Properties of disjunctive domination in product graphs

A. Lekha* and K.S. Parvathy

Abstract
In this paper properties of disjunctive domination in some graph products are studied. We examine whether disjunctive domination number is multiplicative with respect to different graph products, that is,

\[ \gamma_d^2(G_1 \ast G_2) \geq \gamma_d^2(G_1) \gamma_d^2(G_2) \]

for all graphs \( G_1 \) and \( G_2 \) or

\[ \gamma_d^2(G_1 \ast G_2) \leq \gamma_d^2(G_1) \gamma_d^2(G_2) \]

for all graphs \( G_1 \) and \( G_2 \) where \( \ast \) denotes lexicographic, tensor, strong or Cartesian product of graphs. Some other inequalities involving disjunctive domination number of product graphs and the graphs attaining these inequalities are also given.

Keywords
Domination, disjunctive domination, disjunctive domination number, graph product.

AMS Subject Classification
05C69.

1,2 Research Department of Mathematics, St. Mary’s College, Thrissur-680020, Kerala, India.
*Corresponding author: 1 alekharemesh@gmail.com, lekha.a.res@smctsr.ac.in; 2parvathy.math@gmail.com, parvathy.ks@smctsr.ac.in

Article History: Received 13 October 2019; Accepted 27 December 2019

Contents

1 Introduction ........................................... 37
2 Preliminaries ........................................... 37
3 Main Results ........................................... 38
4 Conclusion ........................................... 41
References ........................................... 41

1. Introduction

Various graph products clearly model processor connections in multiprocessor systems. The fast transmission of information between the processors is very important in communication systems. Hence the study of graph theoretic properties of product graphs is important. Dominating number in product graphs has been studied for a long time. Among various products, the Cartesian product is the centre of study in almost all works in literature. These studies are focused largely on Vizing’s conjecture. Here an attempt to determine the disjunctive domination number of different types of graph products is made.

2. Preliminaries

Domination in graphs is an important parameter in graph theory because of its wide applications. Tremendous research has been made by many researchers on this topic. A brilliant survey of studies related to domination is given in [2] by Haynes et al. A variation of classical domination defined as secondary dominations is studied in [3]. Another variation of domination, defined as disjunctive domination, was introduced and studied by Goddard et al. in [4]. For more details on graph products and its applications, we suggest the reader to refer [7].

Definition 2.1. A subset \( S \) of the vertex set \( V \) is a disjunctive dominating set or DD-set, if for any vertex \( u \notin S \) one of the following two conditions are true.

1. there is a vertex \( v \in S \) which is adjacent to \( u \) or
2. there are two vertices \( v_1, v_2 \in S \) such that \( d(u, v_1) = d(u, v_2) = 2 \).

The disjunctive domination number or DD-number, \( \gamma_d^2(G) \) of a graph \( G \) is \( \min\{|S|: S \text{ is a DD-set in } G\} \) [4, 5]. If the above condition is true for every vertex in \( u \in S \), then \( S \) is called a total disjunctive dominating set or TDD-set of \( G \) . Total disjunctive domination number or TDD-number, \( \gamma_d^t(G) \) of \( G \) is \( \min\{|S|: S \text{ is a TDD-set in } G\} \) [6].

Definition 2.2. A vertex \( v \) in a graph is called a universal vertex or full degree vertex if \( N[v] = V(G) \).
Definition 2.3. A graph parameter \( \phi \) is multiplicative with respect to a graph product \( G \times H \) if \( \phi(G \times H) = \phi(G) \phi(H) \) for all graphs \( G \) and \( H \).

For all standard terminology and notation we follow [1]. The terms related to domination in graphs are used as in [2].

3. Main Results

Disjunctive domination in lexicographic products

The Lexicographic product of graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is the graph \( G_1 [G_2] \) whose vertex set is \( V_1 \times V_2 \) in which \( (u_1, v_1), (u_2, v_2) \) is an edge if

- \( u_1 u_2 \in E_1 \)
- \( u_1, u_2 \) are equal and \( v_1 v_2 \in E_2 \).

