On continuous selections of weakly \((1,2)^*\)-\(g\)-closed maps

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Abstract
In this paper, the concepts of weakly \((1,2)^*\)-\(g\)-continuous maps, weakly \((1,2)^*\)-\(g\)-compact spaces and weakly \((1,2)^*\)-\(g\)-connected spaces are introduced and some of their properties are investigated.

Keywords
Weakly \((1,2)^*\)-\(g\)-closed set, Weakly \((1,2)^*\)-\(g\)-continuous maps, Weakly \((1,2)^*\)-\(g\)-compact spaces and Weakly \((1,2)^*\)-\(g\)-connected.

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1. Introduction and Preliminaries

Quite recently, several authors investigated some new maps and notions (see for example [14], [10]. It is the objective of this paper, the concepts of weakly \((1,2)^*\)-\(g\)-continuous maps, weakly \((1,2)^*\)-\(g\)-compact spaces and weakly \((1,2)^*\)-\(g\)-connected spaces are introduced and some of their properties are investigated with addition Examples. Throughout this paper, \((X, \tau_1, \tau_2), (Y, \tau_1, \tau_2)\) and \((Z, \eta_1, \eta_2)\) (briefly, \(X, Y\) and \(Z\)) will denote bitopological spaces.

Definition 1.1. Let \(S\) be a subset of \(X\). Then \(S\) is said to be \(\tau_{1,2}\)-open \([8]\) if \(S = A \cup B\) where \(A \in \tau_1\) and \(B \in \tau_2\).

The complement of \(\tau_{1,2}\)-open set is called \(\tau_{1,2}\)-closed.

Notice that \(\tau_{1,2}\)-open sets need not necessarily form a topology.

Definition 1.2. \([8]\) Let \(S\) be a subset of a bitopological space \(X\). Then

1. the \(\tau_{1,2}\)-closure of \(S\), denoted by \(\tau_{1,2}\)-cl\((S)\), is defined as \(\cap\{F : S \subseteq F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\}\).

2. the \(\tau_{1,2}\)-interior of \(S\), denoted by \(\tau_{1,2}\)-int\((S)\), is defined as \(\cup\{F : F \subseteq S \text{ and } F \text{ is } \tau_{1,2}\text{-open}\}\).

Definition 1.3. A subset \(S\) of a bitopological space \(X\) is called

1. \((1,2)^*\)-\(g\)-generalized closed (briefly, \((1,2)^*\)-\(g\)-closed) set \([11]\) if \(\tau_{1,2}\)-cl\((S) \subseteq U\) whenever \(S \subseteq U\) and \(U\) is \(\tau_{1,2}\)-open in \(X\).

2. \((1,2)^*\)-\(g\)-semi-generalized closed (briefly, \((1,2)^*\)-\(sg\)-closed) set \([9]\) if \((1,2)^*\)-\(cl\)(\(S) \subseteq U\) whenever \(S \subseteq U\) and \(U\) is \((1,2)^*\)-semi-open in \(X\).

3. \((1,2)^*\)-\(alpha\)-generalized closed (briefly, \((1,2)^*\)-\(\alpha-g\)-closed) set \([12]\) if \((1,2)^*\)-\(\alpha-cl\)(\(S) \subseteq U\) whenever \(S \subseteq U\) and \(U\) is \(\tau_{1,2}\)-open in \(X\).

4. \((1,2)^*\)-\(\bar{g}\)-closed set \([3]\) if \(\tau_{1,2}\)-cl\((S) \subseteq U\) whenever \(S \subseteq U\) and \(U\) is \((1,2)^*\)-\(\bar{g}\)-open in \(X\).

5. \((1,2)^*\)-\(pi\)-\(\bar{g}\)-closed set \([10]\) if \(\tau_{1,2}\)-cl\((S) \subseteq U\) whenever \(S \subseteq U\) and \(U\) is \((1,2)^*\)-\(\pi\)-open in \(X\).

The complements of the above mentioned open sets are called their respective closed sets.

