Induced magic labeling of some graphs

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Abstract

Let $G = (V, E)$ be a graph and let $(A, +)$ be an Abelian group with identity element $0$. Let $f : V \rightarrow A$ be a vertex labeling and $f^* : E \rightarrow A$ be the induced labeling of $f$, defined by $f^*(v_1v_2) = f(v_1) + f(v_2)$ for all $v_1v_2 \in E$. Then $f^*$ again induces a labeling say $f^{**} : V \rightarrow A$ defined by $f^{**}(v) = \sum_{vv_1 \in E} f^*(vv_1)$. A graph $G = (V, E)$ is said to be an Induced $A$-Magic Graph (IAMG) if there exists a non zero labeling $f : V \rightarrow A$ such that $f \equiv f^{**}$. The function $f$, so obtained is called an Induced $A$-Magic Labeling (IAML) of $G$.

Keywords

Induced $A$-Magic Labeling of Graphs, Induced $A$-Magic graphs.

AMS Subject Classification

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1 Introduction

This paper deals with only finite, un directed simple and connected graphs. We refer [3] for the phrasing and standard notations related to graph theory. A graph is a pair $G = (V, E)$, where $V, E$ are the vertex set and edge set respectively. The degree of a vertex $v$ in $G$ is the number of edges incident at $v$ and it is denoted as $deg(v)$. Let $(A, +)$ be an Abelian group with identity element $0$. Let $f : V \rightarrow A$ be a vertex labeling and $f^* : E \rightarrow A$ be the induced edge labeling of $f$, defined by $f^*(v_1v_2) = f(v_1) + f(v_2)$ for all $v_1v_2 \in E$. Then $f^*$ again induces a vertex labeling say $f^{**} : V \rightarrow A$ defined by $f^{**}(v) = \sum_{vv_1 \in E} f^*(vv_1)$. A graph $G = (V, E)$ is said to be an Induced $A$-Magic Graph (IAMG) if there exists a non zero labeling $f : V \rightarrow A$ such that $f \equiv f^{**}$. The function $f$, so obtained is called an Induced $A$-Magic Labeling (IAML) of $G$ and a graph which has no such Induced Magic Labeling is called a Non-induced magic graph. In this paper we discuss the existence of Induced Magic Labeling of some special graphs like $P_n$, $C_n$, $K_n$ and $K_{m,n}$.

Main Results

1 Main Results

Lemma 2.1. Let $G = (V, E)$ be a graph and $f$ is an IAML of $G$. If $v_1 \in V$ is a pendant vertex adjacent to $v \in V$, then $f(v_1) = 0$.

Proof. Let $f$ be an IAML of a graph $G$ and $v_1$ be a pendant vertex adjacent to $v$. Then $f^*(vv_1) = f(v) + f(v_1)$ and $v_1$ is a pendant vertex implies that $f^{**}(v_1) = f(v) + f(v_1)$. Also $f$ is an induced magic labeling of $G$ implies that $f(v_1) = f^{**}(v_1) = f(v) + f(v_1)$. Thus $f(v) = 0$.

Corollary 2.2. If $G$ has a pendant vertex, then $G \notin \Gamma_k(A)$ for any Abelian group $A$.

Proof. Proof is indisputable from the lemma 2.1.
Lemma 2.3. Let $f$ be an IAML of a graph $G$ and $wuvz$ be a path in $G$ with $w$ and $z$ are pendant vertices in $G$, then $f^*(uv) = 0$.

Proof. Suppose $f$ is an IAML of a graph $G = (V, E)$ and $wuvz$ is any path in $G$ with $w$ and $z$ are pendant vertices. Then by the lemma 2.1, we have $f(u) = 0 = f(v)$. Hence $f^*(uv) = 0$. □

Theorem 2.4. Let $f$ be a vertex labeling of a graph $G$. Then $f$ is an IAML of $G$, if and only if $|\{\deg(u) - 1 | f(u) + \sum f(v) = 0, \forall u \in V(G)\}|$.

Proof. Let $f$ be an IAML of $G$ and $u$ be a vertex in $G$ with $\deg(u) = m$. Let $v_1, v_2, v_3, \ldots, v_m$ be those vertices adjacent to $u$. Now, $f$ is an IAML if and only if $f(u) = f^*(u) = f^*(v_1) + f^*(v_2) + f^*(v_3) + \cdots + f^*(v_m)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$.