Theorem 3.1. Disjunctive domination number is multiplicative with respect to Lexicographic product.

Proof. Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs with disjunctive domination numbers \( \gamma'_d(G_1) \) and \( \gamma'_d(G_2) \) respectively. We can show that \( S_1 \times S_2 \) is a DD-set of \( G_1[G_2] \).

Claim (i) Let \( (u, v) \) be a vertex in \( G_1[G_2] \) which is not in \( S_1 \times S_2 \).

Case (i) Let \( u \in V_1 \setminus S_1 \) and \( v \in V_2 \). If \( u \) is adjacent to \( u_1 \in S_1 \), then \( (u, v) \) is adjacent to \( (u_1, v) \in S_1 \times S_2 \). If \( u \) is disjunctively dominated by \( u_1, u_2 \in S_1 \), then \( (u, v) \) is disjunctively dominated by \( (u, v') \) for some \( v' \) in \( V_2 \) such that \( d((u, v'), (u_1, v)) = 2 \).

Case (ii) Let \( u \in S_1 \) and \( v \in V_2 \setminus S_2 \). If \( v \) is adjacent to \( v_1 \in S_2 \), then \( (u, v) \) is adjacent to \( (u, v_1) \in S_1 \times S_2 \). If \( v \) is disjunctively dominated by \( v_1, v_2 \in S_2 \), then \( (u, v) \) is disjunctively dominated by \( (u, v') \) for some \( v' \) in \( V_2 \) such that \( d((u, v'), (u_1, v)) = 2 \).

Case (iii) Let \( u \in V_1 \setminus S_1 \) and \( v \in V_2 \setminus S_2 \). If \( u \) is adjacent to \( u_1 \in S_1 \) and \( v \) is adjacent to \( v_1 \in S_2 \), then \( (u, v) \) is adjacent to \( (u_1, v_1) \in S_1 \times S_2 \). If \( u \) is disjunctively dominated by \( u_1, u_2 \in S_1 \), then \( (u, v) \) is disjunctively dominated by \( (u, v') \) for some \( v' \) in \( V_2 \) such that \( d((u, v'), (u_1, v)) = 2 \).

From the above cases it follows that in each case \( (u, v) \) is either dominated or disjunctively dominated by elements of \( S_1 \times S_2 \). Thus \( S_1 \times S_2 \) is a DD-set in \( G_1[G_2] \). Hence \( \gamma'_d(G_1[G_2]) \leq \gamma'_d(G_1) \gamma'_d(G_2) \) for all graphs \( G_1 \) and \( G_2 \).

Remark 3.2. 1. The above bound is sharp. If \( G_1 = P_2 \) and \( G_2 = P_3 \), then \( \gamma'_d(G_1) = 1, \gamma'_d(G_2) = 2, \gamma'_d(G_1[G_2]) = 2 \), and so, \( \gamma'_d(G_1[G_2]) = \gamma'_d(G_1) \gamma'_d(G_2) \).

2. Strict inequality may occur in the above result. For example consider the graphs \( G_1 = P_4 \) and \( G_2 = S_4 \circ K_1 \). Then \( \gamma'_d(G_1) = 1, \gamma'_d(G_2) = 4, \gamma'_d(G_1[G_2]) = 2 \). Here \( \gamma'_d(G_1[G_2]) < \gamma'_d(G_1) \gamma'_d(G_2) \).

Theorem 3.3. 1. \( \gamma'_d(G_1[G_2]) = \gamma'_d(G_1) \) if \( G_2 \) has a universal vertex. In particular for a positive integer \( n \), \( \gamma'_d(G_1[K_n]) = \gamma'_d(G_1) \).

2. \( \gamma'_d(G_1[G_2]) = 2 \), if \( G_1 \) has a universal vertex, but \( G_2 \) has no such vertex. In particular, if \( G_1 = K_n \) and \( G_2 \) has no universal vertex, then \( \gamma'_d(G_1[G_2]) = 2 \).