The family of all \((1,2)^*\)-\(\bar{g}\)-open (resp. \((1,2)^*\)-\(\bar{g}\)-\(\alpha\)-open) sets in \(X\) is denoted by \((1,2)^*\)-\(\bar{g}O(X)\) (resp. \((1,2)^*\)-\(\bar{g}\alpha(X)\)).

Definition 1.4. \([4]\) For every set \(S \subseteq X\), we define the \((1,2)^*\)-\(\bar{g}\)-\(\bar{g}\)-\(\alpha\)-closed of \(S\) to be the intersection of all \((1,2)^*\)-\(\bar{g}\)-\(\alpha\)-closed sets containing \(S\). That is \((1,2)^*\)-\(\bar{g}\)-\(\bar{g}\)-\(\alpha\)-cl\((S) = \cap\{F : S \subseteq F \in (1,2)^*\)-\(\bar{g}\)-\(\alpha\)-closed\}\).
Definition 1.5. Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called
1. completely $(1,2)^\ast$-continuous \cite{10} (resp. $(1,2)^\ast$-$R$-map \cite{10}) if $f^{-1}(V)$ is regular $(1,2)^\ast$-open in $X$ for each $\sigma_1, \sigma_2$-open (resp. regular $(1,2)^\ast$-open) set $V$ of $Y$.
2. perfectly $(1,2)^\ast$-continuous \cite{10} if $f^{-1}(V)$ is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed in $X$ for each $\sigma_1, \sigma_2$-open set $V$ of $Y$.
3. $(1,2)^\ast$-$\bar{g}$-continuous \cite{4} if $f^{-1}(V)$ is $(1,2)^\ast$-$\bar{g}$-closed in $X$ for every $\sigma_1, \sigma_2$-closed set $V$ of $Y$.
4. $(1,2)^\ast$-$\bar{g}$-irresolute \cite{4} if $f^{-1}(V)$ is $(1,2)^\ast$-$\bar{g}$-closed in $X$ for every $(1,2)^\ast$-$\bar{g}$-closed set $V$ of $Y$.
5. $(1,2)^\ast$-$sg$-irresolute \cite{13} if $f^{-1}(V)$ is $(1,2)^\ast$-$sg$-open in $X$ for every $(1,2)^\ast$-$sg$-open set $V$ of $Y$.
6. $(1,2)^\ast$-$\bar{g}$-closed \cite{4} if the image of every $\tau_{1,2}$-closed set in $X$ is $(1,2)^\ast$-$\bar{g}$-closed in $Y$.

Definition 1.6. A subset $S$ of a bitopological space $X$ is called
1. weakly $(1,2)^\ast$-$g$-closed (briefly, $(1,2)^\ast$-$wg$-closed) set \cite{14} if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and $U$ is $\tau_{1,2}$-open in $X$.
2. weakly $(1,2)^\ast$-$\pi g$-closed (briefly, $(1,2)^\ast$-$w\pi g$-closed) set \cite{14} if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and $U$ is $(1,2)^\ast$-$\pi g$-open in $X$.
3. regular weakly $(1,2)^\ast$-generalized closed (briefly, $(1,2)^\ast$-rwcg) set \cite{14} if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and $U$ is regular $(1,2)^\ast$-open in $X$.

Definition 1.7. \cite{4} A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be an $(1,2)^\ast$-$g$-open map if the image $f(S)$ is $(1,2)^\ast$-$\bar{g}$-open in $Y$ for each $\tau_{1,2}$-open set $S$ of $X$.

Remark 1.8. \cite{4} Every $\tau_{1,2}$-open set is $(1,2)^\ast$-$sg$-open but not conversely.

Remark 1.9. \cite{14} For a subset of a bitopological space, we have following implications:

regular $(1,2)^\ast$-open $\rightarrow$ $(1,2)^\ast$-$\pi g$-open $\rightarrow$ $\tau_{1,2}$-open

Definition 1.10. A subset $S$ of a bitopological space $X$ is said to be nowhere dense if $\tau_{1,2}$-$int(\tau_{1,2}$-$cl(S)) = \phi$.

