Some results on smooth fuzzy subspaces and Hausdorff spaces
Murugesan Shakthiganesan¹*

Abstract
In this work, first we prove some interesting results in the context of smooth fuzzy subspaces through bases. In follow, we define the concept of \((\alpha, \ell)\)-Hausdorff spaces and prove that the intersection of finitely many \((\alpha, \ell)\)-Hausdorff topologies is again an \((\alpha, \ell)\)-Hausdorff topology in contrast with the crisp theory; we also prove that product of finitely many \((\alpha, \ell)\)-Hausdorff spaces is \((\alpha, \ell)\)-Hausdorff. Finally, we define and discuss the concept of \(\ell\)-Hausdorffness of space.

Keywords
Smooth fuzzy topological spaces, Subspaces, Hausdorffness.

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¹Department of Applied Mathematics and Computational Sciences, PSG College of Technology, Coimbatore-641004, India.
*Corresponding author: ¹shakthivedha23@gmail.com;
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1. Introduction
Following Chang[3], Šostak[11] redefined the concept of fuzzy topology as a fuzzy subset of \(I^X\), satisfying some properties. Later, Ramadan[7] generalized the concept in the name of “smooth fuzzy topological spaces”. Peeters, Park, Min, Kim, Abbas, Kalaivani and Roopkumar[1, 4–6] are some of the others who studied the concept in Šostak’s sense.

Fang Jin-ming and Yue Yue-li[13] defined the concept of a basis for a given smooth fuzzy topology in 2006. Two necessary and sufficient conditions for a given function \(B : I^X \rightarrow [0, 1]\) to be a basis for a smooth fuzzy topology were given by them. In [12] the concept of basis is defined and discussed in a way different from the one available in [13]. The approach in [12] is much easier and we follow this approach throughout this paper.

In 1992, Ramadan[8] defined the concept of subspace of a smooth fuzzy topological space. Werner Peeters[6] and Abbas[1] are some others who developed the concept further. In 2004, Abbas generalized the concept of subspaces, for a smooth fuzzy topology defined on a general fuzzy subset. The concept of Hausdorffness was studied by Azad, Yueli Yue, Jinming Fang, Rekha Srivastava and many others[2, 9, 10, 14]. But the definition for Hausdorffness which we are going to give is entirely different from ones available in the literature.

In Section 3, we follow the approach of Abbas[1] and prove some interesting results on the context subspaces, using the concept of basis for a smooth fuzzy topology defined in [12]. In Section 4, we define the concept of \((\alpha, \ell)\)-Hausdorff space and prove that the intersection of two \((\alpha, \ell)\)-Hausdorff topologies is again an \((\alpha, \ell)\)-Hausdorff topology in contrast with the crisp theory. In follow, we prove that product of two \((\alpha, \ell)\)-Hausdorff spaces is again \((\alpha, \ell)\)-Hausdorff. Finally, we define and discuss the concept of \(\ell\)-Hausdorffness of space.

2. Preliminaries
For any non empty set \(X\), a function \(\mu : X \rightarrow [0, 1]\) is called a fuzzy subset of \(X\). As usual, \(I^X\) and \(I\) denotes the family of all fuzzy subsets of \(X\) and \([0, 1]\); \(0_X\) and \(1_X\) denotes the characteristic function of \(\emptyset\) and \(X\) respectively. The union \(\bigvee L_{\lambda \in \Lambda} \mu_{\lambda}\) and intersection \(\bigwedge L_{\lambda \in \Lambda} \mu_{\lambda}\) of a collection \(\{\mu_{\lambda} : \lambda \in J\}\) of fuzzy sets of \(X\), where \(J\) is an arbitrary indexing set, are
Definition 2.1. [7, 11] Let μ be a fuzzy subset of a nonempty set X and let Θ = {A ∈ I^X | A ≤ μ}. Let ℱ : Θ → [0, 1] be a mapping that satisfies:

i. ℱ(μ) = 1

ii. ℱ(0_μ) = 1

iii. ℱ(A ∩ B) ≥ ℱ(A) ∧ ℱ(B) for any two fuzzy subsets A, B ∈ Θ.

iv. ℱ(A ∪ B) ≥ ℱ(A) ∨ ℱ(B) for any collection (A_λ)_{λ ∈ Λ}, where A_λ ∈ Θ.

Then ℱ is called a smooth fuzzy topology on μ and the pair (μ, ℱ) is called a smooth fuzzy topological space: for any A ∈ Θ, ℱ(A) is called the degree of openness of the A. Let ℳ : Θ → [0, 1] be the mapping defined by ℳ(A) = ℱ(μ - A).

Then ℳ(μ) is called the degree of closedness of μ.

Definition 2.2. [12] Let ℬ : Θ → [0, 1] be a function. Consider the following conditions:

B1. If x ∈ X, ε > 0 and δ > 0, there exists A ∈ Θ such that A(x) ≥ μ(x) - δ and ℬ(A) ≥ 1 - ε.

B2. If x ∈ X, A, B ∈ Θ, ε > 0 and δ > 0, there exists C ∈ Θ such that C ≤ A ∧ B, C(x) ≥ (μ(x) - δ, and ℬ(C) ≥ ℬ(A) ∧ ℬ(B)) - ε.

