbf{Q}^2$ that satisfy the Lorentzian equation

$$\nabla_{\delta'}\delta' = \psi(\delta').$$

Furthermore, the mixed product of the vector fields $X^*,Y^*,Z^* \in \mathbf{Q}^2$ is the defined by

$$g^*(X^* \times Y^*, Z^*) = dv_{g^*}(X^*,Y^*,Z^*),$$

where dv_{g^*} denotes a volume on $\mathbf{Q}^2$.

If W is a Killing vector in $\mathbf{Q}^2$ and assume that $F_W^* = \iota_W vol_{g^*}$ be the corresponding Killing magnetic field, here the inner product is indicated by ι. Hence the equation Lorentz force of F_W^* is

$$\psi(X^*) = W \times X^*,$$

for all $X^* \in \mathbf{Q}^2$. Corresponding the Lorentz equation can be written as

$$\nabla_{\delta'}\delta' = \psi(\delta') = W \times \delta'.$$

In Minkowski space E_1^3, take account of the Killing vector field $W = a\partial_\delta + b\partial_y + c\partial_z$, with $a,b,c \in \mathbb{R}$, the magnetic trajectories $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ assigned by W are solutions of the Lorentz equation

$$\delta'' = W\delta'.$$

3. Magnetic Curves in the Lightlike Cone $\mathbf{Q}^2 \subset E_1^3$

3.1 $\delta-$magnetic curves

In this part, we consider $\delta-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$.

Definition 3.1. *Let $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ and F_W^* be a magnetic field on $\mathbf{Q}^2 \subset E_1^3$. We call a $\delta-$magnetic curve if its δ vector field satisfies the Lorentz force equation*

$$\nabla_{\delta'}\delta = \nabla_\alpha\delta = \psi(\delta) = W_\delta \times \delta.$$

Theorem 3.2. *Let $\delta(s)$ be a unit speed spacelike $\delta-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{\delta,\alpha,y\}$. The Lorentz force in the Frenet frame are given as follows*

$$\psi^\delta = \begin{bmatrix} 0 & 1 & 0\\ w_3 & 0 & -1\\ 0 & -w_3 & 0 \end{bmatrix}, \tag{3.1}$$

where w_3 is a function defined by $w_3 = g^*(\psi(\alpha),y)$.

Proof. Suppose that $\delta(s)$ be a unit speed spacelike $\delta-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{\delta,\alpha,y\}$. From the definition of the magnetic curve and (2.1), we know that

$$\psi(\delta) = \overrightarrow{\alpha}.$$

Furthermore, since $\psi(\alpha) \in Span\{\delta,\alpha,y\}$, we can write

$$\psi(\alpha) = A_1\overrightarrow{\delta} + B_1\overrightarrow{\alpha} + C_1\overrightarrow{y}.$$

By using the following equalities

$$\begin{aligned} A_1 &= g^*(\psi(\alpha),y) = w_3\\ B_1 &= g^*(\psi(\alpha),\alpha) = 0\\ C_1 &= g^*(\psi(\alpha),\delta) = -g^*(\psi(\delta),\alpha)\\ &= -g^*(\alpha,\alpha) = -1, \end{aligned}$$

we get

$$\psi(\alpha) = w_3 \overrightarrow{\delta} - \overrightarrow{y}.$$

Similar to the above operations, we have

$$\psi(y) = -w_3 \overrightarrow{\alpha}.$$

□

Theorem 3.3. *Assume that $\delta(s)$ be a unit speed spacelike $\delta-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$. The curve δ is then a $\delta-$magnetic trajectory of a magnetic vector field W_δ necessary and sufficient condition the vector field W_δ can be given along the curve δ as the following*

$$W_\delta(s) = \overrightarrow{\delta(s)} + \overrightarrow{y(s)}. \tag{3.2}$$

Proof. Assume that $\delta(s)$ be a unit speed spacelike $\delta-$magnetic curve with $\delta-$magnetic trajectory of a magnetic field W_δ . Considering theorem 3.2 and definition 3.1, we have

$$W_\delta(s) = \overrightarrow{\delta(s)} + \overrightarrow{y(s)}.$$

On the contrary, we suppose that equation (3.2) holds. At the time we have $\psi(\delta) = W_\delta \times \delta$. Therefore the curve δ is a δ-magnetic trajectory of the magnetic vector field W_δ. □

Theorem 3.4. *Let δ be a $\delta-$magnetic trajectory according to the Killing vector field $W_\delta = \overrightarrow{\delta} + \overrightarrow{y}$ in $\mathbf{Q}^2 \subset E_1^3$. Then the curve δ can be expressed as follows*

$$\delta_\delta(s) = \delta(0) + cW_\delta, \tag{3.3}$$

where $c \in \mathbb{R}_0$.