3. If both \( G_1 \) and \( G_2 \) have a universal vertex, then \( \gamma'_d(G_1[G_2]) = 1 \). In particular if \( G_1 = K_n \) and \( G_2 = K_m \), where \( m,n \) are positive integers, then \( \gamma'_d(G_1[G_2]) = 1 \).

Proof. 1. Let \( v \) be a universal vertex of \( G_2 \) and \( S_1 \) be a \( \gamma'_d \)-set of \( G_1 \). Then \( S_1 \times v \) disjunctively dominates \( G_1[G_2] \). The minimality of \( S_1 \times v \) follows from the minimality of the \( \gamma'_d \)-set \( S_1 \) of \( G_1 \). Thus, \( \gamma'_d(G_1[G_2]) = \gamma'_d(G_1) \).

2. Let \( u \) be a universal vertex of \( G_1 \) and \( v_1, v_2 \) are any two vertices in \( G_2 \). Then \( \{ (u_1, v_1), (u_1, v_2) \} \) forms a \( \gamma'_d \)-set of \( G_1[G_2] \), for if \( (u', v') \) is an arbitrary vertex in \( G_1[G_2] \), then \( (u_1, v_1) \) is dominated by \( (u_1, v_1) \) and \( (u_1, v_2) \) whenever \( u \neq u' \) and disjunctively dominated by \( (u_1, v_1) \) \( (u_1, v_2) \) whenever \( u = u' \).

3. Let \( u \) and \( v \) be universal vertices in \( G_1 \) and \( G_2 \) respectively. Then \( (u, v) \) dominates all the vertices in \( G_1[G_2] \). So, \( \gamma'_d(G_1[G_2]) = 1 \).

Corollary 3.4. \( \gamma'_d(G_1[G_2]) = \gamma'_d(G_1) \gamma'_d(G_2) \) if \( G_2 \) has a universal vertex.

Theorem 3.5. Let \( G_1 \) be a graph without isolated vertices and \( G_2 \) be a non-trivial graph. Then,

\[ \gamma'_d(G_1[G_2]) \leq 2 \gamma'_d(G_1). \]

Proof. Let \( S \) be a DD-set of \( G_1 \) and \( x, y \) are any two distinct vertices in \( G_2 \). We can show that \( (S \times x) \cup (S \times y) \) is a DD-set of \( G_1[G_2] \). Clearly, \( S \times x \) dominates or disjunctively dominates all the vertices in \( (G_1 \setminus S) \times G_2 \). Now, let \( (u, v) \) be a vertex in \( S \times G_2 \). Let \( u' \) be a vertex in \( G_1 \) which is adjacent to \( u \) in \( G_1 \). Then \( (u, v) \) is adjacent to \( (u', x) \) which is adjacent to \( (u, x) \) in \( S \times x \) and \( (u, y) \) in \( S \times y \) in \( G_1[G_2] \). It shows that every vertex in \( S \times G_2 \) has at least two vertices in \( (S \times x) \cup (S \times y) \) at a distance 2 from it in \( G_1[G_2] \). Thus \( (S \times x) \cup (S \times y) \) is a DD-set in \( G_1[G_2] \), proving that \( \gamma'_d(G_1[G_2]) \leq 2 \gamma'_d(G_1) \).

Remark 3.6. 1. If \( G_1 \) has a universal vertex, but \( G_2 \) has no such vertex, then equality occurs in the above relation.
2. If both $G_1$ and $G_2$ have a universal vertex then, strict inequality occurs in the above result.

3. If $G_1$ has a $\gamma_2^d$-set in which a pair of vertices are adjacent or if some vertex in $G_1$ is dominated by two different vertices in $S$, then strict inequality occurs in 3.5.

**Theorem 3.7.** If $G_1$ has no isolated vertex, then for all graphs $G_2$, $\gamma_2^d(G_1[G_2]) \leq \gamma_2^d(G_1)$, where $\gamma_2^d(G_1)$ is the total disjunctive domination number of $G_1$.

