Strongly multiplicative labeling of certain tree derived networks

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Abstract
A graph $G = (V(G), E(G))$ with $p$ vertices is said to be strongly multiplicative if the vertices of $G$ can be labeled with $p$ distinct integers $1, 2, ..., p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct [3].

In this paper we prove that the $X$–tree, Hypertree and shuffle Hypertree are strongly multiplicative for all $n \geq 2$.

Keywords
$X$–tree, Hypertree, Shuffle Hypertree and Strongly multiplicative labeling.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

1 Introduction

Graph labeling concerns the assigning of values, usually represented by integers, to the edges and/or vertices of a graph [1]. It plays an important role in Neural Networks, Communication Networks, Circuit Analysis, Coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties and also used in the study of X-ray Crystallography, etc. Graph labeling serves as a frontier between number theory and structure of graphs [2].

In this paper we have considered tree derived networks like $X$–tree, Hypertree and shuffle Hypertree. Hypertree is an interconnection Topology for incrementally expansible multi computer system, which combines the easy expansibility of the tree structure with the compactness of Hypercube, that it combines the best feature of binary tree and the hypercubes [7]. Shuffle Hypertree $SHT(n)$ is a modification of Hypertree $HT(n)$. The basic skeleton of a $X$-tree, hypertree and shuffle hypertree are a complete binary tree $T_n$.

In this paper we prove that $X$-tree, Hypertree and Shuffle Hypertree networks are strongly multiplicative.

2. Preliminaries

Definition 2.1. If every internal vertex of the rooted tree has exactly two children then the tree is called a complete binary tree.

Remark 2.1. For any non negative integer $n$, the complete binary tree of height $n$ denoted by $T_n$, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. $T_n$ has $n$ levels namely $1, 2, 3, ..., n$ and level $i, 1 \leq i \leq n$ contains $2^{i-1}$ vertices.

Remark 2.2. At each level, $T_n$ has exactly $2^n - 1$ vertices and $2^n - 2$ edges.

Definition 2.2. [4]The Slim tree $ST(n)$ is a complete binary tree $T_n$ with the set of edges $L = [(i, i+1)/ 2^{n-1} \leq i \leq 2^n - 2]$ to the complete binary tree $T_n$.
Definition 2.3. The X-tree is a slim tree $ST(n)$ along with edges obtained by joining the left and the right children of every parent at all intermediate levels $2, 3, \ldots, n$ and is denoted by $XT(n)$.[6].

Remark 2.4. The number of vertices and edges in the X-tree $XT(n)$ are $2^n - 1$ and $3(2^{n-1} - 1)$ respectively.

Remark 2.5. For convenience the vertices of the X tree $XT(n)$ are labeled as shown in figure 1.

Definition 2.4. A Hypertree is a graph is a complete binary tree along with the edges obtained by joining the two intermediate nodes in the same level $l$. If the label difference is $2^{l-2}, 2 \leq l \leq n$. We denote the $n-$ level Hypertree as $HT(n)$ [8].

Remark 2.6. The number of vertices and edges in Hypertree graph are $2^n - 1$ and $3(2^{n-1} - 1)$ respectively.

Definition 2.5. The Shuffle Hyper Tree is a Hyper tree where the intermediate edges of $HT(n)$ are removed and replaced by the hyper edges $\{(v_{2^i-1}, v_{2^i-1}) / 2 \leq i \leq n\}$ and $\{(v_{2^i-2}+2k-1, v_{2^i-2}+2k) / 3 \leq i \leq n, 1 \leq k \leq 2^{i-2} - 1\}$ and is denoted by $SHT(n)$. [5]

Remark 2.7. The number of vertices and edges in shuffle Hypertree graph are $2^n - 1$ and $3(2^{n-1} - 1)$ respectively.

### 3. Main Results

**Theorem 3.1.** The X-tree $XT(n)$ is strongly multiplicative for all $n$

**Proof.** To prove that the X-tree $XT(n)$ is strongly multiplicative,

Define the vertex set $V = \{v_i / 1 \leq i \leq 2^n - 1\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$ where $E_1 = \{e_i = (v_i, v_{2i}) / 1 \leq i \leq 2^{n-1} - 1\}$, $E_2 = \{e_i = (v_i, v_{2i+1}) / 1 \leq i \leq 2^{n-1} - 1\}$, and $E_3 = \{e_i = (v_{2i-1}, v_{2i-1}+2l) / 0 \leq l \leq (2^{n-2} - 2), 2 \leq l \leq n\}$, where $l$ denotes each level of the tree.