Any function ℬ satisfying the condition B1 is called a subbasis and any subbasis satisfying the condition B2 is called a basis for a smooth fuzzy topology on μ.

Definition 2.3. [12] A collection (A_λ)_{λ ∈ Λ} of non-zero fuzzy subsets of a fuzzy set A is said to be an inner cover for A if ∨_A_λ = A.

Definition 2.4. [12] Let ℬ be a basis for a smooth fuzzy topology on μ. Define the smooth fuzzy topology from Θ to [0, 1] generated by ℬ as follows:

Define ℱ(A) = 1 if A = 0_μ; otherwise, define

ℱ(A) = sup_λ_∈_Λ {min_ε_∈_E (ℬ(A_λ))},

where (C_λ)_{λ ∈ Λ} is the collection of all possible inner covers C_λ = (A_λ)_{λ ∈ Λ} of A.

The following two theorems together give a characterisation for a function ℬ : Θ → [0, 1] to be a basis for a smooth fuzzy topology.

Theorem 2.5. [12] Let ℱ be a smooth fuzzy topology on μ and let ℬ : Θ → [0, 1] be a function satisfying

i. ℱ(A) ≥ ℬ(A) for all A ∈ Θ.

ii. if δ, ε > 0, x ∈ X, A ∈ Θ, then there exists B ∈ Θ such that B(x) ≥ A(x) - δ, ℬ ≤ A and ℬ(B) ≥ ℱ(A) - ε.

Then ℬ is a basis for ℱ.

Theorem 2.6. [12] If ℬ is a basis for a given smooth fuzzy topology ℱ on μ, then

i. ℱ(A) ≥ ℬ(A) for all A ∈ Θ.

ii. if x ∈ X, A ∈ Θ, δ, ε > 0, there exists B ∈ Θ such that B(x) ≥ A(x) - δ, ℬ ≤ A and ℬ(B) ≥ ℱ(A) - ε.

Definition 2.7. [12] Let ℱ_μ and ℱ_v be smooth fuzzy topologies on μ and v respectively. A basis ℬ for the smooth fuzzy topology on μ × v is defined as a function ℬ from Θ to [0, 1] defined by

Let E ∈ Θ. If E cannot be written as A × B for any A ∈ Θ and B ∈ Θ, then define ℬ(E) = 0. Otherwise define

ℬ(E) = sup_λ_∈_Λ {inf_ε_∈_E (ℬ(A_λ))}

where (A_λ × B_λ)_{λ ∈ Λ} is the collection of all possible ways of writing E as E = A_λ × B_λ, where A_λ ∈ Θ, B_λ ∈ Θ.

The smooth fuzzy topology that ℬ generates is called the smooth fuzzy product topology on μ × v.

Theorem 2.8. [12] Let ℱ_μ and ℱ_v be smooth fuzzy topologies on μ and v respectively. Let ℬ_μ, ℬ_v be bases for the smooth fuzzy topologies ℱ_μ, ℱ_v respectively. Define a function ℬ : Θ → [0, 1] as follows:

Let E ∈ Θ. If E cannot be written as A × B for any A ∈ Θ and B ∈ Θ, then define ℬ(E) = 0. Otherwise define

ℬ(E) = sup_λ_∈_Λ {inf_ε_∈_E (ℬ(A_λ) × ℬ_v(B_λ))}

where (A_λ × B_λ)_{λ ∈ Λ} is the collection of all possible ways of writing E as E = A_λ × B_λ, where A_λ ∈ Θ, B_λ ∈ Θ.

Then ℬ is a basis for the product topology on μ × v.

In [1] S. E. Abbas quoted that for given ℱ_μ and ℱ_v, one can define (ℱ_μ × ℱ_v), the smooth v-topology induced over v by ℱ_μ, in the obvious way. Following him Rooknick and Kalavani gave the definition of a subspace smooth fuzzy topology for general smooth fuzzy topological space (μ, ℱ) in [4].

Definition 2.9. [4] Let (μ, ℱ_μ) be a smooth fuzzy topological space and let v ∈ ℱ_μ. The function ℱ_v : Θ_v → [0, 1] defined by ℱ_v(A) = sup_λ_∈_Λ {min_ε_∈_E (ℬ(A_λ) × ℬ_v(B_λ))} is a smooth fuzzy subspace topology induced over v by ℱ_μ, with this smooth fuzzy topology v is called smooth fuzzy subspace of μ.
3. Some Results on Subspaces

In [1] S. E. Abbas quoted that for given smooth fuzzy topology \( \mathcal{T}_\mu \) on \( \mu \) and \( v \in \mathcal{F}_\mu \), one can define \( (\mathcal{T}_\mu)_v \), the smooth \( v \)-topology induced over \( v \) by \( \mathcal{T}_\mu \), in the obvious way. Following him Roopkumar and Kalaivani gave the definition of a subspace smooth fuzzy topology for general smooth fuzzy topological space \( \mu, \mathcal{T} \) in [4]. We adopt the definition given by Kalaivani and Roopkumar [4] and prove some results.

