Vague positive implicative filter of $BL$-algebras

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Abstract

In this paper, the concept of vague positive implicative filter (VPIF) of $BL$-algebra is introduced. Investigate some important properties of vague positive implicative filter (VPIF) of $BL$-algebras with illustrations. Further, we discuss some equivalent conditions of vague filter (VF) of $BL$-algebras. Finally, we obtain the necessary condition of vague Boolean filter (VBF) is a vague positive implicative filter (VPIF).

Keywords

$BL$-algebra; Filter; Vague set (VS); Vague Filter (VF); Vague Boolean Filter (VBF); Vague positive implicative filter (VPIF).

AMS Subject Classification

03B47, 03G25, 03E70, 03E72.

1 Introduction

L.A. Zadeh [12] introduced the notion of fuzzy set (FS) theory in 1965. The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] in 1986 as an extension of fuzzy set (FS). Hajek [4] introduced the concept of $BL$-algebras as the structures for Basic Logic. Gau and Buehrer [3] proposed the concept of vague set (VS) in 1993, by replacing the value of membership in vague set S is subinterval $[f_S(x), 1 - f_S(x)]$ of $[0, 1]$. The authors [8], [9], and [10] introduced the notions of vague filter (VF), vague prime (VP), Boolean filters (BFs) and vague implicative filter (VIF) of $BL$-algebras and investigate some of their related properties with exemplifications. The aim of this paper, we introduce the definition of vague positive implicative filter (VPIF) of $BL$-algebras, and investigate some important properties with exemplifications.

2 Preliminaries

In this section, we recall some basic knowledge of $BL$-algebras, vague sets and vague filters and their properties which are helpful to develop the main results.

Definition 2.1. [4] The A $BL$-algebra is an algebra $(A, \lor, \land, *, 0, 1)$ of type $(2,2,2,2,0,0)$ such that

(i) $(A, \lor, \land, 0)$ is a bounded lattice,

(ii) $(A, *, 1)$ is a commutative monoid,

(iii) $*$ and $\rightarrow$ form an adjoint pair, that is, $w \leq u \rightarrow v$ if and only if $w * v \leq w$.

(iv) $u \lor v = u * (u \rightarrow v)$,

(v) $(uv) \lor (v \rightarrow u) = 1$ for all $u, v, w \in A$.

Definition 2.2. [4] In a $BL$-algebra $A$, the following properties hold for all $u, v, w \in A$,

(i) $v \rightarrow (u \rightarrow w) = u \rightarrow (v \rightarrow w) = (u * v) \rightarrow w$, 

(ii) $1 \rightarrow u = u, u \rightarrow u = 1, u \rightarrow (v \rightarrow u) = 1, u \rightarrow 1 = 0 \rightarrow u = 1$,

(iii) $u \leq v$ if and only if $u \rightarrow v = 1$,

(iv) $u \lor v = ((u \rightarrow v) \rightarrow v) \land ((v \rightarrow u) \rightarrow u)$.
(v) $u \leq v$ implies $v \mapsto w \leq u \mapsto w$.
(vi) $u \leq v$ implies $w \mapsto u \leq w \mapsto v$.
(vii) $u \mapsto v \leq (w \mapsto u) \mapsto (w \mapsto v)$.
(viii) $u \mapsto v \leq (v \mapsto w) \mapsto (u \mapsto w)$.

**Definition 2.3.** [3] Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. Now we define refined maximum ($\max$) and \( \leq'' \) on elements $D_1 = [p_1,q_1]$ and $D_2 = [p_2,q_2]$ of $D[0,1]$ as $\max(D_1,D_2) = \max\{p_1,p_2\}, \max\{q_1,q_2\}$. Similarly, we can define \( \geq'' \), \( ='' \), and $\min$.

**Definition 2.4.** [8] Let $S$ be a vague set of a BL-algebra $A$ and $\nabla$ a vague filter(VF) of $A$ if it satisfies the following axioms.

(i) $V_S(1) \geq V_S(u)$.
(ii) $V_S(v) \geq \min\{V_S(u \mapsto v), V_S(u)\}$ for all $u,v \in A$.

**Proposition 2.5.** [8] Every VF $S$ of BL-algebra $A$ is order preserving.

**Proposition 2.6.** [8] Let $S$ be a VF of BL-algebra $A$. Let $S$ be a VF of $A$. Then the following hold if for all $u,v,w \in A$.