Definition 1.11. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then $f$ is said to be
1. contra-$(1,2)^\ast$-$\bar{g}$-continuous \cite{14} if $f^{-1}(V)$ is $(1,2)^\ast$-$\bar{g}$-closed in $X$ for every $\sigma_1, \sigma_2$-open set of $Y$.
2. $(1,2)^\ast$-continuous \cite{14} if $f^{-1}(V)$ is $\tau_{1,2}$-closed in $X$ for every $\sigma_1, \sigma_2$-closed set of $Y$.

Remark 1.12. \cite{4} Every $(1,2)^\ast$-continuous function is $(1,2)^\ast$-$\bar{g}$-continuous but not conversely.

Definition 1.13. \cite{5} A subset $S$ of a bitopological space $X$ is called a weakly $(1,2)^\ast$-$\bar{g}$-closed (briefly, $(1,2)^\ast$-$w\bar{g}$-closed) set if $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(S)) \subseteq U$ whenever $S \subseteq U$ and $U$ is $(1,2)^\ast$-$sg$-open in $X$.

Corollary 1.14. \cite{5} If a subset $S$ of a bitopological space $X$ is both $\tau_{1,2}$-open and $(1,2)^\ast$-$w\bar{g}$-closed, then it is both regular $(1,2)^\ast$-open and regular $(1,2)^\ast$-closed in $X$.

Proposition 1.15. \cite{5} Every $(1,2)^\ast$-$\bar{g}$-open set is $(1,2)^\ast$-$w\bar{g}$-open but not conversely.

### 2. Weakly $(1,2)^\ast$-$\bar{g}$-Continuous maps

Definition 2.1. A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(1,2)^\ast$-$\bar{g}$-continuous (briefly $(1,2)^\ast$-$w\bar{g}$-continuous) if the inverse image of each regular $(1,2)^\ast$-open set of $Y$ is $(1,2)^\ast$-$\bar{g}$-closed in $X$.

Definition 2.2. A space $X$ is called $(1,2)^\ast$-$\bar{g}$-connected if $X$ is not the union of two disjoint nonempty $(1,2)^\ast$-$\bar{g}$-open sets.

Definition 2.3. A subset of a bitopological space $X$ is said to be
1. almost $(1,2)^\ast$-connected if $X$ cannot be written as a disjoint union of two non-empty regular $(1,2)^\ast$-open sets.
2. $(1,2)^\ast$-connected if $X$ cannot be written as a disjoint union of two non-empty $\tau_{1,2}$-open sets.

Definition 2.4. Let $X$ and $Y$ be two bitopological spaces. A map $f : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ is called weakly $(1,2)^\ast$-$\bar{g}$-continuous (briefly $(1,2)^\ast$-$w\bar{g}$-continuous) if $f^{-1}(U)$ is a $(1,2)^\ast$-$w\bar{g}$-open set in $X$ for each $\tau_{1,2}$-open set $U$ of $Y$.

Example 2.5. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Then the sets in $\{\phi, \{a\}, \{b, c\}, X\}$ are called $\tau_{1,2}$-open. Let $\sigma_1 = \{\phi, Y, \{a\}\}$ and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, \{a\}, Y\}$ is called $\sigma_{1,2}$-open.

Theorem 2.6. Every $(1,2)^\ast$-$\bar{g}$-continuous function is $(1,2)^\ast$-$w\bar{g}$-continuous.

**Proof.** It follows from Proposition 1.15.

The converse of Theorem 2.6 need not be true as seen in the following example.