**Proof.** Let $S$ be a TDD-set of $G_1$. For any vertex $x \in G_2$, we can show that $S \times x$ is a DD-set in $G_1[G_2]$. It is clear that $S \times x$ dominates or disjunctively dominates $(G_1 \setminus S) \times G_2$. Now let $(u,v)$ be any vertex in $S \times x$. $u$ is either adjacent to $u' \in S$ or has two vertices $u_1$ and $u_2$ in $S$ at a distance 2 from it. Then $(u,v)$ is either dominated by $(u',x) \in S \times x$ or disjunctively dominated by $(u_1,x),(u_2,x) \in S \times x$, showing that $S \times x$ is a disjunctive dominating set in $G_1[G_2]$. This proves that, $\gamma_2^d(G_1[G_2]) \leq \gamma_2^d(G_1)$.

**Remark 3.8.** The bound given in the above theorem is sharp. If $G_1$ has a universal vertex and $G_2$ has no such vertex, then $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1) = 2$. We may also note that strict inequality in the bound can be achieved. Consider the graphs $G_1 = P_5$, $G_2 = P_2$. Then $\gamma_2^d(G_1) = 3$, $\gamma_2^d(G_1[G_2]) = 2$ and hence $\gamma_2^d(G_1[G_2]) < \gamma_2^d(G_1)$.

**Disjunctive domination in tensor products**

Tensor product or Cross Product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2$ whose vertex set is $V_1 \times V_2$ and edge set is $\{((u_1, v_1),(u_2, v_2)) : u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2\}$. There is no consistent relation between the disjunctive domination number of the tensor product of two graphs and the product of their disjunctive domination numbers. There are graphs in which $\gamma_2^d(G_1 \times G_2) > \gamma_2^d(G_1)\gamma_2^d(G_2)$, $\gamma_2^d(G_1 \times G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ and $\gamma_2^d(G_1 \times G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

**Example 3.9.**

1. $\gamma_4^d(P_3 \times P_3) = 4 > \gamma_4^d(P_3)\gamma_4^d(P_3)$.

2. $\gamma_4^d(C_3 \times C_4) = 2 = \gamma_4^d(C_3)\gamma_4^d(C_4)$.

3. If $G_1$ is the graph given in fig.1, then $\gamma_4^d(G_1 \times G_1) = 2 < \gamma_4^d(G_1)\gamma_4^d(G_1)$.

**Theorem 3.10.** For any two graphs $G_1$ and $G_2$ with at least two vertices and $G_2$ having no isolated vertices,

$$\gamma_2^d(G_1 \times G_2) \leq \min \{ \gamma_2^d(G_1)|G_2|, \gamma_2^d(G_2)|G_1| \}$$

**Proof.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs with $\gamma_2^d$-sets $S_1$ and $S_2$ respectively. We can show that $S_1 \times V_2$ and $V_1 \times S_2$ are both DD-sets in $G_1 \times G_2$.

**Disjunctive domination in strong products**

The strong product or normal product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \boxtimes G_2$ whose vertex set is $V_1 \times V_2$ in which $(u_1, v_1)$ is adjacent to $(u_2, v_2)$ if and only if either

- $u_1 = u_2$ and $v_1v_2 \in E_2$ or
- $u_1u_2 \in E_1$ and $v_1 = v_2$ or
- $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

**Theorem 3.12.** For any two non trivial graphs $G_1$ and $G_2$,

$$\gamma_2^d(G_1 \boxtimes G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2).$$

**Proof.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have $\gamma_2^d$-sets $S_1$ and $S_2$ respectively. We can show that $S_1 \times S_2$ is a DD- set of $G_1 \boxtimes G_2$. 
Properties of disjunctive domination in product graphs — 40/41

claim
Let \((u, v) \in S_1 \times S_2\) be a vertex in \(G_1 \square G_2\).