Proof. We prove the theorem according to the W_δ. Since W_δ and $\delta(0)$ are linearly independent and W_δ is spacelike. Let $W_\delta, W_\delta \times \delta(0)$ and W^* be linearly independent and correspond

$$\begin{aligned} \langle W_\delta, W^* \rangle &= 0, \\ \langle W_\delta, W_\delta \times \delta(0) \rangle &= 0, \\ \langle W^*, W_\delta \times \delta(0) \rangle &= 0. \end{aligned}$$

So, we can take

$$W^* = 2\delta(0) - \langle W_\delta, \delta(0) \rangle W_\delta.$$

We can write

$$\begin{aligned} \delta(s) &= \delta(0) + \lambda_\delta(s) W_\delta + \mu_\delta(s) W_\delta \times \delta(0) + \rho_\delta(s) W^*, \\ \delta'(s) &= \delta'(0) + \lambda'_\delta(s) W_\delta + \mu'_\delta(s) W_\delta \times \delta(0) + \rho'_\delta(s) W^* \end{aligned}$$

where $\lambda_\delta(s), \mu_\delta(s), \rho_\delta(s), \lambda'_\delta(s), \mu'_\delta(s), \rho'_\delta(s)$ functions satisfied the following

$$\lambda_\delta(0) = 0, \mu_\delta(0) = 0, \rho_\delta(0) = 0,$$

$$\lambda'_\delta(0) = 0, \mu'_\delta(0) = 0, \rho'_\delta(0) = 0, \tag{3.4}$$

where $s = 0$. The Lorentz equation $\delta'(s) = W_\delta \times \delta(s)$ can be written as

$$\begin{aligned} & \delta'(0) + \lambda'_\delta(s) W_\delta + \mu'_\delta(s) W_\delta \times \delta(0) + \rho'_\delta(s) W^* \\ = \;& W_\delta \times \delta(0) - \mu_\delta(s) W^* - \rho_\delta(s) W_\delta \times \delta(0), \end{aligned}$$

since $\delta'(0) = W_\delta \times \delta(0)$ for $s = 0$, we have

$$\begin{aligned} 0 \;=\; & \lambda'_\delta(s) W_\delta + (\mu'_\delta(s) + \rho_\delta(s)) W_\delta \times \delta_\delta(0) \\ & + (\mu_\delta(s) + \rho'_\delta(s)) W^*, \end{aligned}$$

which is equivalent to

$$\lambda'_\delta(s) = 0, \mu'_\delta(s) + \rho_\delta(s) = 0, \mu_\delta(s) + \rho'_\delta(s) = 0.$$

Solving the previous differential equations and using the initial conditions (3.4), we get

$$\lambda_\delta(s) = c, \mu_\delta(s) = 0, \rho_\delta(s) = 0,$$

where $c \in \mathbb{R}_0$. Hence the curve δ is written as $\delta_\delta(s) = \delta(0) + cW_\delta$. □

3.2 $\alpha-$magnetic curves

In this part, we introduce $\alpha-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$. We also have some characterizations.

Definition 3.5. *Let $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ and F_W^* be a magnetic field on $\mathbf{Q}^2 \subset E_1^3$. We call $\alpha-$magnetic curve if its α vector field satisfies the Lorentz force equation*

$$\nabla_\alpha \alpha = \psi(\alpha) = W_\alpha \times \alpha.$$

Theorem 3.6. *Assume that $\delta(s)$ be a unit speed spacelike $\alpha-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{\delta, \alpha, y\}$. The Lorentz force in the Frenet frame are given as follows*

$$\psi^\alpha = \begin{bmatrix} w_1 & 1 & 0 \\ \kappa & 0 & -1 \\ 0 & -\kappa & -w_1 \end{bmatrix}, \tag{3.5}$$

where w_1 is a function defined by $w_1 = g^*(\psi(\delta), y)$.

Proof. Suppose that δ be a unit speed spacelike $\alpha-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ and let $\{\delta, \alpha, y\}$ be asymptotic orthonormal frame. From the definition of the $\alpha-$magnetic curve and (2.1), we know that

$$\psi(\alpha) = \delta'' = \kappa\delta - y.$$

Furthermore, since $\psi(\delta) \in Span\{\delta, \alpha, y\}$, we can wrıte

$$\psi(\delta) = A_2 \overrightarrow{\delta} + B_2 \overrightarrow{\alpha} + C_2 \overrightarrow{y}.$$

