Some exact solutions of (1+1)-dimensional Kaup-system and seventh-order Kawahara equation

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Abstract
In this paper, we solve a (1+1)-dimensional Kaup system and seventh order Kawahara equation. The Lie symmetry analysis is used to perform the similarity reduction and to obtain the exact solutions of the (1+1)-dimensional Kaup system and seventh order Kawahara equation. Similarity transformation method reduces (1+1)-dimensional Kaup system into a system of ODEs and nonlinear Kawahara equation into nonlinear ordinary differential equation (ODE) and helps to find exact solutions. With the help of reduction equations, we have obtained the exact explicit solutions. Moreover, later by power series method, the exact analytic solutions for seventh order Kawahara equation are obtained.

Keywords
Kaup system, Kawahara equation, Similarity transformation method, Infinitesimal generator, Similarity solutions, Power series solution.

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1. Introduction
It is generally known fact that many phenomena in fluid dynamics, plasma physics, nonlinear optics, biology, chemistry, engineering etc. can be described by nonlinear partial differential equations. Therefore, finding the exact solutions of nonlinear partial differential equations plays an important role in the study of many fields of sciences. In order to help engineers and physicists to better understand the working or to better provide knowledge to the physical problem, many powerful and direct methods for solving nonlinear PDEs have been derived. Some of the most efficient methods to find exact solutions of nonlinear partial differential equations are Darboux and Backlund transformation methods [1], Hyperbolic tangent method [2], Homogeneous balance method [3], Lie-symmetry method [4, 5], Inverse scattering transformation [6], Hirota’s direct method [7], Differential transform Method [8] etc. Similarity transformation method is the most efficient and reliable method to obtain the exact solutions of the nonlinear PDEs. The idea of the Lie symmetry group was developed by Lie [9]. We can study the applications of this method in many problems of fluid dynamics in [10–15].

In this work, we consider the (1+1)-dimensional Kaup-system [16] given by Eq. (1.1).

\[ u_t + v_x + (uv)_x + \frac{u_{xxx}}{3} = 0, \]
\[ u_t + v_x + uu_x = 0. \]  

(1.1)

which is used to model the bi-dimensional propagation of long waves in shallow water. This system is a form of Boussinesq system given in [16]. Later, we will find power series solution of the seventh order Kawahara equation [17] given as follows:

\[ u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \]  

(1.2)

where \( \alpha \) is a nonzero constant and \( u_{3x}, u_{5x} \) and \( u_{7x} \) are the dispersive terms. The seventh-order Kawahara equation is also called the seventh-order KdV equation. To discuss the
structural stability of the KdV equation under the singular perturbation, the seventh-order Kawahara equation was introduced.

The aim of this paper is to obtain the exact analytic solutions of the Kaup system and the Kawahara equation by using the Similarity transformation method [18, 19]. According to our knowledge, Similarity Transformation method is never used before to find the exact solutions of Kaup system (1.1). Similarity transformation method is based on the invariance of the partial differential equations. This method reduces the number of independent variables by one. The main idea of the similarity transformation method is to get the similarity variables. With the help of similarity variables, we find out the nonlinear ODEs. To find the exact solutions of some reduced non-linear ODE’s, we have used tanh method [20].

The rest of the paper is organized in the following manner: Some basic concepts of symmetry method have been presented in Section 2. In Section 3, the vector fields of Kaup system and commutator table are presented using lie symmetry analysis. In Section 4 and 5, symmetry group and exact analytic solutions of system (1.1) have been obtained using tanh method [20]. In section 6, using proposed method, similarity reduction and exact solutions of Kawahara equation have been obtained with the help of power series method [21–23]. A brief conclusion is given in Section 7.

2. Symmetry Method

A system $S$ of $n$-th order partial differential equations in $r$ dependent and $s$ independent variables is as follows:

$$
\Delta_S(x, u^{(n)}) = 0, \quad S = 1, 2, \ldots, l.
$$

(2.1)

where $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^r)$ and $u^{(n)}$ is the derivative of $u$, order of which varies from 0 to $n$.