**case (i)**
Let \(u \in V_1 \setminus S_1\) and \(v \in S_2\). Then either \(u\) is dominated by \(x \in S_1\) or is disjunctively dominated by two different vertices \(x_1, x_2 \in S_1\). If \(u\) is dominated by \(x \in S_1\), then \((u, v)\) is dominated by \((x, v) \in S_1 \times S_2\) in \(G_1 \square G_2\). If \(u\) is disjunctively dominated by two different vertices \(x_1, x_2 \in S_1\), then \((x_1, v), (x_2, v) \in S_1 \times S_2\) and \(d((u, v), (x_1, v)) = d((u, v), (x_2, v)) = 2\) so that \((u, v)\) is disjunctively dominated by \(S_1 \times S_2\) in \(G_1 \square G_2\).

**case (ii)**
Let \(u \in V_1\) and \(v \in V_2 \setminus S_2\). Then either \(v\) is dominated by \(y \in S_2\) or is disjunctively dominated by two different vertices \(y_1, y_2 \in S_2\). If \(v\) is dominated by \(y \in S_2\), \((u, v)\) is dominated by \((u, y) \in S_1 \times S_2\) in \(G_1 \square G_2\). If \(v\) is disjunctively dominated by two different vertices \(y_1, y_2 \in S_2\), then \((u, y_1), (u, y_2) \in S_1 \times S_2\) and \(d((u, v), (u, y_1)) = d((u, v), (u, y_2)) = 2\) so that \((u, v)\) is disjunctively dominated by \(S_1 \times S_2\) in \(G_1 \square G_2\).

**case (iii)**
Let \(u \in V_1 \setminus S_1\) and \(v \in V_2 \setminus S_2\). If \(u\) is dominated by \(x \in S_1\) and \(v\) is dominated by \(y \in S_2\), then \((u, v)\) is dominated by \((x, y) \in S_1 \times S_2\) in \(G_1 \square G_2\).

If \(u\) is disjunctively dominated by two different vertices \(x_1, x_2 \in S_1\) in \(G_1\) and \(v\) is dominated by \(y \in S_2\) in \(G_2\), then \((u, v)\) is adjacent to \((u, y)\) which is adjacent again to \((x_1, y) \in S_1 \times S_2\) in \(G_1 \square G_2\). Similarly, \((u, v)\) is also adjacent to \((u, y_2)\) which is again adjacent to \((x_2, y) \in S_1 \times S_2\) in \(G_1 \square G_2\). Thus \(d((u, v), (x_1, y)) = d((u, v), (x_2, y)) = 2\). In other words, \((u, v)\) is disjunctively dominated by two different vertices \((x_1, y), (x_2, y) \in S_1 \times S_2\). Similarly if \(u\) is dominated by \(x \in S_1\) in \(G\) and \(v\) is disjunctively dominated by \(y_1, y_2 \in S_2\) in \(G_2\), then \((u, v)\) is disjunctively dominated by \((x_1, y), (x_2, y) \in S_1 \times S_2\) in \(G_1 \square G_2\).

Disjunctive domination in cartesian products

The Cartesian Product \(G_1 \square G_2\) of graphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) is the graph with vertex set \(V_1 \times V_2\) in which \((u_1, v_1), (u_2, v_2)\) is an edge if and only if either

- \(u_1 = u_2\) and \(v_1, v_2 \in E_2\) or
- \(u_1, u_2 \in E_1\) and \(v_1 = v_2\)

**Theorem 3.14.** For any two graphs \(G_1\) and \(G_2\),

\[
\gamma^D_{\square}(G_1 \square G_2) \leq \min \{ \gamma^D_{\square}(G_1), \gamma^D_{\square}(G_2) \}
\]

**Proof.** Let \(G_1\) and \(G_2\) are two graphs with \(\gamma^D_{\square}\)-sets \(S_1\) and \(S_2\) respectively. We can show that \(S_1 \times V_2\) and \(V_1 \times S_2\) are both \(DD\)-sets of \(G_1 \square G_2\).