Additionaly, considering the followings,

$$\begin{aligned} A_2 &= g^*(\psi(\delta), y) = w_1, \\ B_2 &= g^*(\psi(\delta), \alpha) = -g^*(\psi(\alpha), \delta) \\ &= -g^*(\kappa\delta - y, \delta) = 1, \\ C_2 &= g^*(\psi(\delta), \delta) = 0, \end{aligned}$$

we have that

$$\psi(\delta) = w_1 \overrightarrow{\delta} + \overrightarrow{\alpha}.$$

In a similar way, we get

$$\psi(y) = -\kappa \overrightarrow{\alpha} - w_1 \overrightarrow{y}.$$

□

Theorem 3.7. *Assume that δ be a unit speed spacelike curve in the $\mathbf{Q}^2 \subset E_1^3$. The curve δ is an $\alpha-$magnetic trajectory of a magnetic field W_α necessary and sufficient condition the vector field W_α is given as*

$$W_\alpha = \mp w_1 \overrightarrow{\alpha} \tag{3.6}$$

and δ is a geodesic curve, where $w_1 = g^*(\psi(\delta), y)$, the cone curvature function $\kappa(s) = -1$.

Proof. Assume that δ be a unit speed spacelike $\alpha-$magnetic curve with $\alpha-$magnetic trajectory of the magnetic field W_α. Considering theorem 3.6 and definition 3.5, we obtain

$$W_\alpha(s) = \mp w_1 \overrightarrow{\alpha(s)}.$$

Conversely, we assume that equation (3.6) hold and δ be a geodesic curve. Then we get $\psi(\alpha) = W_\alpha \times \alpha = 0$, so δ is $\alpha-$magnetic curve. □

Theorem 3.8. *Let δ be a $\alpha-$magnetic trajectory according to the Killing vector field $W_\alpha = \mp w_1 \overrightarrow{\alpha}$ in $\mathbf{Q}^2 \subset E_1^3$. Then the curve δ can be expressed as follows*

$$\delta_\alpha(s) = \delta(0) + cs \tag{3.7}$$

where $c = \mp \frac{1}{w_1} \in \mathbb{R}_0$.

Remark that, if $W_\alpha = \mp w_1 \overrightarrow{\alpha}$ the magnetic curve δ is a straight line in the direction of W_α.

Proof. We prove the theorem according to the W_α. Since $W_\alpha = \mp w_1 \overrightarrow{\alpha}$, we can write

$$\nabla_\alpha \alpha = \alpha'(s) = \delta''(s) = W_\alpha \times \alpha(s) = 0.$$

Hence, we obtain

$$\delta(s) = cs + \delta(0); c \in \mathbb{R}_0.$$

Furthermore, since W_α and $\delta'(0)$ are linearly dependent and W_α is spacelike. Let M,$W_\alpha, M \times W_\delta$ be linearly independent and satisfy

$$\langle W_\alpha, M\rangle = 0, \langle M, M\rangle = 1, \langle M \times W_\alpha, M \times W_\alpha\rangle = 1.$$

We can write

$$\begin{aligned} \delta(s) &= \delta(0) + \lambda_\alpha(s) W_\alpha + \mu_\alpha(s) M + \rho_\alpha(s) M \times W_\alpha, \\ \delta'(s) &= \delta'(0) + \lambda'_\alpha(s) W_\alpha + \mu'_\alpha(s) M + \rho'_\alpha(s) M \times W_\alpha, \end{aligned}$$

where $\lambda_\alpha(s), \mu_\alpha(s), \rho_\alpha(s), \lambda'_\alpha(s), \mu'_\alpha(s), \rho'_\alpha(s)$ are functions satisfying

$$\lambda_\alpha(0) = 0, \mu_\alpha(0) = 0, \rho_\alpha(0) = 0, \lambda'_\alpha(0) = \mp \frac{1}{w_1}, \tag{3.8}$$

$$\mu'_\alpha(0) = 0, \rho'_\alpha(0) = 0,$$

for $s = 0$. The Lorentz equation $\delta''(s) = W_\alpha \times \delta'(s)$ can be written as

$$\begin{aligned} &\lambda''_\alpha(s) W_\alpha + \mu''_\alpha(s) M + \rho''_\alpha(s) M \times W_\alpha \\ = \; & -\mu'_\alpha(s) M \times W_\alpha + \rho'_\alpha(s) M, \end{aligned}$$

which is equivalent to

$$\lambda''_\alpha(s) = 0, \rho''_\alpha(s) = -\mu'_\alpha(s), \mu''_\alpha(s) = \rho'_\alpha(s).$$

Solving the previous differential equations and using the initial conditions of (3.8), we write

$$\begin{aligned} \lambda_\alpha(s) &= \mp \frac{1}{w_1} s \\ \mu_\alpha(s) &= 0 \\ \rho_\alpha(s) &= 0, \end{aligned}$$

where $w_1 = g^*(\psi(\delta), y)$. Hence the curve δ is written as follows

$$\delta(s) = \delta(0) \mp \frac{1}{w_1} s W_\alpha,$$

where $c = \mp \frac{1}{w_1} \in \mathbb{R}_0$. □

3.3 $y-$magnetic curves

In this part, we introduce $y-$magnetic curves in $\mathbf{Q}^2 \subset E_1^3$ and we get some characterizations using $y-$magnetic curves.