We take a one-parameter Lie group of infinitesimal transformations with dependent variables $u$ and independent variables $x$ of the system.

$$
x^I = x^i + \epsilon \xi^I(x,u) + o(\epsilon^2), \quad u^m = u^{m'} + \epsilon \phi^m(x,u) + o(\epsilon^2),
$$

(2.2)

where $\epsilon$ is a parameter of group and $\xi^I$, $\phi^m$ are the infinitesimals of the transformations for the independent and dependent variables, respectively.

The infinitesimals generator $S$ corresponding to Eq. (2.2) can be written as

$$
S = \sum_{i=1}^{s} \xi^I(x,u) \partial_{x^i} + \sum_{\theta=1}^{r} \phi^\theta(x,u) \partial_{u^\theta}.
$$

(2.3)

The invariance condition for infinitesimal generator $S$ of symmetry group under the infinitesimal transformation is given as

$$
Pr^{(n)} \Delta_S(x, u^{(n)}) = 0, \quad S = 1, 2, \ldots, l \quad \text{whenever} \quad \Delta_S(x, u^{(n)}) = 0,
$$

(2.4)

where $Pr^{(n)}$ is said to be prolongation of $n$-th order of the infinitesimal generator $S$, given by

$$
Pr^{(n)}(S) = S + \sum_{\theta=1}^{r} \sum_{m=1}^{l} \phi^\theta_m(x,u^{(n)}) \partial_{u^\theta_m},
$$

(2.5)

where $M = (m_1, \ldots, m_k)$, $1 \leq m_k \leq l$, $1 \leq k \leq n$, and the summation runs over all orders of $M$. If $M = k$, then the coefficients $\phi^\theta_m$ of $\partial_{u^\theta_m}$ will depend only on $k$th and derivatives of lower order of $u$;

$$
\phi^\theta_m(x,u^{(n)}) = D_M(\phi^\theta - \sum_{i=1}^{s} \xi^I u^{\theta_i}) + \sum_{i=1}^{s} \xi^I u^{\theta_i},
$$

(2.6)

where $u^{\theta_i} = \frac{\partial u^\theta}{\partial x^i}$, $u^{\theta_M} = \frac{\partial u^\theta}{\partial u^i}$.

3. Lie symmetry analysis of Kaup-System

In this section, we will use the Lie symmetry method [5] for Eq. (1.1). Firstly, let us consider a one-parameter Lie group of infinitesimal transformations by taking $x^1 = x, x^2 = t, u^1 = u, u^2 = v$:

$$
x = x + \epsilon \xi_1(x,t,u,v) + O(\epsilon^2),
\quad t = t + \epsilon \tau(x,t,u,v) + O(\epsilon^2),
\quad u = u + \epsilon \eta_1(x,t,u,v) + O(\epsilon^2),
\quad v = v + \epsilon \eta_2(x,t,u,v) + O(\epsilon^2),
$$

where $\epsilon$ is a parameter of the group and $\xi_1$, $\xi_2$ are the infinitesimals of the transformations for the variables $x, t, u$ and $v$, respectively. The vector field associated with the above group of transformations can be written as

$$
W = \xi_1 \partial_x + \tau \partial_t + \eta_1 \partial_u + \eta_2 \partial_v.
$$

(3.1)

Applying similarity transformation method on the Kaup-system, we get the following set of infinitesimals:

$$
\xi_1 = \frac{c_1 x}{2} + c_3 t + c_4, \quad \tau = c_1 t + c_2, \\
\eta_1 = -\frac{c_1 u}{2} + c_3, \quad \eta_2 = -c_1 v - c_1,
$$

(3.2)

where $c_1, c_2$ and $c_3$ are the arbitrary constants.