**Definition 3.1.** [4] Let \( (\mu, \mathcal{T}_\mu) \) be a smooth fuzzy topological space and let \( v \in \mathcal{I}_\mu \). The function \( \mathcal{T}_v : \mathcal{I}_v \to [0, 1] \) defined by

\[
\mathcal{T}_v(A) = \sup \{ \mathcal{B}_\mu(B) / B \wedge v = A, B \in \mathcal{I}_\mu \}
\]

is a smooth fuzzy subspace topology over \( v \) by \( \mathcal{T}_\mu \), with this smooth fuzzy topology \( v \) is called smooth fuzzy subspace of \( \mu \).

**Lemma 3.2.** Let \( (v, \mathcal{T}_v) \) be a smooth fuzzy subspace of \( (\mu, \mathcal{T}_\mu) \). If \( \mathcal{B}_\mu \) is a basis for the smooth fuzzy subspace topology \( \mathcal{B}_\mu \) on \( \mu, \mathcal{T}_\mu \), then if \( \mathcal{B}_\mu \) is a smooth fuzzy subspace topology defined as

\[
\mathcal{B}_\mu(A) = \sup \{ \mathcal{B}_\mu(B) / B \wedge v = A, B \in \mathcal{I}_\mu \}
\]

is a basis for the smooth fuzzy subspace topology on \( v \).

**Proof.** First we claim that \( \mathcal{T}_v(A) \geq \mathcal{B}_\mu(A) \) for all \( A \in \mathcal{I}_v \).

Let \( A \in \mathcal{I}_v \), then

\[
\mathcal{T}_v(A) = \sup \{ \mathcal{T}_\mu(B) / B \in \mathcal{I}_\mu, B \wedge v = A \} \\
\geq \sup \{ \mathcal{B}_\mu(B) / B \in \mathcal{I}_\mu, B \wedge v = A \} \\
= \mathcal{B}_\mu(A).
\]

Now we claim that for any \( A \in \mathcal{I}_\mu, x \in X, \delta > 0 \) and \( \varepsilon > 0 \), there exists \( B \in \mathcal{I}_\mu \) such that \( B(x) \geq A(x) - \delta, B \leq A \) and \( \mathcal{B}_\mu(B) \geq \mathcal{T}_v(A) - \varepsilon \). By definition of \( \mathcal{T}_v \), we have

\[
\mathcal{T}_v(A) = \sup \{ \mathcal{T}_\mu(C) / C \in \mathcal{I}_\mu, C \wedge v = A \}.
\]

Then for any given \( \varepsilon > 0 \), there exists \( C \in \mathcal{I}_\mu \) such that \( C \wedge v = A \) and

\[
\mathcal{T}_\mu(C) \geq \mathcal{T}_v(A) - \frac{\varepsilon}{2}.
\]

Now, since \( \mathcal{B}_\mu \) is the basis for the smooth fuzzy topology \( \mathcal{T}_\mu \) we have,

\[
\mathcal{T}_\mu(C) = \sup_\Lambda \{ \inf_{E_k \in \mathcal{E}_\Lambda} \{ \mathcal{B}_\mu(E_k) \} \}.
\]

where \( \{ \mathcal{E}_\Lambda \} \Lambda \in \Gamma \) is the collection of all possible inner covers \( \mathcal{E}_\Lambda = \{ E_k \} \lambda \in \Lambda \) of \( C \). Let \( \mathcal{E}_\Lambda = \{ E_k \} \lambda \in \Lambda \) be an inner cover for \( C \) such that

\[
\inf_{E_k \in \mathcal{E}_\Lambda} \{ \mathcal{B}_\mu(E_k) \} \geq \mathcal{T}_\mu(C) - \frac{\varepsilon}{2}.
\]

Thus there exists some \( E_0 \in \{ E_k \} \mathcal{E}_\Lambda \) such that

\[
E_0(x) \geq C(x) - \delta \text{ and } E_0 \leq C,
\]

Since \( C \wedge v = A \) and \( \{ E_k \} \mathcal{E}_\Lambda \) is an inner cover for \( C \), we have, \( E_0 \wedge v \leq A \). Let \( B = E_0 \wedge v \), then by the Definition of \( \mathcal{T}_v \) we have \( \mathcal{T}_v(B) \geq \mathcal{B}_\mu(E_0) \) and

\[
B(x) = (E_0 \wedge v)(x) = (E_0(x) \wedge v(x)) \geq (C(x) - \delta) \wedge v(x) \geq (C(x) - \delta) \wedge (v(x) - \delta) = (C(x) \wedge (v(x)) - \delta = A(x) - \delta.
\]

Thus \( B(x) \geq A(x) - \delta \) and \( B \leq A \). Now as

\[
\mathcal{T}_v(A) - \frac{\varepsilon}{2} \leq \mathcal{T}_v(C) \leq \inf_{E_k \in \mathcal{E}_\Lambda} \{ \mathcal{B}_\mu(E_k) \} + \frac{\varepsilon}{2},
\]

we have

\[
\mathcal{T}_v(A) \leq \inf_{E_k \in \mathcal{E}_\Lambda} \{ \mathcal{B}_\mu(E_k) \} + \frac{\varepsilon}{2} + \varepsilon
\]

\[
\leq \mathcal{B}_\mu(E_0) + \varepsilon
\]

\[
\leq \mathcal{T}_v(B) + \varepsilon.
\]

Thus

\[
\mathcal{T}_v(B) \geq \mathcal{T}_v(A) - \varepsilon.
\]

Hence by Theorem 2.5, \( \mathcal{T}_v \) is a basis for the smooth fuzzy subspace topology on \( v \).