Definition 3.9. *Let $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ and F_W^* be a magnetic field on $\mathbf{Q}^2 \subset E_1^3$. We call δ an $y-$magnetic curve if its y vector field fulfils the Lorentz force equations*

$$\nabla_\alpha y = \psi(y) = W_y \times y.$$

Theorem 3.10. *Assume that $\delta(s)$ be a unit speed spacelike $y-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{\delta, \alpha, y\}$.The Lorentz force in the Frenet frame are given as follows*

$$\psi^y = \begin{bmatrix} 0 & w_2 & 0 \\ \kappa & 0 & -w_2 \\ 0 & -\kappa & 0 \end{bmatrix}, \tag{3.9}$$

where w_2 is a function given by $w_2 = g^*(\psi(\delta), \alpha), \kappa = -w_2$.

Proof. Let δ be a unit speed spacelike $y-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ and let $\{\delta, \alpha, y\}$ be asymptotic orthonormal frame. From the $y-$magnetic curve definition and equation (2.1), we can write

$$\nabla_\alpha y = \psi(y) = -\kappa \overrightarrow{\alpha}.$$

Since $\psi(\delta) \in Span\{\delta, \alpha, y\}$, we can write

$$\psi(\delta) = A_3 \overrightarrow{\delta} + B_3 \overrightarrow{\alpha} + C_3 \overrightarrow{y}.$$

Then, using the following equalities,

$$\begin{aligned} A_3 &= g^*(\psi(\delta), y) = -g^*(\psi(y), \delta) = -g^*(-\kappa\alpha, \delta) = 0 \\ B_3 &= g^*(\psi(\delta), \alpha) = w_2 \\ C_3 &= g^*(\psi(\delta), \delta) = 0, \end{aligned}$$

we obtain

$$\psi(\delta) = w_2 \overrightarrow{\alpha}.$$

Similarly, we can get that

$$\psi(\alpha) = \kappa \overrightarrow{\delta} - w_2 \overrightarrow{y}.$$

□

Theorem 3.11. *Suppose that δ be a unit speed spacelike curve in the $\mathbf{Q}^2 \subset E_1^3$. The curve δ is a $y-$magnetic trajectory of a magnetic field W_y necessary and sufficient condition the vector field W_y is written as*

$$W_y = \mp\kappa(\overrightarrow{\delta} + \overrightarrow{y}), \tag{3.10}$$

where $\kappa = g^*(\psi(\alpha), \delta)$.

Proof. Suppose that δ be a unit speed spacelike $y-$magnetic curve trajectory of a magnetic field W_y. Considering theorem 3.10 and definition 3.9, we get

$$W_y = \mp\kappa(\overrightarrow{\delta} + \overrightarrow{y}).$$

On the contrary, let's equation (3.10) hold. Then we have $\psi(y) = W_y \times y$, so δ is a $y-$magnetic curve. □

Theorem 3.12. *Let δ be a $y-$magnetic trajectory according to the killing vector field $W_y = \mp\kappa(\overrightarrow{\delta} + \overrightarrow{y})$ in $\mathbf{Q}^2 \subset E_1^3$. Then the curve δ can be expressed as follows*

$$\begin{aligned} \delta_y(s) &= \delta(0) + 2\kappa^2\left(c_2 s + 2c_3 \cosh s\right)\delta''(0) \\ &\quad + \left(cs^2 + c_2\left(1 - s - 2\cosh s\right)\langle W_y, \delta''(0)\rangle\right) W_y \\ &\quad +2c_3(s - \sinh s) W_y \times \delta''(0), \end{aligned} \tag{3.11}$$

where $c, c_3, c_2 \in \mathbb{R}_0^+$.