Thus, the following vector fields span the Lie algebra of infinitesimals symmetries of the system (1.1):

$$
W_1 = \frac{x}{2} \frac{\partial}{\partial x} + \frac{t}{\partial t} - \frac{u}{\partial u} - (v + 1) \frac{\partial}{\partial v},
W_2 = \frac{\partial}{\partial t} \quad \text{(Time translation)},
W_3 = t \frac{\partial}{\partial t} + \frac{\partial}{\partial u} \quad \text{(Galilean boost)},
W_4 = \frac{\partial}{\partial x} \quad \text{(Space translation)}.
$$
Thus, the infinitesimal generators of the system (1.1) can be written in the following form

\[ W = a_1W_1 + a_2W_2 + a_3W_3 + a_4W_4. \]  

(3.3)

For the infinitesimal generators, under the Lie Bracket, the vector fields of the Eq. (3.1) are closed. We have the commutator table, whose \((i, j)\)th entry is \([W_i, W_j] = W_i W_j - W_j W_i\). The commutator table is skew-symmetric with diagonal elements zero.

For the infinitesimal generators (3.2), we have the following commutator table:

<table>
<thead>
<tr>
<th></th>
<th>(W_1)</th>
<th>(W_2)</th>
<th>(W_3)</th>
<th>(W_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(W_1)</td>
<td>0</td>
<td>(-W_2)</td>
<td>(\frac{W_3}{2})</td>
<td>(-\frac{W_4}{2})</td>
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<tr>
<td>(W_2)</td>
<td>(-W_2)</td>
<td>0</td>
<td>(W_4)</td>
<td>0</td>
</tr>
<tr>
<td>(W_3)</td>
<td>(\frac{W_3}{2})</td>
<td>(-W_4)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(W_4)</td>
<td>(\frac{W_4}{2})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### 4. Symmetry group of Kaup-System

This part deals with the one-parameter symmetry groups

\[ h_i : (x, t, u, v) \rightarrow (\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}) \]

of the corresponding infinitesimal generators. To obtain the Lie symmetry groups, we are required to solve the following ordinary differential equations:

\[ \frac{d(x, t, u, v)}{\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}} = \left(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}\right) |_{x=0} = \left(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}\right). \]

For the infinitesimal generator \(W = a_1W_1 + a_2W_2 + a_3W_3 + a_4W_4\), we will take the different values of \(a_i\)'s to obtain the corresponding infinitesimal generators as follows:

**Case 1.** \(a_1 = 1, a_2 = a_3 = a_4 = 0\), the infinitesimal generator

\[ W_1 = \frac{1}{2} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial t} - v \frac{\partial}{\partial u} - (v + 1) \frac{\partial}{\partial v}, \]

**Case 2.** \(a_2 = 1, a_1 = a_3 = a_4 = 0\), the infinitesimal generator

\[ W_2 = \frac{\partial}{\partial t}. \]

**Case 3.** \(a_3 = 1, a_1 = a_2 = a_4 = 0\), the infinitesimal generator

\[ W_3 = \frac{1}{2} \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \]

**Case 4.** \(a_4 = 1, a_1 = a_2 = a_3 = 0\), the infinitesimal generator

\[ W_4 = \frac{\partial}{\partial t}. \]

**Case 5.** \(a_1 = a_3 = 1, a_2 = a_4 = 0\), the infinitesimal generator

\[ W_5 = (t + 1) \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \]

**Case 6.** \(a_2 = a_4 = 1, a_1 = a_3 = 0\), the infinitesimal generator

\[ W_6 = \frac{1}{2} \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \]

**Case 7.** \(a_1 = 0, a_2 = a_3 = a_4 = 1\), the infinitesimal generator

\[ W_7 = (t + 1) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial v}. \]

