Mahgoub transform of Boehmians

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Abstract
In literature many integral transforms are applied for solving the theory of differential equations, from the motivation of classical Sumudu transform, Elzki transform, Mahgoub transform was also developed. We extend this transform for Schwartz space of distribution of compact support and to Boehmian space. More results are also established.

Keywords
Mahgoub Transform, Convolution, Natural Transform, Boehmians.

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1. Introduction

Mohand M.A. Mahgoub [12] introduced a new transform called Mahgoub transform. It is derived from classical Fourier integral. The Mahgoub transform has a strong relation with other integral transform. This relation is given by [9]. Mahgoub transform is used to facilitate the process of solving ordinary and partial differential equation in the time domain. From the motivation of Natural transform [10] the Mahgoub transform of a function \( f(x) \) over the set where the set \( \mathcal{A} \) is

\[ \mathcal{A} = \{ f(x) / \exists K, J_1, J_2 > 0 \ | f(x) | < K e^{\frac{|x|}{J_1}} \} \]

for the given function in the set \( \mathcal{A} \). The constant \( K \) must be finite, \( J_1, J_2 \) may be finite or infinite.

\[ M[f(x)] = \hat{M}(v) = v \int_0^\infty f(x) e^{-vx} \, dx \quad x \geq 0, \quad J_1 \leq v \leq J_2 \]


2. Preliminaries

1. The Mahgoub transform of a constant function i.e. \( f(t) = \alpha \) is

\[ M(\alpha) = \alpha \int_0^\infty \alpha \cdot e^{-vt} \, dt = \alpha \]
2. The Mahgoub transform of \( f(t) = 1 \) is
\[
M(1) = v \int_0^\infty 1 \cdot e^{-vt} \, dt = 1
\]
(2.2)

3. The Mahgoub transform of \( f(t) = t \) is
\[
M(t) = \frac{1}{v}
\]
(2.3)

4. The Mahgoub transform of \( f(t) = e^{at} \) is
\[
M(e^{at}) = \frac{v}{v-a}
\]
(2.4)

5. The Mahgoub transform of \( f(t) = \sin at \), \( f(t) = \cos at \) is
\[
M(\sin at) = \frac{av}{v^2 + a^2}
\]
\[
M(\cos at) = \frac{v^2}{v^2 + a^2}
\]
(2.5)

6. The Mahgoub transform of a delta function
\[
M(\delta) = \hat{M}(v) = v
\]
(2.6)

7. Change of Scale by nonzero integer \( \alpha \)
\[
M(f(\alpha t)) = \hat{M}(\frac{v}{\alpha})
\]
(2.7)

8. The Mahgoub transform of derivative \( F'(t) \), \( F''(t) \)
\[
M(F'(t)) = v(\hat{M}(v) - F(0))
\]
\[
M(F''(t)) = v^2 \hat{M}(v) - vF'(0) - v^2F(0)
\]
(2.8)

9. The inverse Mahgoub transform is related with Bromwich contour integral
\[
f(x) = M^{-1}(f(v)) = \lim_{v \to \infty} \frac{1}{2\pi i} \int_{\gamma - iv}^{\gamma + iv} \hat{M}(f(v))e^{vx} \, dx
\]
(2.9)

3. Relation between Mahgoub transform and other integral transforms

i) Connection between Kamal transform and Mahgoub transform :- If \( \hat{M}(v) \) and \( \hat{K}(v) \) are Mahgoub and Kamal transform of \( f(x) \) then
\[
\hat{K}(v) = v\hat{M}(\frac{1}{v})
\]

ii) Mahgoub Laplace Duality :- If \( \hat{M}(v) \) and \( F(v) \) are the Mahgoub and Laplace transform of \( f(x) \) then
\[
\hat{M}(v) = vF(v)
\]

iii) Similarity Mahgoub-Sumudu Duality is given by
\[
\hat{M}(v) = S\left(\frac{1}{v}\right)
\]

iv) Mahgoub Elzaki Duality is
\[
\hat{M}(v) = v^2E(v)
\]

v) Mahgoub Aboodh Duality is
\[
\hat{M}(v) = v^2A(v)
\]