Proof. We prove the theorem according to the W_y. Since W_y and $\delta''(0)$ linearly independent, we can write $\langle W_y, \delta''(0)\rangle \neq 0$. We consider

$$Z = 2\kappa^2\delta''(0) - \langle W_y, \delta''(0)\rangle) W_y.$$

Hence, $W_y, W_y \times \delta''(0)$ and Z are linearly independent and satisfy

$$\begin{aligned} \langle W_y, Z\rangle &= 0, \\ \langle W_y \times \delta''(0), Z\rangle &= 0, \\ \langle W_y \times \delta''(0), W_y\rangle &= 0, \end{aligned}$$

so, we can write

$$\begin{aligned} \delta(s) &= \delta(0) + \lambda_y(s) W_y + \mu_y(s) W_y \times \delta''(0) \\ &\quad +\rho_y(s) Z, \end{aligned}$$

where $\lambda_y(s), \mu_y(s), \rho_y(s)$ are functions satisfying and for $s = 0$, we have

$$\lambda_y(0) = 0, \mu_y(0) = 0, \rho_y(0) = 0,$$

$$\mu_y'(0) = 0, \rho_y'(0) = c_2, \lambda_y'(0) = c_2\langle W_y, \delta''(0)\rangle. \tag{3.12}$$

The Lorentz equation $\delta'''(s) = W_y \times \delta''(s)$ can be written as follows

$$\begin{aligned} & \lambda_y'''(s) W_y + \mu_y'''(s) W_y \times \delta''(0) + \rho_y'''(s) Z \\ = \; & -\mu_y''(s) Z - \rho_y''(s) W_y \times \delta''(0) \end{aligned}$$

which is equivalent to

$$\lambda_y'''(s) = 0, \rho_y'''(s) = -\mu_y''(s), \mu_y'''(s) = -\rho_y''(s).$$

Solving the previous differential equations and using the initial conditions (3.12) we write

$$\begin{aligned} \lambda_y(s) &= cs^2 + c_2\langle W_y, \delta''(0)\rangle \\ \mu_y(s) &= 2c_3(s - \sinh s) \\ \rho_y(s) &= c_2 s + 2c_3 \cosh s, \end{aligned}$$

where $c, c_3, c_2 \in \mathbb{R}_0^+$. Hence the theorem is proved. □

4. Corollary

Corollary 4.1. *Let $\delta : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ as follows*

$$\delta(s) = \frac{h_s^{-1}}{2}(h^2 - 1, 2h, h^2 + 1),$$

for some non constant function $h(\ s)$. Then we can write the following statements:

1. *If δ is a $\delta-$ magnetic curve, then the Killing vector field W_δ and $\delta-$ magnetic curve can be written as following*

$$\begin{aligned} W_\delta(s) &= (1-\frac{h_s^{-2}h_{ss}^2}{2})\delta(s)+\begin{pmatrix} h_s^{-1}h_{ss}h-h_s, \\ h_s^{-1}h_{ss}, \\ h_s^{-1}h_{ss}h-h_s \end{pmatrix} \\ \delta_\delta(s) &= \delta(0)+c((1-\frac{h_s^{-2}h_{ss}^2}{2})\delta(s) \\ &\quad +\left(h_s^{-1}h_{ss}h-h_s,\ h_s^{-1}h_{ss}, h_s^{-1}h_{ss}h-h_s\right)), \end{aligned}$$

where $c\in\mathbb{R}_0^+$.

2. *If δ is an $\alpha-$ magnetic curve, then the Killing vector field W_α and $\alpha-$ magnetic curve can be written as following*

$$\begin{aligned} W_\alpha(s) &= \mp w_1\left\{(-h_s^{-1}h_{ss})\delta(s)+(h,1,h)\right\} \\ \delta_\alpha(s) &= \delta(0)\mp cw_1 s\begin{pmatrix} (-h_s^{-1}h_{ss})\delta(s) \\ +(h,1,h) \end{pmatrix}, \end{aligned}$$

where w_1 is a function defined by $w_1=g^*(\psi(\delta),y), c\in\mathbb{R}_0^3$.

3. *If δ is a $y-$ magnetic curve, then the Killing vector field W_y and $y-$ magnetic curve can be written as*

$$\begin{aligned} W_y(s) &= \mp\kappa\left\{\begin{array}{c} (1-\frac{h_s^{-2}h_{ss}^2}{2})\delta(s) \\ +\begin{pmatrix} h_s^{-1}h_{ss}h-h_s, \\ h_s^{-1}h_{ss}, \\ h_s^{-1}h_{ss}h-h_s \end{pmatrix} \end{array}\right\} \\ \delta_y(s) &= \delta(0)+2\kappa^2\left(c_2s+2c_3\cosh s\right)\delta''(0) \\ &\quad +\left(cs^2+c_2\left(1-s-2\cosh s\right)\langle W_y,\delta''(0)\rangle\right)W_y \\ &\quad +2c_3(s-\sinh s)W_y\times\delta''(0), \end{aligned}$$

where $\delta''(0)$ is as follows

$$\delta''(0) = -\frac{1}{2}h_s^{-2}h_{ss}^2\mid_0\delta(0)$$
$$+\begin{pmatrix} h_s^{-1}h_{ss}h-h_s, \\ h_s^{-1}h_{ss}, \\ h_s^{-1}h_{ss}h-h_s \end{pmatrix}\quad\mid\ _0.$$

5. Examples

We can give the Example 1 and 2 held δ,α and y- magnetic curves in 2-dimensional lightlike cone $\mathbf{Q}^2$.