**Theorem 3.3.** Let \( (v, \mathcal{T}_v) \) be a smooth fuzzy subspace of \( (\mu, \mathcal{T}_\mu) \) and let \( A \in \mathcal{I}_v \).

i. If \( \mathcal{T}_v(A) > \alpha \) and \( \mathcal{T}_v(v) > \alpha \), then \( \mathcal{T}_v(A) > \alpha \).

ii. If \( \mathcal{T}_v(A) = \alpha \) and \( \mathcal{T}_v(v) = \alpha \), then \( \mathcal{T}_v(A) = \alpha \).

iii. If \( \mathcal{T}_v(A) = \alpha \) and \( \mathcal{T}_v(v) > \alpha \), then \( \mathcal{T}_v(A) = \alpha \).

iv. If \( \mathcal{T}_v(A) > \alpha \) and \( \mathcal{T}_v(v) = \alpha \), then \( \mathcal{T}_v(A) \geq \alpha \).

**Proof.** (i) Let \( \mathcal{T}_v(A) = \beta > \alpha \). Since \( \beta > \alpha \), there exists \( B \in \mathcal{I}_\mu \) such that \( B \wedge v = A \) and \( \mathcal{T}_\mu(B) > \beta \). Thus we have

\[
\mathcal{T}_\mu(A) = \mathcal{T}_\mu(B \wedge v) \geq \mathcal{T}_\mu(B) \wedge \mathcal{T}_\mu(v) > \alpha \wedge \alpha = \alpha.
\]

(ii) Let \( A \in \mathcal{I}_v \) be such that

\[
\mathcal{T}_v(A) = \sup \{ \mathcal{T}_\mu(B) / B \wedge v = A, B \in \mathcal{I}_\mu \} = \alpha.
\]

Let \( \varepsilon > 0 \). Then there exists \( E \in \mathcal{I}_\mu \) such that \( B \wedge v = A \) and \( \mathcal{T}_\mu(B) \geq \mathcal{T}_v(A) - \varepsilon = \alpha - \varepsilon \). Then it follows that

\[
\mathcal{T}_\mu(A) = \mathcal{T}_\mu(B \wedge v) \geq \mathcal{T}_\mu(B) \wedge \mathcal{T}_\mu(v) = (\alpha - \varepsilon) \wedge \alpha = \alpha - \varepsilon.
\]

Since this is true for every \( \varepsilon > 0 \), we have \( \mathcal{T}_\mu(A) \geq \alpha \). Now suppose \( \mathcal{T}_\mu(A) > \alpha \), then \( \mathcal{T}_v(A) > \alpha \), which leads to a contradiction. Thus it follows that \( \mathcal{T}_v(A) = \alpha \).

Similarly, we can prove (iii) and (iv). Note that both strict inequality and equality may hold in (iv).
For example, let $X = \{a, b, c\}$ and let $\mu(x) = 1$ for all $x \in X$. Define $\mathcal{T}_\mu : \mathcal{J}_\mu \to [0,1]$ as follows:

$$
\mathcal{T}_\mu(A) = \begin{cases} 
1 & \text{if } A = 1_X \text{ or } A = 0_X \\
\frac{3}{4} & \text{if } A = \chi_{\{a,b\}} \text{ or } A = \chi_{\{a\}} \text{ or } A = \chi_{\{b\}} \\
0 & \text{otherwise}
\end{cases}
$$

Let $v(x) = \chi_{\{a,c\}}$ and let $\mathcal{T}_v$ be the subspace smooth fuzzy topology on $v$. Then

$$
\mathcal{T}_v(A) = \begin{cases} 
1 & \text{if } A = v \text{ or } A = 0_X \\
\frac{3}{4} & \text{if } A = \chi_{\{a\}} \\
0 & \text{otherwise}
\end{cases}
$$

Let $A = \chi_{\{a\}}$ and $\alpha = \frac{1}{2}$, then $\mathcal{T}_v(A) > \alpha$, $\mathcal{T}_v(v) = \alpha$ and $\mathcal{T}_v(A) > \alpha$.