Linearity
If \( f(x) \) and \( g(x) \) have Mahgoub transform as \( M(f) \) and \( M(g) \) then
\[
M(\alpha f(x) + \beta g(x)) = \alpha M(f) + \beta M(g) \quad \text{where} \quad \alpha, \beta \in \mathbb{R}_+
\]
(3.1)

Proof.
\[
M(\alpha f(x) + \beta g(x))
\]
\[
= v \int_0^\infty (\alpha f(x) + \beta g(x)) \cdot e^{-vx} \, dx
\]
\[
= v \int_0^\infty \alpha f(x) \cdot e^{-vx} \, dx + v \int_0^\infty \beta g(x) \cdot e^{-vx} \, dx
\]
\[
= \alpha \int_0^\infty v f(x) \cdot e^{-vx} \, dx + \beta \int_0^\infty v g(x) \cdot e^{-vx} \, dx
\]
(3.2)

Convolution
The Convolution of two functions \( f \) and \( g \) is given by
\[
(f \ast g)(x) = \int_0^\infty f(u)g(x-u) \, du
\]
(3.3)

The Convolution theorem of two function \( f, g \) of Mahgoub transform is given by,
\[
M(f \ast g)(v) = \frac{1}{v} M(f) \cdot M(g)
\]
(3.4)

\[
M(f \ast g)(v)
\]
\[
= \int_0^\infty (f \ast g)(x) [ve^{-vx}] \, dx
\]
\[
= \int_0^\infty \int_0^\infty f(u)g(x-u) \, du \cdot (ve^{-vx}) \, dx
\]
(3.5)
Substitute \( x - u = y \), we have
\[
M(f \ast g)(v) = \int_0^\infty f(u) \int_0^\infty g(y) v e^{-(y+u)v} dy du
\]
\[
= \int_0^\infty f(u) e^{-uv} du \int_0^\infty g(y) v e^{-vy} dy
\]
\[
= \frac{1}{v} \left\{ v \int_0^\infty f(u) e^{-uv} du \cdot v \int_0^\infty g(y) v e^{-vy} dy \right\}
\]
\[
= \frac{1}{v} M(f) \cdot M(g)
\]
\[
M(f \ast g)(v) = \frac{1}{v} M(f) \cdot M(g)
\]  \( (3.7) \)

\section{4. Mahgoub Transform of Distribution}

Let \( H(\mathbb{R}) \) be the space of infinitely smooth function on \( \mathbb{R} \) and \( H'(\mathbb{R}) \) be the dual of \( H(\mathbb{R}) \); i.e. \( H'(\mathbb{R}) \) is dual space of distribution of compact support.

Let \( D(\mathbb{R}) \) denote the subspace of \( H(\mathbb{R}) \) of testing function space of compact support and \( D'(\mathbb{R}) \) is its dual consists of Schwartz space of distributions. Now \( D \subset H, H' \subset D' \).

The kernel of Mahgoub transform \( v e^{-vx} \) is clearly a member of \( H(\mathbb{R}) \). Hence we define the generalized Mahgoub transform \( M \) on \( H'(\mathbb{R}) \) by
\[
M(f(x))(v) = \langle f(x), ve^{-vx} \rangle
\]
for every distribution \( f \in H'(\mathbb{R}) \)

**Theorem 4.1.** \( M \) is well defined mapping in the space \( H(\mathbb{R}) \)

**Proof.** Proof is immediate since, \( ve^{-vx} \in H(\mathbb{R}) \)

**Theorem 4.2.** \( M \) is infinitely smooth and
\[
\frac{d^k}{dv^k} M(f)(v) = \langle f(x), \frac{d^k}{dv^k} ve^{-vx} \rangle \quad f \in H'(\mathbb{R})
\]