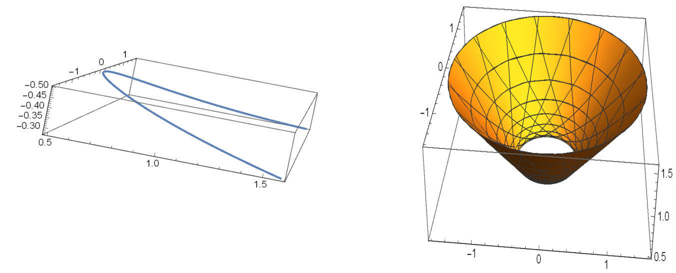
Example 5.1. *We consider a spacelike curve with arc length $\delta=\delta(s):I\to\mathbf{Q}^2\subset E_1^3$ defined by*

$$\delta(s)=\left(-\frac{\cos s}{2},\tan s,\frac{1}{\cos s}-\frac{\cos s}{2}\right).$$

The magnetic trajectories with to the Killing vector field of this curve are given as follows:

1. *The $\delta-$magnetic trajectory with the killing vector field W_δ is given as*

$$\delta_\delta(s)=\begin{pmatrix} -\frac{1}{2}+c(-\frac{1}{\cos s}-\frac{\cos s}{2}+\frac{\sin^2 s}{4\cos s}), \\ c\frac{\tan^3 s}{2}, \\ \frac{1}{2}+c(-\frac{\cos s}{2}+\frac{\tan s-\tan^2 s}{2\cos s}) \end{pmatrix},$$



(a) The curve δ (b) The rotated surface formed by the δ
Figure 1. Graphics of δ curve

where

$$W_\delta(s)=\begin{pmatrix} -\frac{1}{\cos s}-\frac{\cos s}{2}+\frac{\sin^2 s}{4\cos s},\frac{\tan^3 s}{2}, \\ -\frac{\cos s}{2}+\frac{\tan s-\tan^2 s}{2\cos s} \end{pmatrix},$$

$c\in\mathbb{R}_0^+$.

2. *The $\alpha-$magnetic trajectory with the killing vector field V_α is given as*

$$\delta_\alpha(s)=\left(-\frac{1}{2}+\frac{1}{2}s\sin s,\frac{s}{\cos^2 s},s(\frac{\tan s}{\cos s}+\frac{\sin s}{2})\right)$$

where

$$W_\alpha(s)=\mp w_1(\frac{\sin s}{2},\frac{1}{\cos^2 s},\frac{\tan s}{\cos s}+\frac{\sin s}{2}).$$

3. *The $y-$magnetic trajectory with the killing vector field V_y is given as*

$$\begin{aligned} \delta_y(s)=(-\frac{1}{2},0,\frac{1}{2})+ &\quad 2\kappa(c_2s+2c_3\cosh s).(-1,0,-1) \\ &\quad +\begin{pmatrix} (cs^2+c_2(1-s-2\cos hs) \\ \langle W_y,(-1,0,-1)\rangle)W_y \end{pmatrix} \\ &\quad +2c_3(s-\sinh s)W_y\times(-1,0,-1) \end{aligned}$$

or

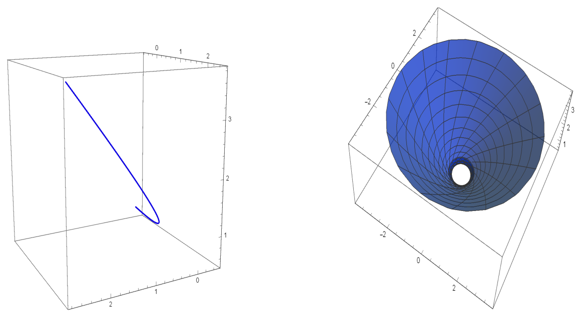
$$\begin{aligned} \delta_y(s) &= (-\frac{1}{2}-2\kappa(c_2s+2c_3\cosh s) \\ &\quad -3\kappa^2(cs^2+c_2(1-s-2\cos hs)), \\ &\quad -4c_3(s-\sinh s), \\ &\quad \frac{1}{2}-2\kappa(c_2s+2c_3\cosh s) \\ &\quad -\kappa(cs^2+c_2(1-s-2\cos hs)), \end{aligned}$$

where $W_y(s)=\mp\kappa(s)W_\delta(s)$.