Then, $f$ is an IAML of $P_n$. Conversely suppose $n$ is not a multiple of 3, then $n = 3m + 1$ or $n = 3m + 2$ for some positive integer $m$. Let $f: V \to A$ be a vertex labeling function with $f \equiv f^*$. Then for $1 \leq k \leq n - 3$ and any path $v_k v_{k+1} v_{k+2} v_{k+3}$ in $P_n$, we have $f(v_{k+1}) = f^*(v_{k+1})$ implies that $f(v_k) + f(v_{k+1}) + f(v_{k+2}) = 0$. Also $f(v_{k+2}) = f^*(v_{k+2})$ implies that $f(v_k) + f(v_{k+1}) + f(v_{k+3}) = 0$. Therefore we should have $f(v_k) = f(v_{k+3})$. Let us deal with the following cases:

Case 1 : $n = 3m + 1$

In this context, from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m-1}) = f(v_{n-2})$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_6) = f(v_9) = 0$. Thus $f(v_3) = 0$ and $f(v_1) + f(v_4) = 0$ implies that $f(v_1) = 0$, which again implies that $0 = f(v_1) + f(v_4) = f(v_7) = \cdots = f(v_{3m+1}) = f(v_n.)$ Hence $f \equiv 0$. Therefore $f$ is not an IAML.

Case 2 : $n = 3m + 2$

In this context from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m+2}) = f(v_{n-2})$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_6) = f(v_9) = 0$. Thus $f(v_1) = 0$ and $f(v_1) + f(v_4) = 0$ implies that $f(v_3) = 0$, which implies $0 = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3m}) = f(v_{n-2})$. Hence $f \equiv 0$. Therefore, $f$ is not an IAML.

Hence if $n$ is not a multiple of a 3, then $P_n \notin \Gamma(A)$. □

Theorem 2.6. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $C_n$. Then for any path $v_{k-1} v_k v_{k+1}$ mod $n$, $f$ is an IAML of $C_n$ if and only if $f(v_{k-1}) + f(v_k) + f(v_{k+1})$ mod $n = 0$, where $1 \leq k \leq n$. Moreover any IAML $f$ of $C_n$ satisfies $f(v_k) = f(v_{k+3})$ mod $n$ for $1 \leq k \leq n$.

Proof. For $k = 1, 2, 3, \ldots, n$, consider the path $v_{k-1} v_k v_{k+1}$ mod $n$ in $C_n$. Observe that $f$ is an IAML of $C_n$ if and only if $f(v_k) = f^*(v_k)$, which holds if and only if $f(v_{k-1}) + f(v_k) + f(v_{k+1})$ mod $n = 0$.

Also for any $0 \leq k \leq n - 1$, let $v_k v_{k+1} v_{(k+2)}$ mod $n$, is a path in $C_n$, we have $f(v_k) + f(v_{k+1}) + f(v_{k+2})$ mod $n = 0$ and $f(v_{k+1}) + f(v_{k+2})$ mod $n + f(v_{(k+3)})$ mod $n = 0$.

Thus $f(v_k) = f(v_{(k+3)})$ mod $n$.

Corollary 2.7. $C_n \in \Gamma_k(A)$ if and only if $O(k) = 3$, where $O(k)$ denotes the order of $k$ in $A$.

Proof. Consider $C_n$ with $V(C_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n = v_0\}$. Suppose $C_n \in \Gamma_k(A)$, that is there exist an IAML $f$ of $C_n$ with $f(v_i) = k$ for $i = 1, 2, 3, \ldots, n$. Then by theorem 2.6 we have $3k = 0$ in $A$, which implies $O(k) = 3$. Conversely suppose $O(k) = 3$. Then consider the vertex label $f(v_i) = k$ for $i = 1, 2, 3, \ldots, n$. Since $f(v_i) = k$ for all $i$ and $O(k) = 3$, we have, $f^*(v_{i+1}) = 2k$ for all $i$, and which implies $f^*(v_i) = f^*(v_{i+1}) + f^*(v_{i-1}) = 4k = f(v_i)$, for all $i$. Thus $f$ is an IAML of $C_n$, that is $C_n \in \Gamma_k(A)$. Hence the proof.

Corollary 2.8. $C_n$ has a non-constant IAML if and only if $n$ is a multiple of 3.

Proof. Consider $C_n$ with vertex set $\{v_1, v_2, \ldots, v_{n-1}, v_n = v_0\}$. Suppose $n = 3k$, for some integer $k$. Let $a, b, c$ be any three distinct elements in $A$, such that $a + b + c = 0$, then define $f: V(C_n) \to A$ as follows:

$$f(v_i) = \begin{cases} a & \text{if } i = 1, 4, 7, \ldots, 3k - 2 \\ b & \text{if } i = 2, 5, 8, \ldots, 3k - 1 \\ c & \text{if } i = 3, 6, 9, \ldots, 3k. \end{cases}$$

Then clearly $f$ is a non constant IAML of $C_n$. Conversely assume that $n$ is not a multiple of 3. Then either $n = 3k + 1$ or $3k + 2$ for some integer $k$. Let $f$ be an IAML of $C_n$ and $f(v_1) = w$.

Case 1: $n = 3k + 1$

In this context, by the theorem 2.6 we have:

$$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_6) = f(v_9) = \cdots = f(v_{3k}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1}).$$

Thus $f(v_i) = w$, for $i = 1, 2, 3, \ldots, n$.