**Proof.** This theorem can be proved with the help of [[13], Theorem 2.9.1].

**Theorem 4.3.** The generalized Mahgoub transform \( M \) is Linear.

**Proof.** Let \( f, g \in H'(\mathbb{R}) \) and \( \alpha, \beta \) non negative real number then
\[
M(\alpha f + \beta g)(v) = \langle \alpha f + \beta g, ve^{-vx} \rangle
\]
\[
= \langle \alpha f, ve^{-vx} \rangle + \langle \beta g, ve^{-vx} \rangle
\]
\[
= \alpha < f, ve^{-vx} > + \beta < g, ve^{-vx} >
\]
\[
= \langle \alpha M(f)(v) + \beta M(g)(v) \rangle
\]

**Theorem 4.4.** If \( f \in H'(\mathbb{R}) \) and \( g(x) = \begin{cases} f(x - \tau) & x \geq \tau \\ 0 & x < \tau \end{cases} \) then
\[
M(g)(u) = e^{-uv} M(f)(v)
\]

**Proof.** Let \( f \in H'(\mathbb{R}) \) and \( g \) is defined as above then by translation property of distributional Zemanian [14]
\[
M(g)(v) = \langle g(x), ve^{-vx} \rangle
\]
\[
= \langle f(x - \tau), ve^{-vx} \rangle
\]
\[
= \langle f(x), ve^{-v(x+\tau)} \rangle
\]
\[
= \langle g(x), ve^{-vx} \cdot e^{-vt} \rangle
\]
\[
= e^{-uv} < f(x), ve^{-vx} >
\]
\[
M(g)(v) = e^{-uv} M(f)(v)
\]

Let us define the Convolution of two functions. Let \( f, g \in H'(\mathbb{R}) \) we define the generalized convolution between \( f \) and \( g \) by
\[
< (f \ast g) , \psi(x) > = < f(x), < g(t), \psi(t+x) > >
\]
for every \( \psi \in H(\mathbb{R}) \)

**Theorem 4.5.** Let \( f, g \in H'(\mathbb{R}) \) then
\[
M(f \ast g)(v) = \frac{M(f)(v)M(g)}{v}
\]

**Proof.** By definition of \( M \)
\[
M(f \ast g)(v) = \langle (f \ast g), ve^{-vx} \rangle
\]
\[
= \langle f(x), < g(t), ve^{-v(x+t)} > >
\]
\[
= \langle f(x), < g(t), ve^{-vx} \cdot e^{-vt} > >
\]
\[
= \langle f(x), ve^{-vx} > < g(t), e^{-vt} >
\]
\[
M(f \ast g)(v) = \frac{1}{v} < f(x), ve^{-vx} > < g(t), ve^{-vt} >
\]
\[
M(f \ast g)(v) = \frac{1}{v} M(f) \cdot M(g)
\]

**Theorem 4.6.** Let \( f \in H'(\mathbb{R}) \) then the following holds
\[
M(\alpha f)(v) = \frac{1}{v} [M(f)(v) - \frac{d}{dv} M(f)(v)]
\]

**Proof.** By using the properties of Mahgoub transform and
Theorem (2.2) we get
\[
\frac{d}{dv} M(f) = \frac{d}{dv} < f(x), ve^{-vx} >
\]
\[
= < f(x), \frac{dv}{dv} ve^{-vx} >
\]
\[
= < f(x), ve^{-vx} > < (-x) + e^{-vx} >
\]
\[
= < f(x), -xve^{-vx} + e^{-vx} >
\]
\[
= < f(x), e^{-vx} > - < f(x), xve^{-vx} >
\]
\[
= \frac{1}{v} < f(x), ve^{-vx} > - < \frac{1}{v} f(x), ve^{-vx} >
\]
\[
= \frac{d}{dv} M(f)(v) - M(xf)(v)
\]
\[
M(xf)(v) = \frac{1}{v} M(f)(v) - \frac{d}{dv} M(f)
\]

\[\square\]