We give the graphics of δ curve in Figure 1. In addition, we show δ, $\alpha, y-$magnetic curves of δ curve and their surfaces and W_δ, W_α, W_y trajectory curve and their surfaces in Figure 3.

Example 5.2. *We study a spacelike curve with arc length $\delta^*=\delta^*(s):I\to\mathbf{Q}^2\subset E_1^3$ defined by*

$$\delta^*(s)=\left(\frac{s^2}{2}+s,s+1,\frac{s^2}{2}+s+1\right).$$



(a) The curve δ^* (b) The rotated surface formed by δ^*

Figure 2. Graphics of δ^* curve

The magnetic trajectories with to the Killing vector field of this curve are given as follows:

1. *The $\delta-$magnetic trajectory with the Killing vector field W^*_δ is given as*

$$\delta^*_\delta(s)=\begin{pmatrix} c(\frac{s^2}{2}+s+1),\\ 1+c(s+1),\\ 1+c(\frac{s^2}{2}+s)\end{pmatrix},$$

where

$$W^*_\delta(s)=\left(\frac{s^2}{2}+s-1,s+1,\frac{s^2}{2}+s\right),$$

$c\in\mathbb{R}^+_0$.

2. *The $\alpha-$magnetic trajectory with the Killing vector field V^*_a is given as*

$$\delta^*_\alpha(s)=\left(s^2+s,1+s,s^2+s+1\right),$$

where $W^*_\alpha(s)=\mp w_1(s+1,1,s+1)$.

3. *The $y-$magnetic trajectory with the Killing vector field W^*_y is given as*

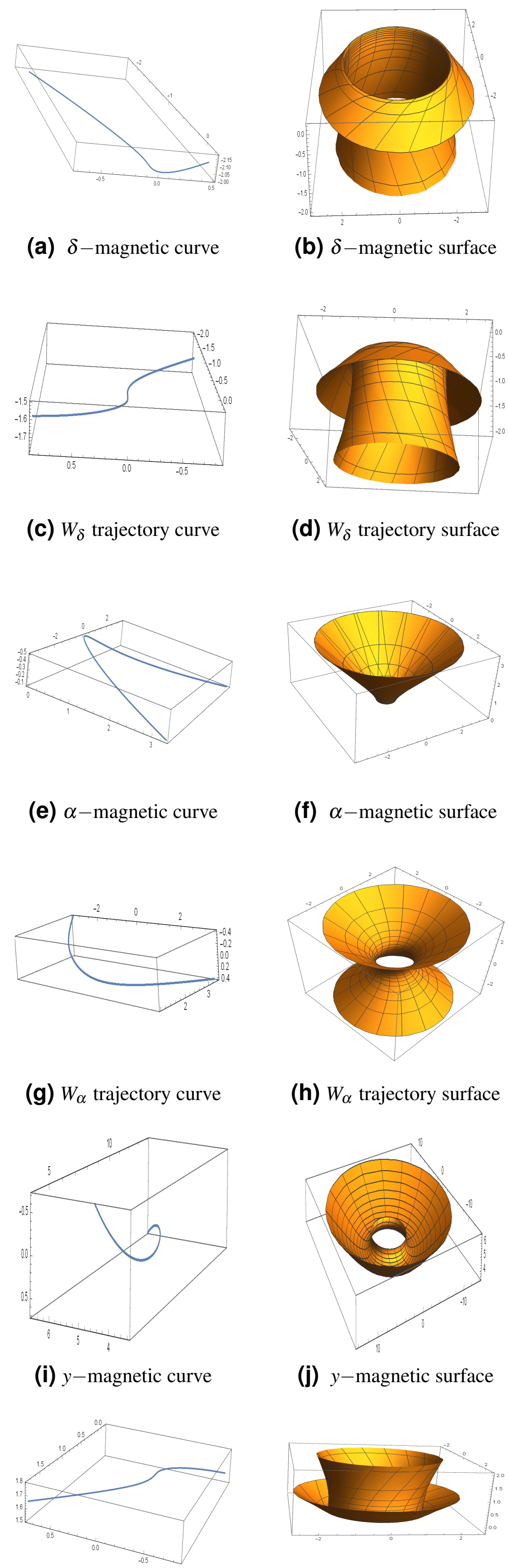
$$\begin{aligned}\delta^*_y(s) &= (0,1,1)+2\kappa(c_2s+2c_3\cosh s)(-1,0,-1)\\ &\quad+\begin{pmatrix}(cs^2+c_2(1-s-2\cos hs)\\ \langle W^*_y,(-1,0,-1)\rangle)W^*_y\end{pmatrix}\\ &\quad+2c_3(s-\sinh s)W^*_y\times(-1,0,-1)\end{aligned}$$

or

$$\delta^*_y(s)=\left\{\begin{array}{c}\begin{pmatrix}-2\kappa(c_2s+2c_3\cosh s)\\ \pm\kappa(cs^2+\kappa c_2(1-s-2\cos hs)\\ -2c_3(s-\sinh s)\end{pmatrix},\\ \begin{pmatrix}1\mp(cs^2+\kappa c_2(1-s-2\cos hs)\\ -2c_3(s-\sinh s)\end{pmatrix},\\ \begin{pmatrix}1-2\kappa(c_2s+2c_3\cosh s)\\ -2c_3(s-\sinh s)\end{pmatrix}\end{array}\right\}$$

where $W^*_y(s)=\mp\kappa W^*_\delta(s)$.