Example 2.7. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Then the sets in $\{\phi, X, \{a\}, \{b, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a\}, \{b, c\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\phi, Y, \{a, b\}\}$ and $\sigma_2 = \{\phi, Y\}$. Then the sets in $\{\phi, Y, \{a, b\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\phi, Y, \{c\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then $f$ is $(1,2)^\ast$-$w\bar{g}$-continuous but not $(1,2)^\ast$-$\bar{g}$-continuous.
Theorem 2.8. A map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\)-wg-continuous if and only if \( f^{-1}(U) \) is a \((1,2)^*\)-wg-open set in \( X \) for each \( \sigma_{1,2}\)-closed set \( U \) of \( Y \).

Proof. Let \( U \) be any \( \sigma_{1,2}\)-closed set of \( Y \). According to the assumption \( f^{-1}(U^c) = X \setminus f^{-1}(U) \) is \((1,2)^*\)-wg-open in \( X \), so \( f^{-1}(U) \) is \((1,2)^*\)-wg-closed in \( X \).

The converse can be proved in a similar manner.

Theorem 2.9. Suppose that \( X \) and \( Y \) are bitopological spaces and \((1,2)^*\)-GC(X) is closed under arbitrary intersection. If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is contra \((1,2)^*\)-\(g\)-continuous and \( Y \) is \((1,2)^*\)-\(g\)-regular, then \( f \) is \((1,2)^*\)-\(g\)-continuous.

Proof. Let \( x \) be an arbitrary point of \( X \) and \( V \) be an \( \sigma_{1,2}\)-open set of \( Y \) containing \( f(x) \). Since \( Y \) is \((1,2)^*\)-\(g\)-regular, there exists an \( \sigma_{1,2}\)-open set \( G \) in \( Y \) containing \( f(x) \) such that \( \sigma_{1,2}-cl(G) \subseteq V \). Since \( f \) is contra \((1,2)^*\)-\(g\)-continuous, there exists \( U \in (1,2)^*\)-GO(X) containing \( x \) such that \( f(U) \subseteq \sigma_{1,2}-cl(G) \). Then \( f(U) \subseteq \sigma_{1,2}-cl(G) \subseteq V \). Hence, \( f \) is \((1,2)^*\)-\(g\)-continuous.

Theorem 2.10. Suppose that \( X \) and \( Y \) are bitopological spaces and the family of \((1,2)^*\)-\(g\)-sets in \( X \) is closed under arbitrary intersection. If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is contra \((1,2)^*\)-\(g\)-continuous and \( Y \) is \((1,2)^*\)-\(g\)-regular, then \( f \) is \((1,2)^*\)-\(g\)-continuous.

Proof. The proof is obvious from Theorem 2.9.

Definition 2.11. A bitopological space \( X \) is said to be locally \((1,2)^*\)-\(g\)-indiscrete if every \((1,2)^*\)-\(g\)-open set of \( X \) is \( \tau_{1,2}\)-closed in \( X \).

Theorem 2.12. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a function. If \( f \) is \((1,2)^*\)-\(g\)-continuous and \( X \) is locally \((1,2)^*\)-\(g\)-indiscrete, then \( f \) is \((1,2)^*\)-\(g\)-continuous.

Proof. Let \( V \) be an \( \sigma_{1,2}\)-open set in \( Y \). Since \( f \) is \((1,2)^*\)-\(g\)-continuous, \( f^{-1}(V) \) is \((1,2)^*\)-\(g\)-open in \( X \). Since \( X \) is locally \((1,2)^*\)-\(g\)-indiscrete, \( f^{-1}(V) \) is \( \tau_{1,2}\)-closed in \( X \). Hence \( f \) is \((1,2)^*\)-\(g\)-continuous.

Theorem 2.13. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a function. If \( f \) is contra \((1,2)^*\)-\(g\)-continuous and \( X \) is locally \((1,2)^*\)-\(g\)-indiscrete, then \( f \) is \((1,2)^*\)-\(g\)-continuous.

Proof. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be contra \((1,2)^*\)-\(g\)-continuous and \( X \) is locally \( \sigma_-\)-indiscrete. By Theorem 2.12, \( f \) is \((1,2)^*\)-\(g\)-continuous, then \((1,2)^*\)-\(g\)-continuous by Remark 1.12 and Theorem 2.6.