Case 2: $n = 3k + 2$

In this context, by the theorem 2.6 we have:

$$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1}) = f(v_{3k+2}) = f(v_n).$$

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Thus in either case, we have \( f(v_i) = w \) for \( i = 1, 2, 3, \ldots, n \).

Thus if \( n \not\equiv 0 \pmod{3} \) then every IAML of \( C_n \) is a constant IAML of \( C_n \).

**Theorem 2.9.** \textbf{The complete graph \( K_n \in \Gamma(A, f) \) if and only if \((n - 3)f(v_i) = (n - 3)f(v_2) = \cdots = (n - 3)f(v_3) = -f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n) \) where \( v_1, v_2, v_3, \ldots, v_n \) are the vertices of \( K_n \).}

**Proof.** For \( 1 \leq i, j \leq n \), we have \( f(v_i) = f^{**}(v_j) \) holds if and only if \( f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{i-1}) + (n - 2)f(v_i) + f(v_{i+1}) + \cdots + f(v_n) = 0 \), similarly the condition \( f(v_j) = f^{**}(v_j) \) is equivalent to the condition \( f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{j-1}) + (n - 2)f(v_j) + f(v_{j+1}) + \cdots + f(v_n) = 0 \). Thus we have \( f \) is an IAML if and only if \((n - 3)f(v_i) = (n - 3)f(v_j) = -f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n) \), for \( 1 \leq i, j \leq n \). Hence the proof.

**Corollary 2.10.** \( K_n \in \Gamma_k(A) \) if and only if \( O(k) \) divides \( 2n - 3 \), where \( O(k) \) denotes the order of \( k \) in \( A \).

**Proof.** Let \( K_n \) be the complete graph with vertex set \( \{v_1, v_2, v_3, \ldots, v_n\} \). We have \( K_n \in \Gamma_k(A) \), means there exist an IAML \( f \) with \( f(v) = k \), for all \( v \in V(K_n) \). Also by the theorem 2.9, we have \( f \) is an IAML of \( K_n \) if and only if \((n - 3)f(v) = -f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n) \), for all \( v \in V(K_n) \). Thus \( K_n \in \Gamma_k(A) \) if and only if \((n - 3)k = nk \), that is if and only if \((2n - 3)k = 0 \), that is if and only if \( O(k) \) divides \( 2n - 3 \) in \( A \). Completes the proof.

**Theorem 2.11.** \( K_{m,n} \in \Gamma_k(A) \) if and only if \( O(k) \) divides \( 2m - 1 \) and \( O(k) \) divides \( 2n - 1 \), where \( O(k) \) denotes the order of \( k \) in \( A \).

**Proof.** Let \( V(K_{m,n}) = \{v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\} \) with each \( (v_i, u_j) \in E(K_{m,n}) \), for \( 1 \leq i \leq m, 1 \leq j \leq n \). Suppose \( K_{m,n} \in \Gamma_k(A) \), then we have there exist an IAML \( f \) with \( f(v_i, u_j) = k \), for \( 1 \leq i \leq m, 1 \leq j \leq n \). Now \( f \) is an IAML of \( K_{m,n} \) implies \( k = f(v_1) = f^{**}(v_1) = 2nk \), since \( f^{**}(v_1, u_j) = 2k \) for \( 1 \leq j \leq n \), that is \((2n - 1)k = 0 \) in \( A \), which implies \( O(k) \) divides \( 2n - 1 \). Similarly by considering the equation \( f(u_1) = f^{**}(u_1) \) we get \( k = f(u_1) = f^{**}(u_1) = 2mk \), that is \((2m - 1)k = 0 \) in \( A \), which implies \( O(k) \) divides \( 2m - 1 \). Conversely suppose that \( O(k) \) divides \( 2m - 1 \) and \( O(k) \) divides \( 2n - 1 \). Consider the vertex label \( f(v_i) = k = f(u_j) \), for \( v_i, u_j \in V(K_{m,n}) \), \( 1 \leq i \leq m, 1 \leq j \leq n \). Then \( f^{**}(v_i, u_j) = 2k \) for \( 1 \leq i \leq m, 1 \leq j \leq n \). There for \( i = 1, 2, 3, \ldots, m \), \( f^{**}(v_i) = \sum_{j=1}^{n} f^{**}(v_i, u_j) = 2nk = k \), since \( O(k) \) divides \( 2n - 1 \). Thus we have \( f^{**}(v_i) = f(v_i) = k \) for \( i = 1, 2, 3, \ldots, m \). In a similar way, we have \( f^{**}(u_j) = f(u_j) = k \) for \( j = 1, 2, 3, \ldots, n \). Hence we have \( f = f^{**} \), Thus we get \( K_{m,n} \in \Gamma_k(A) \). This concludes the proof.