Let $X = \{a, b, c\}$ and let $\mu(x) = 1$ for all $x \in X$. Define $\mathcal{T}_\mu : \mathcal{J}_\mu \to [0,1]$ as follows:

$$
\mathcal{T}_\mu(A) = \begin{cases} 
1 & \text{if } A = 1_X \\
\frac{3}{4} & \text{if } A = \chi_{\{a,b\}} \\
\frac{1}{2} & \text{if } A = \chi_{\{a\}} \text{ or } A = \chi_{\{b\}} \text{ or } A = \chi_{\{a,c\}} \\
0 & \text{otherwise}
\end{cases}
$$

Let $v(x) = \chi_{\{a,c\}}$ and let $\mathcal{T}_v$ be the subspace smooth fuzzy topology on $v$. Then

$$
\mathcal{T}_v(A) = \begin{cases} 
1 & \text{if } A = v \text{ or } A = 0_X \\
\frac{3}{4} & \text{if } A = \chi_{\{a\}} \\
0 & \text{otherwise}
\end{cases}
$$

If $A = \chi_{\{a\}}$ and $\alpha = \frac{1}{2}$, then $\mathcal{T}_v(A) > \alpha$, $\mathcal{T}_v(v) = \alpha$ and $\mathcal{T}_v(A) > \alpha$. \hfill \Box

**Corollary 3.4.** Let $(v, \mathcal{T}_v)$ be a smooth fuzzy subspace of $(\mu, \mathcal{T}_\mu)$ and let $A \in \mathcal{J}_v$. Then $\mathcal{T}_\mu(A) \supseteq \mathcal{T}_v(A) \wedge \mathcal{T}_\mu(v)$.

**Theorem 3.5.** Let $(A, \mathcal{T}_A)$ and $(B, \mathcal{T}_B)$ be smooth fuzzy subspaces of $(\mu, \mathcal{T}_\mu)$ and $(v, \mathcal{T}_v)$. Then the smooth fuzzy product topology on $A \times B$ is same as the smooth fuzzy topology on $A \times B$ inherits as the subspace of $\mu \times v$.

**Proof.** Let $\mathcal{B}_A$, $\mathcal{B}_B$ be bases for the topologies $\mathcal{T}_A$, $\mathcal{T}_B$. Let $\mathcal{B}_A(G)$ be $\sup \{ \mathcal{B}_\mu(C) / C \wedge A = G, C \in \mathcal{J}_\mu \}$ and $\mathcal{B}_B(H) = \sup \{ \mathcal{B}_v(D) / D \wedge B = H, D \in \mathcal{J}_v \}$, then by Lemma 3.2 $\mathcal{B}_A$, $\mathcal{B}_B$ are the bases for the topologies $\mathcal{T}_A$, $\mathcal{T}_B$. Let $\mathcal{B}_{A \times B}$ be a function from $\mathcal{J}_{A \times B}$ to $[0,1]$ defined as follows: Let $E \in \mathcal{J}_{A \times B}$. If $E$ cannot be written as $A \times B$ for any $A \in \mathcal{J}_A$ and $B \in \mathcal{J}_B$, then define $\mathcal{B}_{A \times B}(E) = 0$. Otherwise define

$$
\mathcal{B}_{A \times B}(E) = \sup_{\lambda \in \Lambda} \{ \mathcal{B}_\mu(A_\lambda), \mathcal{B}_B(B_\lambda) \}
$$

where $\{A_\lambda \times B_\lambda\}_{\lambda \in \Lambda}$ is the collection of all possible representations of $E$ as $E = A_\lambda \times B_\lambda$, where $A_\lambda \in \mathcal{J}_A$, $B_\lambda \in \mathcal{J}_B$. Then by Theorem 2.8, we have $\mathcal{B}_{A \times B}$ is a basis for the product topology on $A \times B$. Let $\mathcal{B}_{\mu \times v}(E)$ be a function from $\mathcal{J}_{\mu \times v}$ to $[0,1]$ defined as follows: Let $E \in \mathcal{J}_{\mu \times v}$. If $E$ cannot be written as $C \times D$ for any $C \in \mathcal{J}_\mu$ and $D \in \mathcal{J}_v$, then define $\mathcal{B}_{\mu \times v}(E) = 0$. Otherwise define

$$
\mathcal{B}_{\mu \times v}(E) = \sup \{ \inf \{ \mathcal{B}_\mu(C_\lambda), \mathcal{B}_v(D_\lambda) \} \}_{\lambda \in \Lambda}
$$

where $\{C_\lambda \times D_\lambda\}_{\lambda \in \Lambda}$ is the collection of all possible representations of $E$ as $E = C_\lambda \times D_\lambda$, where $C_\lambda \in \mathcal{J}_\mu$, $D_\lambda \in \mathcal{J}_v$. Then again by Theorem 2.8, $\mathcal{B}_{\mu \times v}$ is a basis for the product topology on $\mu \times v$. Let $\mathcal{B}_{A \times B}$ be a function from $\mathcal{J}_{A \times B}$ to $[0,1]$ defined as

$$
\mathcal{B}_{A \times B}(E) = \sup \{ \mathcal{B}_{\mu \times v}(F) / F \wedge (A \times B) = E, F \in \mathcal{J}_{\mu \times v} \}.
$$

Then by Lemma 3.2 $\mathcal{B}_{A \times B}$ is a basis for the smooth fuzzy subspace topology on $A \times B$.

Now we prove that, $\mathcal{B}_{A \times B}(E) = \mathcal{B}_{A \times B}(E)$ for all subsets $E$ in $\mathcal{J}_{A \times B}$.