Theorem 4.7 (Shifting Theorem). Let \( f \in H'(\mathbb{R}) \) then
\[
M(e^{ax} f(x))(v) = < e^{ax} f(x), ve^{-vx} >
\]
\[
= < f(x), ve^{-(v-a)x} >
\]
\[
= \frac{v}{(1-a/v)} e^{-\left(\frac{v}{1-a/v}\right)}
\]
\[
= \frac{1}{v} \left(1-a/v\right) ve^{-(v-a)x} >
\]
\[
= \frac{v}{(v-a)} < f(x), ve^{-(v-a)x} >
\]
\[
= \frac{v}{(v-a)} M f(v-a)
\]

5. Mahgoub transform of Boehmians

Distributions or generalized function are the objects that generalize functions. To differentiate functions whose derivative do not exist in classical sense is possible in distributional sense. In 1983 Boehmians are the objects obtained by abstract algebraic construction to generalized distribution [11]. The original construction was motivated by regular operators [8]. Boehmians are defined as equivalence classes of convolution quotients of functions that are subclass of Mikusinski’s operator. The most youngest generalization of functions is the theory of Boehmians.

For Linear Space \( G \) and subspace \( F \) of \( G \) assume to all pair \((f, \phi), (g, \psi)\) of elements, \( f, g \in G \) \( \phi, \psi \in F \) the product \( f \ast \phi, g \ast \psi \) such that the following conditions are satisfied:

1. \( \phi \ast \psi \in F \) and \( \phi \ast \psi = \psi \ast \phi \)
2. \( (f \ast \phi) \ast \psi = f \ast (\phi \ast \psi) \)
3. \( (f + g) \ast \phi = (f \ast \phi) + (g \ast \phi) \)
4. \( \lambda (f \ast \phi) = (\lambda f) \ast \phi = f \ast (\lambda \phi) \) \( \lambda \in \mathbb{R} \)
5. If \( (\varepsilon_n) \in \Delta \) and \( (f \ast \varepsilon_n) = (g \ast \varepsilon_n) \) \( n=1,2,\cdots \) then \( f=g \)
6. \( (\varepsilon_n), (\mu_n) \in \Delta \Rightarrow (\varepsilon_n \ast \mu_n) \in \Delta \)

Elements of \( \Delta \) are called \( \Delta \) sequences.

Now consider the class \( U \) of pairs of sequences defined by \( U = \{((f_n), (\varepsilon_n)) : (f_n) \subseteq G^N, \varepsilon_n \in \Delta \} \) for each \( n \).

The pair \(((f_n), (\varepsilon_n)) \in U \) is said to be quotient of sequences denoted by \( f_n/\varepsilon_n \chi \) if \( f_n \ast \varepsilon_m = f_m \ast \varepsilon_n \) for \( \forall m, n \in \mathbb{N} \).

Two quotients of sequences \( f_n/\phi_n \) and \( g_n/\psi_n \) are equivalent \( f_n/\phi_n \sim g_n/\psi_n \) if \( f_n \ast \phi_m = g_m \ast \phi_n \) for \( \forall m, n \in \mathbb{N} \).

The relation \( \sim \) is an equivalence relation on \( U \) and hence splits \( U \) into equivalence classes. The equivalence class containing \( f_n/\phi_n \) is denoted by \( [f_n/\phi_n] \). These equivalence classes are called Boehmians and is denoted by \( B(G, F, \Delta, \ast) \).

The sum and multiplication by a scalar of two Boehmians can be defined as,
\[
[f_n/\phi_n] + [g_n/\psi_n] = [(f_n \ast \psi_n) + (g_n \ast \phi_n)]/[(\phi_n \ast \psi_n)]
\]
and
\[
[\gamma f_n/\phi_n] = [\gamma f_n]/[\phi_n] \quad \gamma \in \mathbb{C}
\]
The operation \( \ast \) and differentiation are given by,
\[
[f_n/\phi_n] \ast [g_n/\psi_n] = [(f_n \ast g_n)]/[(\phi_n \ast \psi_n)]
\]
and
\[
D^\alpha[f_n/\phi_n] = [D^\alpha f_n]/[\phi_n]
\]

\( G \) is equipped with notion of convergence.