Corollary 2.14. Let \( Y \) be a \((1,2)^*\)-regular space and \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a function. Suppose that the collection of \((1,2)^*\)-\(g\)-closed sets in \( X \) is closed under arbitrary intersections. Then if \( f \) is \(((1,2)^*\)-\(g\)),\(s\)-continuous, \( f \) is \((1,2)^*\)-\(g\)-continuous.
Proof. (1) ⇒ (ii). Let \( S \subseteq X \) be any proper subset, which is both \((1,2)^*\)-\(g\)-open and \((1,2)^*\)-\(g\)-closed. Its complement \( X \setminus S \) is also \((1,2)^*\)-\(g\)-open and \((1,2)^*\)-\(g\)-closed. Then \( X = S \cup (X \setminus S) \) is a disjoint union of two non-empty \((1,2)^*\)-\(g\)-open sets which is a contradiction with the fact that \( X \) is \((1,2)^*\)-\(g\)-connected. Hence, \( S = \emptyset \) or \( X \).

(2) ⇒ (1). Let \( X = A \cup B \) where \( A \cap B = \emptyset \), \( A \neq \emptyset \), \( B \neq \emptyset \) and \( A, B \) are \((1,2)^*\)-\(g\)-open. Since \( A = X \setminus B \), \( A \) is \((1,2)^*\)-\(g\)-closed. According to the assumption \( A = \emptyset \), which is a contradiction.

(2) ⇒ (3). Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \((1,2)^*\)-\(g\)-continuous function where \( Y \) is a discrete bitopological space with at least two points. Then \( f^{-1}(y) \) is \((1,2)^*\)-\(g\)-closed and \((1,2)^*\)-\(g\)-open for each \( y \in Y \) and \( X = \cup \{ f^{-1}(y) \mid y \in Y \} \). According to the assumption, \( f^{-1}(y) = \emptyset \) or \( f^{-1}(\{ y \}) = X \). If \( f^{-1}(\{ y \}) = \emptyset \) for all \( y \in Y \), \( f \) will not be a function. Also there is no exist more than one \( y \in Y \) such that \( f^{-1}(\{ y \}) = X \). Hence, there exists only one \( y \in Y \) such that \( f^{-1}(\{ y \}) = X \) and \( f^{-1}(\{ y \}) = \emptyset \) if \( y \neq y_1 \in Y \). This shows that \( f \) is a constant function.

(3) ⇒ (2). Let \( S \neq \emptyset \) be both \((1,2)^*\)-\(g\)-open and \((1,2)^*\)-\(g\)-closed in \( X \). Let \( f: X \to Y \) be a \((1,2)^*\)-\(g\)-continuous function defined by \( f(S) = \{ a \} \) and \( f(X \setminus S) = \{ b \} \) where \( a \neq b \). Since \( f \) is constant function we get \( S = X \).

**Theorem 2.23.** Let \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \((1,2)^*\)-\(g\)-continuous surjective function. If \( X \) is \((1,2)^*\)-\(g\)-connected, then \( Y \) is \((1,2)^*\)-\(g\)-connected.

**Proof.** We suppose that \( Y \) is not \((1,2)^*\)-\(g\)-connected. Then \( Y = A \cup B \) where \( A \cap B = \emptyset \), \( A \neq \emptyset \), \( B \neq \emptyset \) and \( A, B \) are \((1,2)^*\)-\(g\)-open sets in \( Y \). Since \( f \) is \((1,2)^*\)-\(g\)-continuous surjective function, \( X = f^{-1}(A) \cup f^{-1}(B) \) is disjoint union of two non-empty \((1,2)^*\)-\(g\)-open subsets. This is contradiction with the fact that \( X \) is \((1,2)^*\)-\(g\)-connected.

### 3. Conclusion

The notions of sets and functions in bitopological spaces and fuzzy topological spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences. By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all bitopological sets and functions defined will have many possibilities of applications in digital topology and computer graphics.

**References**

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