**Suppose,** $E$ is of the form $G \times H$ for some fuzzy subsets $G \in \mathcal{J}_A$, $H \in \mathcal{J}_B$.

Now to compute $\mathcal{B}_{A \times B}(E)$, first we collect all pairs $(G, H)$ such that $G \in \mathcal{J}_A$, $H \in \mathcal{J}_B$ and $E = G \times H$ and compute $\inf \{ \mathcal{B}_G(H), \mathcal{B}_H(G) \}$ and then we find the supremum of all these numbers obtained from all such pairs $(G, H)$. To compute $\mathcal{B}_G(H)$, we collect all possible members $\{C_{G,H} \}_{\lambda \in \Lambda}$ in $\mathcal{J}_\mu$ such that $C_{G,H}$ $\wedge A = G$ and compute $\sup \{ \mathcal{B}_\mu(C_{G,H}) \}$. Similarly, to compute $\mathcal{B}_H(G)$, we collect all possible members $\{D_{H,G} \}_{\gamma \in \Gamma}$ in $\mathcal{J}_\mu$ such that $D_{H,G} \wedge B = H$ and compute $\sup \{ \mathcal{B}_v(D_{H,G}) \}$.

$$
\mathcal{B}_{A \times B}(E) = \sup \{ \inf \{ \mathcal{B}_G(H), \mathcal{B}_H(G) \} \}
$$

Thus we have

$$
\mathcal{B}_{A \times B}(E) = \sup \{ \mathcal{B}_\mu(C) \wedge \mathcal{B}_v(D) \},
$$

where the supremum is taken over all possible pairs $(C, D)$ such that $E = (C \wedge A) \times (D \wedge B)$.

Now to compute $\mathcal{B}_{A \times B}(F)$: first we collect all possible members $\{F\} \in \mathcal{J}_{\mu \times v}$ such that $F \wedge (A \times B) = E$ and find
Suppose $F$ is not of the form $C \times D$ for any $C \in \mathcal{C}_\mu$, $D \in \mathcal{D}_\nu$ then by definition of $\mathcal{B}_{\mu \times \nu}$ we have,

$$\mathcal{B}_{\mu \times \nu}(F) = 0.$$ 

So, $\mathcal{B}_{\mu \times \nu}(E) = \sup \{\mathcal{B}_{\mu \times \nu}(C \times D)\}$, where the supremum is taken over all pairs $(C,D)$ such that $E = (C \times D) \cap (A \times B)$. But since $\mathcal{B}_{\mu \times \nu}(C \times D)$ is equal to $\sup \{\mathcal{B}_\mu(C_q \cap \mathcal{V}_\nu(D_r))\}$ where the supremum is taken over all pairs $(C_q,D_r)$ such that $C_q \times D_r = C \times D$. It follows that,

$$\mathcal{B}_{\mu \times \nu}(E) = \sup \{\mathcal{B}_\mu(C') \cap \mathcal{B}_\nu(D')\},$$

where the supremum is taken over all possible pairs $(C',D')$ such that $(C' \times D') \cap (A \times B) = E$. Now since $(C' \times D') \cap (A \times B)$ is equal to $(C' \times A) \cap (D' \times B)$, it is clear to see that $\mathcal{B}_{\mu \times \nu}(E) = \sup \{\mathcal{B}_\mu(C') \cap \mathcal{B}_\nu(D')\}$, where the supremum is taken over all possible pairs $(C',D')$ such that $E = (C' \times A) \times (D' \times B)$, as desired.

**Theorem 3.6.** Let $(\mu, \mathcal{I}_\mu)$ be a smooth fuzzy topological space and let $B < A < \mu$. Then the smooth fuzzy subspace topology induced over $B$ by the smooth fuzzy topology on $A$ is same as the smooth fuzzy subspace topology induced over $B$ by the smooth fuzzy topology on $\mu$.

**Proof.** Let $\mathcal{B}_\mu$ be a basis for $\mathcal{I}_\mu$. Let $\mathcal{B}_{\mu|B}$ be the smooth fuzzy subspace topology induced over $B$ by $\mathcal{B}_\mu$ with basis $\mathcal{B}_{\mu|B}$ as defined in Lemma 3.2 and let $\mathcal{I}_{\mu|B}$ and $\mathcal{I}_{\mu|A}$ be the smooth fuzzy subspace topologies induced over $B$ by $\mathcal{I}_\mu$ and $\mathcal{I}_A$ respectively. Let $\mathcal{B}_{\mu|B}$ and $\mathcal{B}_{\mu|A}$ be the bases for smooth fuzzy subspace topology $\mathcal{I}_{\mu|B}$ and $\mathcal{I}_{\mu|A}$ as defined in Lemma 3.2. Now we prove that $\mathcal{B}_{\mu|B} = \mathcal{B}_{\mu|A}$. Let $E \in \mathcal{I}_{\mu|B}$, then by the definitions of $\mathcal{B}_{\mu|B}$ and $\mathcal{B}_{\mu|A}$ we have

$$\mathcal{B}_{\mu|B}(E) = \sup \{\mathcal{B}_\mu(F) \mid F \cap B = E, F \in \mathcal{I}_\mu\}$$

and

$$\mathcal{B}_{\mu|A}(E) = \sup \{\mathcal{B}_{\mu|A}(G) \mid G \cap B = E, G \in \mathcal{I}_A\}.$$ 