Define the vertex labeling by a bijective map $f : V \rightarrow N$ such that $f(v_i) = i$ for all $1 \leq i \leq 2^n - 1$.

We shall prove that all the edge labelings in $E_1$ are distinct.

Define an edge induced function $g : E_1 \rightarrow N$ such that for all $v_i \in E_1$, $g(e_i) = f(v_i)f(v_{2i}), 1 \leq i \leq 2^{n-1} - 1$.

For $i \neq p$, let $e_i, e_p \in E_1$ be distinct edges, we claim that $g(e_i) \neq g(e_p)$.

Assume that

$$g(e_i) = g(e_p)$$

$$g(v_i, v_{2i}) = g(v_p, v_{2p})$$

$$f(v_i)f(v_{2i}) = f(v_p)f(v_{2p})$$

$$i(2i) = p(2p)$$

$$i^2 = p^2 \Rightarrow i = p.$$
We shall show that the labeling of edges $E$ is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$, for all $0 \leq i \leq (2^{l-2} - 1)$, $2 \leq l \leq n$. Hence all the labelings of edges in $E_3$ is distinct.

We shall show that the labeling of edges $E_1$ and $E_2$ are distinct.

If $e_i \in E_1$ and $e_p \in E_2$ be two distinct edges then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 1 \leq i, p \leq 2^n - 1$, assume that $g(e_i) = g(e_p)$.

$$g(v_i, v_{2i}) = g(v_p, v_{2p+1})$$
$$f(v_i)f(v_{2i}) = f(v_p)f(v_{2p+1})$$
$$i(2i) = p(2p + 1) \Rightarrow i = \sqrt{\frac{p(2p + 1)}{2}}$$

This is a contradiction for $i$.

Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq 2^n - 1$.

Hence all the edge labelings set in $E_1$ and $E_2$ are distinct.

We shall show that the labeling of edges $E_1$ and $E_3$ are distinct.

If $e_i \in E_1$ and $e_p \in E_3$ be two distinct edges at levels $l_i$ and $l_p$ in $E_3$, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 2^{l_i} - 1 \leq i \leq 2^{l_i} - 2, 2 \leq l \leq n$.

Assume that $g(e_i) = g(e_p)$.

$$g(v_i, v_{2i}) = g(v_{2^{l_i-2}+i}, v_{2^{l_i-2}+2p+1})$$
$$f(v_i)f(v_{2i}) = f(v_{2^{l_i-2}+i}, f(v_{2^{l_i-2}+2p+1})$$
$$i(2i) = (2^{l_i-1} + 2p)(2^{l_i-1} + 2p + 1)$$

$$\Rightarrow i = \sqrt{\frac{(2^{l_i-1} + 2p)(2^{l_i-1} + 2p + 1)}{2}}$$

This is a contradiction for $i$.

Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq 2^{l_i-1} - 1$.

Therefore all the induced edge labeling in $E_1$ and $E_3$ are distinct.

Hence all the induced edge labelings in $E$ are distinct.

Hence the X-tree $XT(n)$ is strongly multiplicative for all positive $n \geq 2$.

**Theorem 3.2.** The Hypertree as $HT(n)$ is strongly multiplicative for all $n \geq 2$.

**Proof.** Let $HT(n)$ be the Hypertree, with the vertex set $V = \{v_i | 1 \leq i \leq 2^n - 1\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{e_i = (v_i, v_{2i}) | 1 \leq i \leq 2^n - 1\}$

$E_2 = \{e_i = (v_i, v_{2i+1}) | 1 \leq i \leq 2^n - 1\}$

$E_3 = \{e_i = (v_i, v_{2i+1}) | 2^{l_i} - 1 \leq i \leq 2^{l_i} - 2, 2 \leq l \leq n\}$.

To prove that $HT(n)$ is strongly multiplicative.

Define a bijective mapping $f : V \rightarrow N$ such that $f(v_i) = i$ for all $1 \leq i \leq 2^n - 1$.

Proving that all the edge labelings in $E_1$ and $E_2$ are distinct is similar to that of $X$-tree.

We shall prove the edge labelings in $E_3$ are distinct.

Define an edge induced function $g : E_3 \rightarrow N$ such that for any $e_i \in E_3$, $g(e_i) = f(v_{2^{l_i-2}+i})$, where $2^{l_i-1} \leq i \leq 2^{l_i-1} + 2^{l_i-2} - 1, 2 \leq l \leq n$.

