δg Closed sets in grill topological spaces

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Abstract
In this paper we introduce a new type of closed sets called ζδg closed sets in a topological space X, defined in terms of grills. The relations between ζδg closed sets and already available closed sets are discussed.

Keywords
δ closed sets, ω closed sets, ζω closed sets, r closed sets, ζg closed sets.

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1. Introduction

2. Preliminaries
Let (X, τ) be a topological space with no separation properties assumed. A sub-collection ζ of P(X) is called a grill on X if ζ satisfies the conditions

(i) φ ∉ ζ
(ii) A ∈ ζ and A ⊆ B ⊆ X ⇒ B ∈ ζ
(iii) A, B ⊆ X & A ∪ B ∈ ζ implies that A ∈ ζ or B ∈ ζ.

For any point x of a topological space (X, τ), τ(x) denotes the collection of all open neighborhoods of x.

Definition 2.1. Let [X, τδ, ζ] be a grill delta space. A mapping ψδ : P(X) → P(X) denoted by ψδ(A) is called the operator associated with grill ζ and the topology τδ and is defined by ψδ = {x ∈ X/A \ U ∈ ζ for all U ∈ δO(X, τ)}. A mapping ψδ : P(X) → P(X) is defined as A ∪ ψδ(A) for all A ∈ P(X).

Definition 2.2. To a grill delta space (X, τδ, ζ) there exists a unique topology τζδ on X given by

τζδ = {U ⊆ X : ψδ(X\U) = (X\U)}

where for any A ⊆ X, ψδ(A) = A ∪ ψδ(A) = τζδ - clδ(A).

3. ζδg-closed sets

Definition 3.1. Let (X, τ) be a topological space and ζ be a grill on X. Then a subset A of X is said to be a ζδg closed if ψδ(A) ⊆ U whenever A ⊆ U and U is ζω open in (X, τ, ζ).

Proposition 3.2. Every δ closed set is ζδg closed.

Proof. Let A be a δ closed set and U be any ζω open set containing A. Since A is δ closed then ψδ(A) ⊆ clδ(A) = A for every subset A of X. Therefore ψδ(A) = A ⊆ U and hence A is ζδg closed.

Remark 3.3. The converse of the above theorem is false as shown in the following example.

Example 3.4. Let X = {p, q, r}, τ = {φ, X, {p}, {q, r}},
ζ = {X, {p}, {r}, {p, q}, {q, r}, {p, r}}
δ closed = {φ, X, {p}, {q, r}}
ζδg closed = {φ, X, {p}, {q}, {p, q}, {q, r}}.
Here {p, q} is ζδg closed but not δ closed in (X, τ).
Proposition 3.5. Every regular closed set is $\zeta \delta \bar{g}$ closed.

Proof. Assume $A$ is regular closed set and $U$ be any $\zeta \omega$ open set containing $A$. Here $A$ is regular closed set. This implies $A$ is $\delta$ closed and $\text{cls}(A) = A$ for any subset $A$ in $(X, \tau)$ and $\varphi_\delta(A) \subseteq \text{cls}(A) \subseteq U$.

Hence $A$ is $\zeta \delta \bar{g}$ closed.

Remark 3.6. The converse of the above theorem is false as shown in the following example.

Example 3.7. Let $X = \{p, q, r\}$, $\tau = \{\phi, X, \{p\}, \{q, r\}\}$, $\zeta = \{X, \{p\}, \{q, p, \}, \{q, r\}\} \zeta \delta \bar{g}$ closed $= \{\phi, X, \{p\}, \{q, r\}, \{q, r\}, \{p, r\}\}$ reclassification $= \{\phi, X, \{p\}, \{q, r\}\}$.
Here $\{q\}$ is $\zeta \delta \bar{g}$ closed but not regular closed.

Proposition 3.8. Every $\zeta \delta \bar{g}$ closed set is $\zeta g$ closed set.