Thus, we have the one-parameter group \(h_i\) of the infinitesimal generators given as follows:

\[ h_1 : (x, t, u, v) \rightarrow (xe^{t/2}, te^{t/2}, ue^{-t/2}, ve^{-t/2} - 1), \]

\[ h_2 : (x, t, u, v) \rightarrow (x, t + e, u, v), \]

\[ h_3 : (x, t, u, v) \rightarrow (x + te, t + u + e, v), \]

\[ h_4 : (x, t, u, v) \rightarrow (x + e, t, u, v), \]

\[ h_5 : (x, t, u, v) \rightarrow (x + (t + 1)e, t + e, u + e, v), \]

\[ h_6 : (x, t, u, v) \rightarrow (x + e, t + e, u, v), \]

\[ h_7 : (x, t, u, v) \rightarrow (x + (t + 1)e, t + e, u + e, v). \]

If \(u = g(x,t)\) and \(v = f(x,t)\) is the solution of system (1.1), then by using above groups, the corresponding new solutions can be given as

\[ u_1 = e^{t/2}g(xe^{-t/2}, te^{-t/2}), \quad v_1 = e^{t/2}f(xe^{-t/2}, te^{-t/2}) + 1, \]

\[ u_2 = g(x, t - e), \quad v_2 = f(x, t - e), \]

\[ u_3 = g(x - te, t - e), \quad v_3 = f(x - te, t - e), \]

\[ u_4 = g(x - e, t), \quad v_4 = f(x - e, t), \]

\[ u_5 = g(x - (t + 1)e, t - e), \quad v_5 = f(x - (t + 1)e, t - e), \]

\[ u_6 = g(x - e, t - e), \quad v_6 = f(x - e, t - e), \]

\[ u_7 = g(x - (t + 1)e, t - e) - e, \quad v_7 = f(x - (t + 1)e, t - e). \]

### 5. Reduction of symmetry and exact solutions of Kaup system

In this section, we will find the reduction equations with the help of similarity variables. From reduction equations, we will obtain similarity solutions.

**Case 1:** For the infinitesimal generator \(W_1 = \frac{1}{2x} \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{1}{2} u \frac{\partial}{\partial u} - (v + 1) \frac{\partial}{\partial v}\), the characteristic equations are given by

\[ \frac{dx}{x} = \frac{dt}{t} = \frac{du}{u} = \frac{dv}{v - 1}. \]

After solving the characteristic equations (5.1), we find the group invariant solutions \(u = \frac{F(X)}{t^2}\) and \(v = \frac{G(X)}{t^2} - 1\) with similarity variable \(X = \frac{x}{\sqrt{t}}\). Now, substituting the value of \(u\) and \(v\) in system (1.1), we find the following reduction equations:

\[ \frac{F''}{3} + 2F' + 2G' - 2FG' - G = 0, \]

\[ 2FF' + 2G' - XF' - F = 0. \]

**Case 2:** For the infinitesimal generator \(W_2 = \frac{\partial}{\partial t}\), the characteristic equations are

\[ \frac{dx}{x} = \frac{dt}{t} = \frac{du}{u} = \frac{dv}{v}. \]

The similarity variable is \(X = x\) and the group invariant solutions are \(u = F(X)\) and \(v = G(X)\). Substituting the value of \(u\) and \(v\) in the system (1.1), we obtain the following reduction equations:

\[ F' + FG' + GF' + \frac{F''}{3} = 0, \]

\[ G' + FF' = 0. \]

(5.3)

Solving the system (5.3) by using the tanh method [20] and then substituting into group invariant solutions, we determined the following similarity solutions of the system (1.1)

\[ u = \sqrt{2} + \frac{\sqrt{2} \tanh(x)}{3}, \]

\[ v = \frac{-1}{2} \left( \sqrt{2} + \frac{\sqrt{2} \tanh(x)}{3} \right)^2. \]

(5.4)
After solving the characteristic equations (5.5), we find the as follows:

group invariant solutions, we obtain the similarity solutions and

Solving Eq. (5.6), we obtain

In the system (1.1), we obtained the following solutions

The characteristic equations are given by

After solving the characteristic equations (5.8), we find the group invariant solutions \( u = F(T) \) and \( v = G(T) \) with similarity variable \( T = t \) and putting the value of \( u \) and \( v \) in system (1.1), we find the following reduction equations:

Therefore, the system (1.1) has a solution \( u = a_1, \ v = a_2 \), where \( a_1 \) and \( a_2 \) are the arbitrary constants.