Let $\mathcal{B}_{\mu|B}(E) = \lambda$ and $\epsilon > 0$, then there exists a $F \in \mathcal{I}_\mu$ such that $\mathcal{B}_\mu(F) \geq \lambda - \epsilon$ and $F \cap B = E$. As $F \cap B = E$ it clearly follows that $(F \cap A) \cap B = E$ and $F \cap A \leq A$. Now since $\mathcal{B}_\mu(F) \geq \lambda - \epsilon$ and by the definition of $\mathcal{B}_{\mu|A}$, it follows that $\mathcal{B}_{\mu|A}(F \cap A) \geq \lambda - \epsilon$. Let $G = F \cap A$, then $\mathcal{B}_{\mu|A}(G) \geq \lambda - \epsilon$, $G \cap B = E$ and $G \leq A$. Thus for each $\epsilon > 0$, there exists a $G \in \mathcal{I}_A$ such that $\mathcal{B}_{\mu|A}(G) \geq \lambda - \epsilon$, $G \cap B = E$ and $G \leq A$. This implies,

$$\mathcal{B}_{\mu|B}(E) = \sup \{\mathcal{B}_{\mu|A}(G) \mid G \cap B = E, G \in \mathcal{I}_A\} \geq \lambda.$$ 

Now suppose that

$$\mathcal{B}_{\mu|B}(E) = \sup \{\mathcal{B}_{\mu|A}(G) \mid G \cap B = E, G \in \mathcal{I}_A\} > \lambda.$$ 

Let $\mathcal{B}_{\mu|B}(E) = \delta > \lambda$. Choose $\epsilon > 0$ such that $\delta > \lambda - \epsilon > \lambda$, then by definition of $\mathcal{B}_{\mu|A}$ we can find a $G \in \mathcal{I}_A$ such that $\mathcal{B}_{\mu|A}(G) \geq \delta - \frac{\epsilon}{2} > \lambda$ and $G \cap B = E$. Now by the definition of $\mathcal{B}_{\mu|A}$, we can find a $H \in \mathcal{I}_\mu$ such that $H \cap A = G$ and

$$\mathcal{B}_{\mu|A}(H) = \delta - \frac{\epsilon}{2} > \lambda.$$ 

But since $H \cap B = E$, it follows that, there exists a $H \in \mathcal{I}_\mu$ such that $H \cap B = E$ and

$$\mathcal{B}_{\mu|A}(H) = \delta - \epsilon > \lambda.$$ 

This leads to a contradiction that $\mathcal{B}_{\mu|A}(E) = \lambda$, and hence our assumption $\mathcal{B}_{\mu|B}(E) > \lambda$ is wrong. Thus it follows that $\mathcal{B}_{\mu|B}(E) = \lambda$ and therefore $\mathcal{B}_{\mu|B} = \mathcal{B}_{\mu|A}$. \hfill \qed

**Result 3.7.** Let $\mathcal{I}_\mu$ and $\mathcal{I}_\mu'$ be two smooth fuzzy topologies on $\mu$ and let $\nu < \mu$. Let $(\mu, \mathcal{I}_\nu)$ be smooth fuzzy subspace topological spaces of $(\mu, \mathcal{I}_\mu)$ and $(\mu, \mathcal{I}_\mu')$ respectively. If $\mathcal{I}_\mu \leq \mathcal{I}_\mu'$ then $\mathcal{I}_\nu \leq \mathcal{I}_\nu'$.

**Proof.** Let $E \in \mathcal{I}_\nu$, then

$$\mathcal{I}_\nu(E) = \sup \{\mathcal{I}_\mu(G) \mid G \cap \nu = E, G \in \mathcal{I}_\mu\} \leq \sup \{\mathcal{I}_\mu'(G) \mid G \cap \nu = E, G \in \mathcal{I}_\mu\} = \mathcal{I}_\nu(E).$$ 

In the above result the equality may also hold.

**Example 3.8.** Let $X = \{1,2,3,4\}$ and let $\mu(x) = 1$ for all $x \in X$. Now, define a function $\mathcal{I}_\mu$ from $\mathcal{I}_\mu$ to $[0,1]$ as follows:

$$\mathcal{I}_\mu(E) = \begin{cases} 
1 & \text{if } E = \mu \text{ or } E = 0_X \\
\frac{1}{2} & \text{if } E = \chi_{\{1\}} \\
\frac{1}{4} & \text{if } E = \chi_{\{1,2\}} \\
0 & \text{otherwise}
\end{cases}$$