**Case 1.** If $e_i$ and $e_p$ are distinct edges at different levels $l_i$ and $l_p$ in $E_3$, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 2^{l_i} - 1 \leq i \leq 2^{l_i} - 2 - 1, 2 \leq l \leq n$.

Assume that $g(e_i) = g(e_p)$.

$$g(v_i, v_{2^{l_i-2}+i}) = g(v_p, v_{2^{l_i-2}+p})$$
$$f(v_i)f(v_{2^{l_i-2}+i}) = f(v_p)f(v_{2^{l_i-2}+p})$$
$$i(2^{l_i-2} + i) = p(2^{l_i-2} + p)$$

$$i = \sqrt{\frac{(2^{l_i-2} + p)(2^{l_i-2} + p)}{2}}$$

This is a contradiction for $i$.

Hence $g(e_i) \neq g(e_p)$, for all $2^{l_i-1} \leq i \leq 2^{l_i-1} + 2^{l_i-2} - 1, 2 \leq l \leq n$.

**Case 2.** If $e_i$ and $e_p$ are distinct edges at the same level in $E_3$ then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 2^{l_i} - 1 \leq i \leq 2^{l_i} - 2 - 1, 2 \leq l \leq n$.

Assume that $g(e_i) = g(e_p)$.

$$g(v_i, v_{2^{l_i-2}+i}) = g(v_p, v_{2^{l_i-2}+p})$$
$$f(v_i)f(v_{2^{l_i-2}+i}) = f(v_p)f(v_{2^{l_i-2}+p})$$
$$i(2^{l_i-2} + i) = p(2^{l_i-2} + p)$$

$$i = -p - 2^{l_i-2},$$ which is a contradiction for $i$.

Hence $g(e_i) \neq g(e_p)$, for all $2^{l_i-1} \leq i \leq 2^{l_i-1} + 2^{l_i-2} - 1, 2 \leq l \leq n$.

Hence all the edge labelings in $E_3$ are distinct.

We shall show that the labeling of edges $E_1$ and $E_3$ are distinct.

If $e_i$ and $e_p$ are distinct edges in $E_1$ and $E_3$, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 1 \leq i \leq 2^{n-1} - 1, 2 \leq l \leq n$.

Assume that $g(e_i) = g(e_p)$.

$$g(v_i, v_{2^{l_i-2}+i}) = g(v_p, v_{2^{l_i-2}+p})$$
$$f(v_i)f(v_{2^{l_i-2}+i}) = f(v_p)f(v_{2^{l_i-2}+p})$$
$$i(2^{l_i-2} + i) = p(2^{l_i-2} + p)$$

This is a contradiction for $i$.

Hence $g(e_i) \neq g(e_p)$, for all $2^{l_i-1} \leq i \leq 2^{l_i-1} + 2^{l_i-2} - 1, 2 \leq l \leq n$.
This is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$, for all $1 \leq i \leq 2^{n-1} - 1$.
Therefore all the induced edge labelings in $E_1$ and $E_2$ are distinct. The induced edge labeling in $E_1$ and $E_2$ can be similarly proved to be distinct as in the case of $X$-Tree.

Hence all the induced edge labelings in $E$ are distinct. Hence the hypertree $HT(n)$ is strongly multiplicative for all positive $n \geq 2$.

**Theorem 3.3.** The Shuffle Hypertree $SHT(n)$ is strongly multiplicative for all $n \geq 2$.

**Proof.** Let $SHT(n)$ be the Shuffle Hypertree, with the vertex set $V = \{v_i/ 1 \leq i \leq 2^n - 1\}$ and the edge set $E = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{ e_1 = (v_i, v_{2i}) / 1 \leq i \leq 2^{n-1} - 1 \}$, $E_2 = \{ e_2 = (v_i, v_{2^i+1}) / 1 \leq i \leq 2^{n-1} - 1 \}$ and

$E_3 = \{ e_3 = (v_{2i}, v_{2^i+1}) / 3 \leq i \leq n, 1 \leq k \leq 2^{n-2} - 1 \} \cup \{ (v_{2i}, v_{2^i+1}) / 2 \leq i \leq n \}$.

To prove that $SHT(n)$ is strongly multiplicative, define a bijective mapping $f : V \rightarrow N$ such that $f(v_i) = i$ for all $1 \leq i \leq 2^n - 1$.

![Diagram](image)

**Fig. 3.** Shuffle Hypertree $SHT(4)$

To prove in shuffle hypertree that all the edge labelings in $E_1$ and $E_2$ are distinct is similar to that $X$-tree.

We shall prove the edge labelings in $E_3$ are distinct.