Case 5: For the infinitesimal generator \( W_5 = (t + 1) \frac{\partial}{\partial t} + \frac{\partial}{\partial u} \), the characteristic equations are given by

Solving the characteristic equations (5.10), we obtain the group invariant solutions \( u = \frac{x + F(T)}{t + 1} \) and \( v = G(T) \) with the similarity variable \( T = t \). Substituting the values of \( u \) and \( v \) in the system (1.1), we determined the reduction equations as follows:

Solving the equations (5.11), we obtain \( F = a_1 \) and \( G = \frac{-t + a_2}{t + 1} \). Therefore, the group invariant solutions are given as

where \( a_1 \) and \( a_2 \) are the arbitrary constants.

Case 6: For the infinitesimal generator \( W_6 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \), the characteristic equations are

Solving the characteristic equations (5.13), we obtain similarity variable \( X = x - t \) and the group invariant solutions are \( u = F(X) \) and \( v = G(X) \). Substituting the values of \( u \) and \( v \) in the system (1.1), we obtain the following reduction equation:

Solving the system (5.14) by using tanh method [20] and then substituting the value of \( \frac{F'''}{3} + FG' + GF' + F' - G' = 0 \) and \( FF' + G' - F' = 0 \) in the group invariant solutions, we obtain the following exact solitary wave solution of the system (1.1):

\[
\begin{align*}
u(x,t) &= 1 + \frac{2tanh(x-t)}{\sqrt{3}}, \\
v(x,t) &= \frac{1}{2} \left( 1 + \frac{4tanh(x-t)}{\sqrt{3}} \right) - \frac{1}{2} \left( 1 + \frac{2tanh(x-t)}{\sqrt{3}} \right)^2.
\end{align*}
\]
Case 7: For the infinitesimal generator \( W_7 = (t+1) \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \), the characteristic equations are given by

\[
\frac{dx}{t+1} = \frac{dt}{1} = \frac{du}{1} = \frac{dv}{0}.
\]  

(5.16)

Solving the characteristic equations (5.16), we obtain the group invariant solutions \( u = t + \frac{a_3}{3} \) and \( v = G(X) \) with similarity variable \( X = x - \frac{(t+1)^2}{2} \). Putting the value of \( u \) and \( v \) in the system (1.1), we determined the reduction equations as follows:

\[
\frac{F'''}{3} + FG' + GF' - G' + F' = 0,
\]

\[
FF' - F' + G' + 1 = 0.
\]  

(5.17)

Solving the equations (5.17), we obtain the solutions of the system (1.1) as follows:

\[ u = t + 1, \quad v = \frac{(t+1)^2}{2} - x + a_1, \]

(5.18)

where \( a_1 \) is the arbitrary constant.

In this section, we apply similarity transformation method to the Eq. (1.2). We take the infinitesimal generator of Kawahara equation as follows:

\[ W = \xi(x,t,u) \frac{\partial}{\partial x} + \tau(x,t,u) \frac{\partial}{\partial t} + \eta(x,t,u) \frac{\partial}{\partial u} \]

where \( \xi(x,t,u) \), \( \tau(x,t,u) \) and \( \eta(x,t,u) \) are the infinitesimals for the transformations for the variables \( x \), \( t \) and \( u \), respectively.