Then $\mathcal{I}_\mu$ is a smooth fuzzy topology on $\mu$. Let $\mathcal{I}_\mu'$ be a function from $\mathcal{I}_\mu$ to $[0,1]$ defined as

$$\mathcal{I}_\mu'(E) = \begin{cases} 
1 & \text{if } E = \mu \text{ or } E = 0_X \\
\frac{1}{2} & \text{if } E = \chi_{\{2,3,4\}} \text{ or } \chi_{\{1\}} \\
\frac{1}{4} & \text{if } E = \chi_{\{3,4\}} \text{ or } \chi_{\{1,2\}} \\
0 & \text{otherwise}
\end{cases}$$

Then $\mathcal{I}_\mu'$ is a smooth fuzzy topology on $\mu$. Let

$$v(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{otherwise}
\end{cases}$$

then clearly it can be seen that $v < \mu$. Let $\mathcal{I}_v$ be the smooth fuzzy subspace topology induced over $v$ by $\mu$. Then for $E \in \mathcal{I}_v$ we have

$$\mathcal{I}_v(E) = \begin{cases} 
1 & \text{if } E = v \text{ or } E = 0_X \\
0 & \text{otherwise}
\end{cases}$$

and

$$\mathcal{I}_v'(E) = \begin{cases} 
1 & \text{if } E = v \text{ or } E = 0_X \\
0 & \text{otherwise}
\end{cases}$$

Thus $\mathcal{I}_v = \mathcal{I}_v'$. 

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4. \((\alpha, \ell)-\)Hausdorff Spaces

In this section, we define the concept of \((\alpha, \ell)-\)Hausdorff spaces and prove some interesting results; in particular we prove that the intersection of two \((\alpha, \ell)-\)Hausdorff topologies is again an \((\alpha, \ell)-\)Hausdorff topology in contrast with the crisp theory.

**Definition 4.1** \((\alpha, \ell)-\)Hausdorff. Let \(\alpha \in (0, 1]\) and \(\ell \in [0, 1]\). A smooth fuzzy topological space \((\mu, T)\) is said to be \((\alpha, \ell)-\)Hausdorff if \(\forall x, y \in X\) with \(x \neq y\), there exist \(A, B \in \mathcal{T}_\mu\) such that \(T_\mu(A) > \alpha, T_\mu(B) > \alpha, A(x) \geq \ell, B(y) \geq \ell\). Note that, \(C(x_1, y_1) = (A \times \mu_2)(x_1, y_1) = A(x_1) \wedge \mu_2(y_1) = \ell_1 \wedge \ell_2 = \ell\).

**Theorem 4.5.** Let \(\alpha \in (0, 1]\) and \(\ell \in [0, 1]\). A smooth fuzzy topological space \((\mu, T)\) is \((\alpha, \ell)-\)Hausdorff if and only if for all \(x \neq y\) in \(X\), there exist \(A, B \in \mathcal{T}_\mu\) such that \(T_\mu(A) > \alpha, T_\mu(B) > \alpha, A(x) \geq \ell, B(y) \geq \ell\). Note that, \(C(x_1, y_1) = (A \times \mu_2)(x_1, y_1) = A(x_1) \wedge \mu_2(y_1) = \ell_1 \wedge \ell_2 = \ell\).

**Proof.** As the result can be proved in a usual way, we skip the proof.

Now we prove an interesting result, that the intersection of finitely many \((\alpha, \ell)-\)Hausdorff topologies is a \((\alpha, \ell)-\)Hausdorff topology in contrast with the crisp theory.

**Theorem 4.3.** Let \((\mu, T_\mu)\) be \((\alpha, \ell)-\)Hausdorff for all \(i = 1, 2, \ldots, n\) and let \(T = \bigwedge_{i=1}^n T_\mu\). Then \((\mu, T)\) is an \((\alpha, \ell)-\)Hausdorff space, where \(\alpha = \min\{\alpha_i\}\) and \(\ell = \min\{\ell_i\}\).

**Proof.** Let \(x \neq y \in X\). Now since \((\mu, T_\mu)\) is \((\alpha, \ell)-\)Hausdorff for all \(i\), there exist \(A_i, B_i \in \mathcal{T}_\mu\) such that \(T_\mu(A_i) > \alpha, T_\mu(B_i) > \alpha, A_i(x) \geq \ell, B_i(y) \geq \ell\) and \((A_i \wedge B_i)(z) < \ell_i\) for all \(z \in X\). Let \(A = \min\{A_i\}\) and \(B = \min\{B_i\}\), then it follows that \(T(A) = \bigwedge_{i=1}^n T_\mu(A) \geq \bigwedge_{i=1}^n \{T_\mu(A_i)\} > \min\{\alpha_i\} = \alpha\) and analogously, it is easy to see that \(T(B) > \alpha\). Now, by construction of \(A\) and \(B\), we have \(A(x) \geq \ell, B(y) \geq \ell\) and \((A \wedge B)(z) < \ell\) for all \(z \in X\). Thus \((\mu, T)\) is an \((\alpha, \ell)-\)Hausdorff space.

**Corollary 4.4.** The intersection of finitely many \((\alpha, \ell)-\)Hausdorff topologies is \((\alpha, \ell)-\)Hausdorff.

**Theorem 4.6.** The product of two \((\alpha, \ell)-\)Hausdorff spaces is \((\alpha, \ell)-\)Hausdorff.

**References**


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