**claim**
Let \((u, v)\) be a vertex in \(G_1 \square G_2\). If \(u \in S_1\), then \((u, v)\) is dominated by \(x \in S_1\) or disjunctively dominated by two different vertices \(x_1, x_2 \in S_1\). If \(u\) is dominated by \(x \in S_1\), then \((u, v)\) is adjacent to \((x, v) \in S_1 \times S_2\) in \(G_1 \square G_2\). If \(u\) is disjunctively dominated by two different vertices \(x_1, x_2 \in S_1\), then \((x_1, v), (x_2, v) \in S_1 \times S_2\) and \(d((u, v), (x_1, v)) = d((u, v), (x_2, v)) = 2\) so that \((u, v)\) is disjunctively dominated by \(S_1 \times S_2\) in \(G_1 \square G_2\).

**Remark 3.15.**
1. Equality comes in the above theorem if \(G_1 = P_2\) or \(P_3\) and \(G_2 = P_2\).
2. Strict inequality occurs if \(G_1 = P_2\) and \(G_2 = P_3\).

**Remark 3.16.** The Vizing’s like inequality \(\gamma^D_{\square}(G_1 \square G_2) \geq \gamma^D_{\square}(G_1) \gamma^D_{\square}(G_2)\) is not true in disjunctive domination. There are graphs in which \(\gamma^D_{\square}(G_1 \square G_2) > \gamma^D_{\square}(G_1) \gamma^D_{\square}(G_2)\).

For example,

1. If \(G_1 = P_3\) and \(G_2 = P_2\), then \(\gamma^D_{\square}(G_1 \square G_2) = 3 > \gamma^D_{\square}(G_1) \gamma^D_{\square}(G_2)\).
2. If \(G_1 = C_4\) and \(G_2 = P_2\), then \(\gamma^D_{\square}(G_1 \square G_2) = \gamma^D_{\square}(G_1) \gamma^D_{\square}(G_2) = 2\).
3. If \(G_1 = C_2\) and \(G_2 = C_4\), then \(\gamma^D_{\square}(G_1) = \gamma^D_{\square}(G_2) = 2\) and \(\gamma^D_{\square}(G_1 \square G_2) = 2\). Hence \(\gamma^D_{\square}(G_1 \square G_2) < \gamma^D_{\square}(G_1) \gamma^D_{\square}(G_2)\).

**Theorem 3.17.** For any two graphs \(G_1\) and \(G_2\), where \(G_1\) has a \(\gamma\)-set which is such that the vertices not in this set are twice dominated, \(\gamma^D_{\square}(G_1 \square G_2) \leq \gamma(G_1) \gamma(G_2)\).

**Proof.** Let \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) be two graphs with \(\gamma\)-sets \(S_1\) and \(S_2\) respectively. Let the elements of \(V_1 \setminus S_1\) are dominated by two different vertices in \(S_1\). We can show that \(S_1 \times S_2\) is a disjunctive dominating set of \(G_1 \square G_2\). Let \((u, v)\) be a vertex in \(G_1 \square G_2\).
case (i)
If \( u \in S_1 \) and \( v \in S_2 \), then \((u,v) \in S_1 \times S_2 \).

case (ii)
Let \( u \in S_1 \) and \( v \in V_2 \setminus S_2 \). If \( v \) is dominated by \( x \in S_2 \) in \( G_2 \), then \((u,v)\) is dominated by \((u,x) \in S_1 \times S_2 \) in \( G_1 \sqcap G_2 \). Similar is the case when \( u \in V_1 \setminus S_1 \) and \( v \in S_2 \).

case (iii)
Let \( u \in V_1 \setminus S_1 \) and \( v \in V_2 \setminus S_2 \). By hypothesis \( u \) is adjacent to two different vertices \( x_1, x_2 \in S_1 \) in \( G_1 \) and \( v \) is adjacent to \( y \in S_2 \) in \( G_2 \). Then in \( G_1 \sqcap G_2 \), \((u,v)\) is adjacent to \((u,y)\) which is adjacent to \((x_1,y)\) and \((x_2,y)\) \( \in S_1 \times S_2 \). Thus there are two different vertices \((x_1,y)\) , \((x_2,y)\) \( \in S_1 \times S_2 \) such that \( d((u,v),(x_1,y)) = d((u,v),(x_2,y)) = 2 \). Hence \((u,v)\) is disjunctively dominated by \( S_1 \times S_2 \).