The intrinsic relationship between the notion of convergence and the product \( \ast \) is given by,

1. If \( f_n \rightarrow f \) as \( n \rightarrow \infty \) in \( G \) and \( \phi \in F \) is any fixed element then \( f_n \ast \phi \rightarrow f \ast \phi \) in \( G \) as \( n \rightarrow \infty \)
2. If \( f_n \rightarrow f \) as \( n \rightarrow \infty \) in \( G \) and \( (\delta_n) \in \Delta \) then \( f_n \ast \delta_n \rightarrow f \) in \( G \) as \( n \rightarrow \infty \)
This operation \( \ast \) can be extended to \( B(G, F, \Delta, \ast) \times F \)
3. If \( [f_n/\delta_n] \in B(G, F, \Delta, \ast) \) and \( \phi \in F \) then \( [f_n/\delta_n] \ast \phi = [(f_n \ast \phi)/\delta_n] \)

In \( B(G, F, \Delta, \ast) \) two types of convergences, \( \delta \)-convergence and \( \Delta \)-convergence are defined as:

(\( \delta \)-convergence) A sequence of Boehmians \( (\gamma_n) \) in \( B(G, F, \Delta, \ast) \) is said to be \( \delta \) convergent to a Boehmian \( \gamma \) in
Proof. For delta sequence $(\delta_n)$ we have,

$$M(\delta_n) \to M(\delta)(v)$$

$$\Rightarrow M(\delta_n) \to v \text{ as } n \to \infty \text{ by (2.6)}$$

Let $\phi \in D(\mathbb{R})$ be such that $M(\phi_k) > 0$ on the support of $\phi$, $k \in \mathbb{N}$.

$$f_n/\psi_n \text{ is quotient of sequences implies that } f_n * \psi_m = f_m * \psi_n$$

$$M(f_n) \cdot M(\psi_n) = M(f_m) \cdot M(\psi_m) \quad (5.1)$$

Therefore $M(f_n)(\phi) = M(f_n) \phi M(\psi_n)/M(\psi_k)$

$$= M(f_n) \phi M(\psi_n)/M(\psi_k)$$

$$= M(f_k) (M(\psi_n)) \cdot \phi$$

$$= M(f_k) (M(\psi_n)) \cdot \phi$$

Now $\frac{\phi M(\psi_n)}{M(\psi_k)} \to \phi \cdot \frac{v}{M(\psi_k)} \text{ in } D(\mathbb{R})$

Hence the sequence $M(f_n)$ converges in $D'(\mathbb{R})$

Let $\frac{f_n}{\psi_n} = \frac{\delta_n}{\phi_n}$ in $B(H,D,\triangle,*)$ and define

$$h_n = \begin{cases} f_{n+1} * \psi_{n+1} & n \text{ is odd} \\ g_2 * \psi_2 & n \text{ is even} \end{cases}$$

$\delta_n = \begin{cases} \psi_{n+1} * \phi_{n+1} & n \text{ is odd} \\ \psi_2 * \phi_2 & n \text{ is even} \end{cases}$

then $\frac{h_n}{\delta_n} = \frac{f_n}{\psi_n} = \frac{\delta_n}{\phi_n}$

The sequence $M(h_n)$ converges in $D'$ Moreover.

$$\lim_{n \to \infty} M(h_{n+1}) \text{ and } M(f_n)$$

therefore $M(h_{n+1})$ and $M(f_n)$ converge to the same limit.

Similarly $M(h_{n+1})$ and $M(g_n)$ converge to the same limit.

So $M$ maps $B(H,D,\triangle,*) \to D'(\mathbb{R})$

We define Mahgoub transform of Boehmians

$$\gamma = [f_n/\psi_n] \in B(H,D,\triangle,*) \text{ by }$$

$$M(\gamma) = \lim_{n \to \infty} M(f_n) \quad (5.2)$$

Theorem 5.2. The Mahgoub transform $\tilde{M} : B(H,D,\triangle,*) \to D'(\mathbb{R})$ is linear.