Proof. Consider $A$ to be a $\zeta \delta \bar{g}$ closed set and $U$ be any open set containing $A$ in $(X, \tau)$. Since every open set is $\zeta \omega$ open and $A$ is $\zeta \delta \bar{g}$ closed. Then $\varphi_\delta(A) \subseteq U$ for every subset $A$ of $X$. $\varphi(A) \subseteq \varphi_\delta(A) \subseteq U \Rightarrow \varphi(A) \subseteq U$ and hence $A$ is $\zeta g$ closed.

Remark 3.9. The converse of the above theorem is false as shown in the following example.

Example 3.10. Let $(X, \tau)$ be a topological space with $X = \{p, q, r\}$, $\tau = \{\phi, X, \{p\}\}$, $\zeta = \{X, \{p\}, \{q, p, \}, \{q, r\}\} \zeta \delta \bar{g}$ closed $= \{\phi, X, \{q, r\}, \{r, \}, \{p, r\}\}$ $\zeta \delta \bar{g}$ closed $= \{\phi, X, \{q, r\}, \{r, \}, \{p, r\}\}$. Here $\{q\}$ is $\zeta \delta \bar{g}$ closed but not $\zeta \delta \bar{g}$ closed.

Proposition 3.11. Every $\zeta \delta \bar{g}$ closed set is $\zeta g \delta s$ closed set.

Proof. Let $A$ be a $\zeta \delta \bar{g}$ closed set and $U$ be any $\delta$-open set containing $A$ in $(X, \tau)$. Since $\delta$-open set is $\zeta \omega$ open and $A$ is $\zeta \delta \bar{g}$ closed. Then $\varphi_\delta(A) \subseteq U$ for every subset $A$ of $X$.

Therefore $A$ is $\zeta g \delta s$ closed.

Remark 3.12. The converse of the above theorem is not true as shown in the following example.

Example 3.13. Let $X = \{p, q, r\}$, $\tau = \{\phi, X, \{q\}\}$, $\zeta = \{X, \{p\}, \{q, p, \}, \{q, r\}\} \zeta \delta \bar{g}$ closed $= \{\phi, X, \{q, r\}, \{r, \}, \{p, r\}\}$ $\zeta g \delta s$ closed $= \{\phi, X, \{q, r\}, \{r, \}, \{p, r\}\}$. Here $\{q\}$ is $\zeta g \delta s$ closed but not $\zeta g \delta s$ closed.

Remark 3.14. The following example shows that $\zeta \delta \bar{g}$ closedness is independent from closedness, $\omega$-closedness, $\zeta \omega$ closedness, $\delta g$ closedness and $\delta g^*$ closedness.

Example 3.15. Let $X = \{p, q, r\}$, $\tau = \{\phi, X, \{q\}\}$, $\zeta = \{X, \{p\}, \{q, p, \}, \{q, r\}\} \zeta \delta \bar{g}$ closed $= \{\phi, X, \{q, r\}, \{r, \}, \{p, r\}\}$ $\delta g$ closed $= \delta g^*$ closed $= \{\phi, X, \{p, q, r\}, \{q, r\}, \{p, r\}\}$ $\zeta \omega$ closed $= \{\phi, X, \{p\}, \{q, r\}, \{q, r\}, \{p, r\}\}$. Here $\{p\}$ is $\delta g$ closed, $\delta g^*$ closed and $\zeta \omega$ closed but not $\zeta \delta \bar{g}$ closed.

Example 3.16. Let $X = \{p, q, r\}$, $\tau = \{\phi, X, \{r\}, \{q, r\}\}$, $\zeta = \{X, \{p\}, \{r\}, \{q, p, \}, \{q, r\}\}$, $\zeta \delta \bar{g}$ closed but not $\delta g $ closed, $\delta g^*$ closed and $\zeta \omega$ closed in $(X, \tau)$.