Using the invariance condition given by Eq. (2.4) and applying the seventh-order prolongation of \( S \) to Eq. (1.2), we find the following set of infinitesimals:

\[
\xi_1 = a_2t + a_3, \quad \tau = a_1, \quad \eta = \frac{a_2}{6}, \]

(6.1)

where \( a_1 \), \( a_2 \) and \( a_3 \) are the arbitrary constants. If we take \( a_1 = a_2 = 1 \), and \( a_3 = 0 \), then the characteristic equations corresponding to the infinitesimals (6.1) are as follows:

\[
\frac{dx}{t} = \frac{dt}{1} = \frac{du}{1/6}.
\]  

(6.2)

Solving the characteristic equations (6.2), we obtain the invariant solution \( u = \frac{1}{6} + f(\xi) \), where \( \xi = x - \frac{t}{2} \) is the similarity variable. Substituting this value of \( u \) in Eq. (1.2), we obtain
the following reduced ODE:
\[
\frac{1}{6} + 6f^{(1)} + f^{(3)} - f^{(5)} + \alpha f^{(7)} = 0. \tag{6.3}
\]

Now, we will solve the nonlinear ordinary differential equation (ODE) (6.3) by the power series method [26]. Generally, the exact explicit solutions of nonlinear ODE can not be found out by using integrals and elementary functions. But, to solve nonlinear ordinary differential equations which includes various complicated nonlinear terms with non constant coefficients, the power series method can be used. Now, we seek the solution for the reduced Eq. (6.3) in the form of power series as:
\[
f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{6.4}
\]

From Eqs. (6.3) and (6.4), we have
\[
\frac{1}{6} + 6c_0 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n} \frac{(n-k)}{k!} c_k c_{n+1-k} \xi^n + 6c_3 \right)
+ \sum_{n=1}^{\infty} (n+3)(n+2)(n+1)c_{n+3} \xi^n - 120c_5
- \sum_{n=1}^{\infty} (n+5)(n+4)(n+3)(n+2)(n+1)c_{n+5} \xi^n
+ 5040\alpha c_7 + \sum_{n=1}^{\infty} (n+7)(n+6)(n+5)
(n+4)(n+3)(n+2)(n+1)c_{n+7} \xi^n = 0. \tag{6.5}
\]

Comparing the coefficients for \( n = 0 \) in Eq. (6.5), we obtain
\[
\frac{1}{6} + 6c_0 + 6c_3 - 120c_5 + 5040\alpha c_7 = 0, \tag{6.6}
\]
and for \( n \geq 1 \), we have
\[
c_{n+7} = \frac{1}{G} \left[ -(n+3)(n+2)(n+1)c_{n+3}
+ (n+5)(n+4)(n+3)(n+2)(n+1)c_{n+5}
- \sum_{k=0}^{n} (n+1-k) c_k c_{n+1-k} \right],
\]
\[
n = 1, 2, \ldots. \tag{6.7}
\]

where
\[
G = \alpha(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1).
\]

Thus, for arbitrary chosen value of the constants such as \( c_0 = a, c_1 = b, c_2 = c, c_3 = d, c_4 = e, c_5 = h, c_6 = i \), we have,
\[
c_7 = \frac{120h-6ab-6d-1/6}{5040\alpha} \text{ and the other terms of the sequence } \{c_n\}_{n=8}^{\infty} \text{ can be determined successively from Eq. (6.7) in a unique manner. For Eq. (6.3), there exists a solution given by Eq. (6.4) in the form of power series with the coefficients given by Eqs. (6.6) and (6.7).}

In fact, the solution of Eq. (6.3) in the form of power series can be written as:
\[
f(\xi) = c_0 + c_1 \xi + c_2 \xi^2 + c_3 \xi^3 + c_4 \xi^4 + c_5 \xi^5 + c_6 \xi^6
+ c_7 \xi^7 + \sum_{n=1}^{\infty} c_{n+7} \xi^{n+7}. \tag{6.8}
\]