The above cases show that \( S_1 \times S_2 \) is a disjunctive dominating set of \( G_1 \sqcap G_2 \). Hence \( \gamma_2^d(G_1 \sqcap G_2) \leq \gamma(G_1)\gamma(G_2) \). \(\square\)

Remark 3.18. The above result is not true in general. The following examples show this.

1. If \( G_1 = G_2 = P_6 \), \( \gamma(G_1) = \gamma_2^d(G_1) = 2 \), \( \gamma(G_2) = \gamma_2^d(G_2) = 2 \), \( \gamma_2^d(G_1 \sqcap G_2) = 6 > \gamma_2^d(G_1)\gamma_2^d(G_2) = \gamma(G_1)\gamma(G_2) \).

2. If \( G_1 = G_2 = P_7 \), \( \gamma(G_1) = \gamma(G_2) = 3 \), \( \gamma_2^d(G_1) = \gamma_2^d(G_2) = 2 \), \( \gamma_2^d(G_1 \sqcap G_2) = 8 \), \( \gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma_2^d(G_1 \sqcap G_2) < \gamma(G_1)\gamma(G_2) \).

3. If \( G_1 = G_2 = P_{10} \), \( \gamma(G_1) = \gamma(G_2) = 4 \), \( \gamma_2^d(G_1) = \gamma_2^d(G_2) = 3 \), \( \gamma_2^d(G_1 \sqcap G_2) = 15 \). Here, \( \gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma_2^d(G_1 \sqcap G_2) < \gamma(G_1)\gamma(G_2) \).

4. If \( G_1 = G_2 = P_{11} \), \( \gamma_2^d(G_1) = \gamma_2^d(G_2) = 3 \), \( \gamma(G_1) = \gamma(G_2) = 4 \), \( \gamma_2^d(G_1 \sqcap G_2) = 18 \). Here \( \gamma_2^d(G_1)\gamma_2^d(G_2) < \gamma(G_1)\gamma(G_2) < \gamma_2^d(G_1 \sqcap G_2) \).

Theorem 3.19. For any two positive integers \( m, n \), \( \gamma_2^d(K_m \sqcap K_n) = 2 \).

Proof. Let \((u_1, v_1), (u_2, v_2)\) are two distinct vertices in \( K_m \sqcap K_n \). A vertex \((x, y) \in K_m \sqcap K_n\) which not dominated by these vertices is such that \( d((u_1, v_1), (x, y)) = d((u_2, v_2), (x, y)) = 2 \). Hence \( \{(u_1, v_1), (u_2, v_2)\} \) is a \( DD \)-set in \( K_m \sqcap K_n \) which gives \( \gamma_2^d(K_m \sqcap K_n) \leq 2 \). If \( u_1 \neq u_2 \) and \( v_1 \neq v_2 \) then \((u_1, v_1)\) and \((u_2, v_2)\) are not adjacent in \( K_m \sqcap K_n \). So there does not exist a universal vertex in \( K_m \sqcap K_n \) which implies that \( \gamma_2^d(K_m \sqcap K_n) \geq 2 \). Therefore \( \gamma_2^d(K_m \sqcap K_n) = 2 \). \(\square\)

4. Conclusion

In this paper we have tried to find properties of disjunctive domination in certain product of graphs. Further investigations are possible to find \( DD \)-number of product of important classes of graphs. The problem of determining \( \gamma_2^d(G_1 \ast G_2) \) precisely for different classes of graphs would be interesting.

Acknowledgment

The first author wishes to thank St. Mary’s College, Thrissur and University of Calicut, Kerala for providing necessary facilities to pursue her research.

References


ISSN(P):2319–3786
Malaya Journal of Matematik
ISSN(O):2321–5666

**********