Example 3.17. Let $X = \{p, q, r\}$, $\tau = \{\phi, X, \{q\}\}$, $\zeta = \{X, \{p\}, \{q, p, \}, \{q, r\}, \{r, \}\}$, $\zeta \delta \bar{g}$ closed but not $\zeta \omega$ closed and closed in $(X, \tau)$.

Remark 3.18. The relationship of $\zeta \delta \bar{g}$ closed sets with known existing sets is given below: $A \rightarrow B$ represents $A$ implies $B$ but not conversely.

1. $\zeta \delta \bar{g}$ closed, 2. $\delta$ closed, 3. $r$ closed, 4. $\delta g^*$ closed, 5. $\delta g$ closed, 6. $\zeta g \delta s$ closed, 7. $\zeta g$ closed, 8. $\zeta \omega$ closed, 9. $\omega$ closed, 10. closed.

Theorem 3.19. Let $(X, \tau^\delta, \zeta)$ be a grill delta space. If a subset $A$ of $X$ is $\zeta \delta \bar{g}$ closed then $\tau^\delta - cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\zeta \omega$ open.

Proof. Let $A$ be a $\zeta \delta \bar{g}$ closed set and $U$ be $\zeta \omega$ open in $X$ such that $A \subseteq U$ then $\varphi_\delta(A) \subseteq U \Rightarrow A \cup \varphi_\delta(A) \subseteq U \Rightarrow \tau^\delta - cl_\delta(A) \subseteq U$.

Thus $\tau^\delta - cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\zeta \omega$ open.

Theorem 3.20. Let $(X, \tau^\delta, \zeta)$ be a grill delta space. If a subset $A$ of $X$ is $\zeta \delta \bar{g}$ closed then for all $x \in \tau^\delta - cl_\delta(A) \subseteq U cl_\delta(\{x\}) \cap A \neq \emptyset$.

Proof. Let $x \in \tau^\delta - cl_\delta(A)$. If $cl_\delta(\{x\}) \cap A = \emptyset \Rightarrow A \subseteq X \setminus cl_\delta(\{x\})$ then by Theorem 3.19 $\tau^\delta - cl_\delta(A) \subseteq X \setminus cl_\delta(\{x\})$ which is a contradiction to our assumption that $x \in \tau^\delta - cl_\delta(A)$. Therefore, $cl_\delta(\{x\}) \cap A \neq \emptyset$.

Lemma 3.21. Let $(X, \tau^\delta)$ be a space and $\zeta$ be a grill on $X$. If $A \subseteq X$ is $\tau^\delta -$ dense in itself, then $\varphi_\delta(A) = cl_\delta(\varphi_\delta(A)) = \tau^\delta - cl_\delta(A) = cl_\delta(A)$.

Proof. Assume $A$ to be $\tau^\delta -$ dense in itself. $A \subseteq \varphi_\delta(A)$.

Thus $cl_\delta(A) \subseteq cl_\delta(\varphi_\delta(A)) = \varphi_\delta(A) \subseteq cl_\delta(A)$.

This implies $cl_\delta(A) = \varphi_\delta(A) = cl_\delta(\varphi_\delta(A))$.

Now by definition $\tau^\delta - cl_\delta(A) = A \cup \varphi_\delta(A) = A \cup cl_\delta(A) = cl_\delta(A)$.
Therefore

\[ \varphi_\delta(A) = cl_\delta(\varphi_\delta(A)) = \tau_\zeta^\delta - cl_\delta(A) = cl_\delta(A). \]

\[ \square \]

**Theorem 3.22.** Let \( \zeta \) be a grill on a space \( (X, \tau^\delta) \). If \( A(\subseteq X) \) is \( \tau_\zeta^\delta \)-dense in itself and \( \zeta \delta^\hat{g} \) closed, then \( A \) is \( \delta \omega \) closed.

*Proof.* From Lemma 3.21

\[ \square \]

#### References