In view of Eqs. (6.4), (6.6) and (6.7), the solution of Eq. (1.2) in the form of power series is given by
\[
u = \frac{t}{6} + a + b \left(x - \frac{t^2}{2}\right) + c \left(x - \frac{t^2}{2}\right)^2 + d \left(x - \frac{t^2}{2}\right)^3
+ e \left(x - \frac{t^2}{2}\right)^4 + h \left(x - \frac{t^2}{2}\right)^5 + i \left(x - \frac{t^2}{2}\right)^6
+ \left(\frac{120h-6ab-6d-1/6}{5040\alpha}\right) \left(x - \frac{t^2}{2}\right)^7
\]

\[
+ \sum_{n=1}^{\infty} \left[ - (n+3)(n+2)(n+1)c_{n+3}
+ (n+5)(n+4)(n+3)(n+2)(n+1)c_{n+5}
- \sum_{k=0}^{n} (n+1-k) c_k c_{n+1-k} \right] \left(x - \frac{t^2}{2}\right)^{n+7}, \tag{6.9}
\]

where \( c_{n+7} (n = 1, 2, \ldots) \) is given by Eq. (6.7).

Now, we will show that the power series solution given by Eq. (6.9) is convergent by using the implicit function theorem.

Firstly, we will show convergence of Eq. (6.4). For this, from Eq. (6.7), we have
\[
|c_{n+7}| \leq M \left[ |c_{n+3}| + |c_{n+5}| + \sum_{k=0}^{n} |c_k||c_{n+1-k}| \right], \tag{6.10}
\]

where \( M = \max \left\{ \frac{1}{\alpha}, 1 \right\} \). If we define a power series \( \mu = Q(\xi) = \sum_{n=0}^{\infty} q_n \xi^n \) by \( q_0 = |c_0| = |a|, q_1 = |c_1| = |b|, q_2 = |c_2| = |c|, q_3 = |c_3| = |d|, q_4 = |c_4| = |e|, q_5 = |c_5| = |h|, q_6 = |c_6| = |i|, q_7 = |c_7| = |\frac{120h-6ab-6d-1/6}{5040\alpha}| \) and
\[
q_{n+7} = M \left[ q_{n+3} + q_{n+5} + \sum_{k=0}^{n} |q_k|q_{n+1-k} \right], \quad n = 1, 2, \ldots.
\]

Now, it can easily seen that \( |c_n| \leq q_n, \quad n = 0, 1, \ldots \)

In other words, the series \( \mu = Q(\xi) = \sum_{n=0}^{\infty} q_n \xi^n \) is a majorant series of Eq. (6.4). Now, we will show that this series \( \mu = Q(\xi) \) is convergent and has a positive radius of
We can see that $F$ is analytic in the neighborhood of the point $Q_0, q_0$, $Q_0$ is analytic with a positive radius. Thus, in the $\xi$-plane and $F$ is analytic in $(\xi, \mu)$-plane and $F(0, q_0) = 0$, $F'_\mu(0, q_0) = 1 \neq 0$, so by using implicit function theorem, we see that in a neighborhood of the point $(0, q_0)$ of the plane, $\mu = Q(\xi)$ is analytic with a positive radius. Thus, in the neighborhood of the point $(0, q_0)$ of the plane, the power series (6.9) converges. This completes the proof.

### 7. Conclusion

In this work, we have performed Lie symmetry analysis to the Kaup-system (1.1) and Kawahara equation (1.2) and investigated the algebraic structure of the symmetry groups for Kaup-system. After applying similarity transformation method on Kaup-system, we get the reduction equations and after solving reduction equations using tanh method, we have obtained similarity solutions of Eq.(1.1). Moreover the solutions of the reduced equation of Kawahara equation is given in the form of power series (6.9). This is the new solution of the Kawahara equation and it converges rapidly. By the results of this paper, it can be seen that the similarity transformation method is a very powerful, effective and direct method to find exact analytic solutions of nonlinear partial differential equations.

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