Proof. Let $\beta_1, \beta_2 \in B(H,D,\triangle,*)$ such that $\beta_1 = [f_n/\phi_n], \beta_2 = [g_n/\psi_n]$ then

$$(\beta_1 + \beta_2) = [(f_n * \psi_n) + (g_n * \phi_n)]/(\phi_n * \psi_n)$$

$$\tilde{M}(\beta_1 + \beta_2) = \lim_{n \to \infty} M(f_n * \psi_n) + \lim_{n \to \infty} M(g_n * \phi_n)$$

$$= \lim_{n \to \infty} M(f_n) + \lim_{n \to \infty} M(g_n)$$

$$= \tilde{M}\beta_1 + \tilde{M}\beta_2.$$
Theorem 5.4. $\tilde{M}$ is one-one mapping from $B(H,D,\triangle,\ast)$ to $D'(\mathbb{R})$

Proof. Let $\tilde{M}\beta_1 = \tilde{M}\beta_2$, $\beta_1, \beta_2 \in B(H,D,\triangle,\ast)$ then by above theorem (4.2) and (4.3)

$$\tilde{M}(\beta_1 - \beta_2) = 0 \text{ in } D'(\mathbb{R})$$

$$\beta_1 - \beta_2 = 0 \Rightarrow \beta_1 = \beta_2$$

Theorem 5.5 (Convolution Theorem). Let $\beta_1 = [f_n/\phi_n], \beta_2 = [g_n/\psi_n] \in \beta$ then we have

$$\tilde{M}(\beta_1 \ast \beta_2) = \frac{1}{v} \tilde{M}(\beta_1) \cdot \tilde{M}(\beta_2).$$

Proof.

$$\tilde{M}([f_n/\phi_n] \ast [g_n/\psi_n]) = \tilde{M}([f_n \ast g_n]/(\phi_n \ast \psi_n))$$

$$= \lim_{n \to \infty} \tilde{M}(f_n \ast g_n)$$

$$= \lim_{n \to \infty} \frac{1}{v} \tilde{M}(f_n) \cdot (g_n)$$

$$= \frac{1}{v} \lim_{n \to \infty} \tilde{M}(f_n) \cdot \lim_{n \to \infty} M(g_n)$$

$$= \frac{1}{v} \tilde{M}(\beta_1) \cdot \tilde{M}(\beta_2)$$

Theorem 5.6. The Mahgoub transform $\tilde{M}(\beta)$ is infinitely smooth.

Proof. Let $\beta = [f_n/\psi_n] \in B(H,D,\triangle,\ast)$ and let J be bounded set in $\mathbb{R}$ then for $m \in \mathbb{N}$ we have $M(\psi_m) > 0$ on J.

$$\tilde{M}(\beta) = \lim_{n \to \infty} M(f_n)$$

$$= \lim_{n \to \infty} \frac{M(f_n) \cdot M(\psi_m)}{M(\psi_m)}$$

$$= v \lim_{n \to \infty} \frac{M(f_n \ast \psi_m)}{M(\psi_m)}$$

$$= v \lim_{n \to \infty} \frac{M(f_m \ast \psi_m)}{M(\psi_m)}$$

$$= \lim_{n \to \infty} \frac{M(f_m) \cdot M(\psi_m)}{M(\psi_m)}$$

$$= \frac{M(f_m)}{M(\psi_m)} \lim_{n \to \infty} M(\psi_m)$$

$$= M(f_m) \cdot v \text{ on J}$$

but $M(f_m) \cdot M(\psi_m) \in H(\mathbb{R})$ and $M(\psi_m) > 0$ on J

$$\Rightarrow \tilde{M}(\beta) \text{ is infinitely smooth.}$$

6. Conclusion

In this paper we have obtained the relation between Mahgoub transform and other integral transforms. We defined the Mahgoub transform in distributional sense. Also convolution and Boehmian space for Mahgoub transform is defined.

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