We give the graphics of δ^* curve in Figure 2. Furthermore, we show $\delta,\alpha,y-$magnetic curves of δ^* curve and their surfaces and $W^*_\delta,W^*_\alpha,W^*_y$ trajectory curves and their surfaces in Figure 4.



(a) $\delta-$magnetic curve **(b)** $\delta-$magnetic surface

(c) W_δ trajectory curve **(d)** W_δ trajectory surface

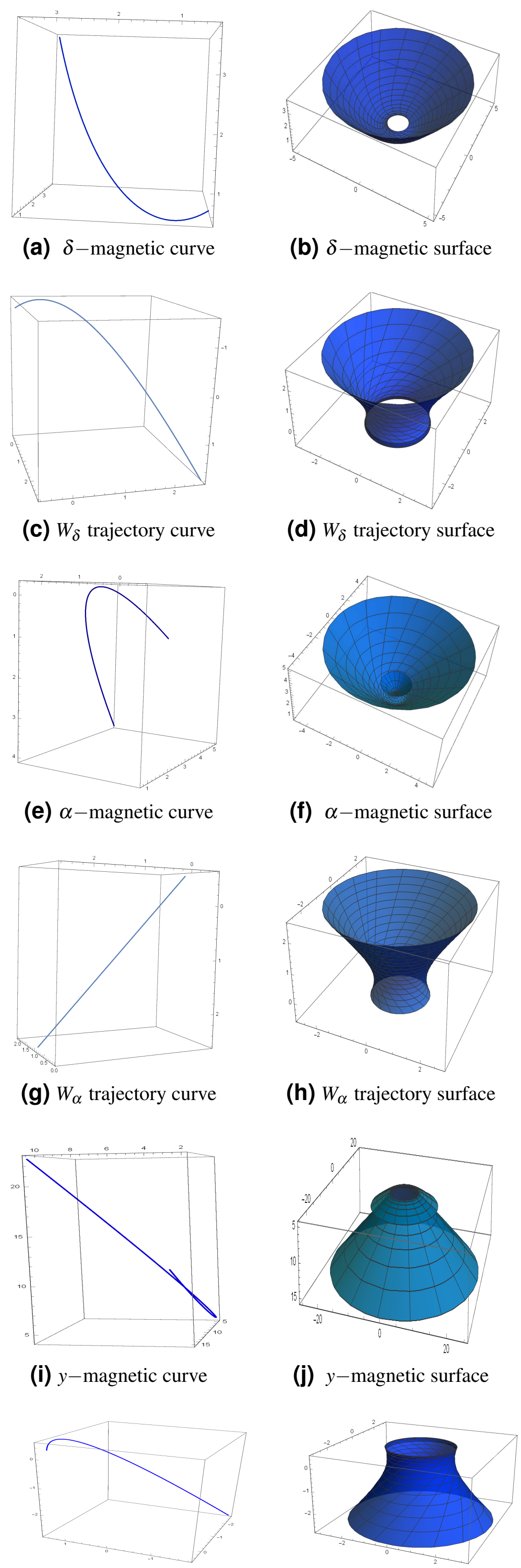
(e) $\alpha-$magnetic curve **(f)** $\alpha-$magnetic surface

(g) W_α trajectory curve **(h)** W_α trajectory surface

(i) $y-$magnetic curve **(j)** $y-$magnetic surface

(k) W_y trajectory curve **(l)** W_y trajectory surface

Figure 3. The magnetic curves for δ curve and their rotated surfaces



(a) $\delta-$magnetic curve **(b)** $\delta-$magnetic surface

(c) W_δ trajectory curve **(d)** W_δ trajectory surface

(e) $\alpha-$magnetic curve **(f)** $\alpha-$magnetic surface

(g) W_α trajectory curve **(h)** W_α trajectory surface

(i) $y-$magnetic curve **(j)** $y-$magnetic surface

(k) W_y trajectory curve **(l)** W_y trajectory surface

Figure 4. The magnetic curves for δ^* curve and their rotated surfaces

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