**Case 1:** Define an edge induced function $g : E_3 \rightarrow N$ such that for all $e_i \in E_3$,

$g(e_i) = f(v_{2^i-1,2^i+1})f(v_{2^i+1,2i+1})$, $3 \leq i \leq n, 1 \leq k \leq 2^{n-2} - 1$.

If $e_i$ and $e_p$ are distinct edges in $E_3$ then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 3 \leq i, p \leq n, 1 \leq k, r \leq 2^{n-2} - 1$ Assume that $g(e_i) = g(e_p)$.

$g(v_{2^i-1,2^i+1}) = g(v_{2^i-1,2^i+1},v_{2^i-1,2r})$
$f(v_{2^i-1,2^i+1})f(v_{2^i+1,2i+1}) = f(v_{2^i-1,2^i+1})f(v_{2^i+1,2r})$
$(2^{i-1} + 2k - 1)(2^{i-1} + 2k) = (2^{p-1} + 2r - 1)(2^{p-1} + 2r)$

This is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$.

**Case 2:** Define an edge induced function $g : E_3 \rightarrow N$ such that for all $e_i \in E_3$, $g(e_i) = f(v_{2^i-1})f(v_{2^i-1})$, $2 \leq i \leq n$.

If $e_i$ and $e_p$ are distinct edges in $E_3$, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p, 2 \leq i, p \leq n$, assume that $g(e_i) = g(e_p)$.

$g(v_{2^i-1},v_{2^i-1}) = g(v_{2^i-1},v_{2p-1})$
$f(v_{2^i-1})f(v_{2^i-1}) = f(v_{2^i-1})f(v_{2p-1})$
$2^{i-1}(2^{i-1} - 1) = 2^{p-1}(2^{p-1} - 1)$

This is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$.

**Case 3:** If $e_i = (v_{2^i-1,2^i+1})$ and $e_p = (v_{2^i-1,2^p-1})$ be distinct edges in $E_3$, $3 \leq i \leq n, 1 \leq k \leq 2^{n-2} - 1, 2 \leq p \leq n$, then to prove $g(e_i) \neq g(e_p)$.

For $i \neq p$, assume that $g(e_i) = g(e_p)$.

This is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$, for all $3 \leq i \leq n$.

Hence all the edge labeling set in $E_3$ is distinct.

The induced edge labeling in $E_1$ and $E_2$ can be similarly proved to be distinct as in the case of $X$-Tree.

We shall show that the labeling of edges $E_1$ and $E_3$ are distinct. If $e_i$ and $e_p$ are distinct edges in $E_1$ and $E_3$, to prove $g(e_i) \neq g(e_p)$.

**Case 1:** For $i \neq p, 1 \leq i \leq 2^{n-1}, 3 \leq p \leq n, 1 \leq r \leq 2^{n-2} - 1$

Assume that $g(e_i) = g(e_p)$.

$g(v_1,v_2) = g(v_{2p-1,2r-1},v_{2p-1,2r})$
$f(v_1)f(v_2) = f(v_{2p-1,2r-1})f(v_{2p-1,2r})$
$i(2i) = (2^{p-1} + 2r - 1)(2^{p-1} + 2r)$

This is a contradiction for $i$. Hence $g(e_i) \neq g(e_p)$.

**Case 2:** For $i \neq p, 1 \leq i, p \leq 2^{n-1}$,

Assume that $g(e_i) = g(e_p)$.

$g(v_1,v_2) = g(v_{2p-1},v_{2p-1})$
$f(v_1)f(v_2) = f(v_{2p-1})f(v_{2p-1})$
\[ i(2i) = 2^{p-1}(2^p - 1) \]
\[ \Rightarrow i = \sqrt{\frac{2^{p-1}(2^p - 1)}{2}}. \]

This is a contradiction for \( i \). Hence \( g(e_i) \neq g(e_p) \).

Therefore all the induced edge labeling in \( E_1 \) and \( E_3 \) are distinct.

Thus all the edge labeling of \( E_1, E_2 \) and \( E_3 \) are distinct. Also the induced edge labeling of \( E_1 \& E_2 \) and \( E_2 \& E_3 \) are distinct. Hence the Shuffle Hypertree Network has strongly multiplicative labeling for all \( n \geq 2 \). \( \square \)

### 4. Conclusion

In this paper we have proved that \( X\text{-Tree } XT(n) \), Hypertree \( HT(n) \) and Shuffle Hypertree \( SHT(n) \) are strongly multiplicative for all \( n \geq 2 \). Finding strongly multiplicative labeling for other tree derived networks is quite challenging.

### References