(i) If $V_S(u \mapsto v) = V_S(1)$ then $V_S(u) \leq V_S(v)$.
(ii) $V_S(u \land v) = \min\{V_S(u), V_S(v)\}$.
(iii) $V_S(u \lor v) = \max\{V_S(u), V_S(v)\}$.
(iv) $V_S(0) = \min\{V_S(u), V_S(u')\}$.
(v) $V_S(u \rightarrow w) \geq \min\{V_S(u \rightarrow v), V_S(v \rightarrow w)\}$.
(vi) $V_S(u \rightarrow w) \leq V_S(u \rightarrow w \rightarrow v \rightarrow w)$.
(vii) $V_S(u \rightarrow v) \leq V_S((v \rightarrow w) \mapsto (u \rightarrow w))$.
(viii) $V_S(u \rightarrow v) \leq V_S((w \mapsto u) \mapsto (w \mapsto v))$.
(ix) $u \mapsto v = v \mapsto u = u' \mapsto v = (u \mapsto v)$.

### 3. Vague positive implicative filter

In this part, we introduce a notion of VPIIF and investigate some related properties with exemplifications.

**Definition 3.1.** Let $S$ be a VF of BL-algebra $A$. $S$ is called a VPIIF, if it satisfies.

(i) $V_S(1) \geq V_S(u)$.
(ii) $V_S(u \rightarrow w) \geq \min\{V_S(u \rightarrow (v \rightarrow w)), V_S(u \rightarrow v)\}$ for all $u,v,w \in A$.

**Example 3.2.** Let $A = \{0, p, q, r, 1\}$. Define $u \land v = \min\{u,v\}$, $u \lor v = \max\{u,v\}$ and \( \leq'' \) and \( \rightarrow'' \) given by the following tables I and II.

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Table I: \( \leq'' \) operator

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Table II: \( \rightarrow'' \) operator

Then $(A, \lor, \land, \leq, \rightarrow, 0, 1)$ is a BL-algebra. Define $V_S$ as follows:

$S = \{(0,[0,0.5]),(p,[0,0.5]),(q,[0,0.5]),(r,[0.4,0.7]),(1,[0.7,0.7])\}$.

It is easily verify that $S$ is a VPIIF of $A$.

**Proposition 3.3.** Every VPIIF is a vague filter.

*Proof.* Let $S$ be a VPIIF of $A$. Then taking $u = 1$ in (ii) of definition 3.1, we have $V_S(1 \mapsto w) \geq \min\{V_S(1 \mapsto (v \mapsto w)), V_S(1 \mapsto v)\}$ for all $u,v,w \in A$.

$V_S(w) \geq \min\{V_S((v \mapsto w), V_S(v)\}$

From (ii) proposition 2.6 and (i) of definition 3.1 is exist. Thus, $S$ be a VF of $A$.

\[\square\]

Converse of the proposition 3.3 may not be true. We prove this by the example as shown below.

**Example 3.4.** Let $A = \{0, p, q, r, 1\}$. Define $u \land v = \min\{u,v\}$, $u \lor v = \max\{u,v\}$ and \( \leq'' \) and \( \rightarrow'' \) given by the following tables III and IV.

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Table III: \( \leq'' \) operator

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Table IV: \( \rightarrow'' \) operator
Then, \((A, \vee, \wedge, *, \rightarrow, 0, 1)\) is a BL-algebra. Define a VSS of \(A\) as follows:
\[
S = \{(0, [0.1, 0.2]), (p, [0.3, 0.4]), (q, [0.3, 0.4]), (1, [0.5, 0.9])\}
\] .
It is easily verify that \(S\) is a VF, but \(S\) is not a VPIF of \(A\).
Since
\[
V_S(u \rightarrow p) = V_S(q) = [0.3, 0.4] < \text{rmin}\{V_S(q \rightarrow (q \rightarrow p)), V_S(q \rightarrow q)\} = V_S(1) = [0.5, 0.9].
\]
Next, we obtain some characteristics of VPIFs as follows.

**Proposition 3.5.** Let \(S\) be VF of \(A\). The following are equivalent for all \(u, v, w \in A\).

(i) \(S\) is a VPIF.

(ii) \(V_S(u \rightarrow v) \geq V_S(u \rightarrow (u \rightarrow v))\).

(iii) \(V_S(u \rightarrow v) = V_S(u \rightarrow (u \rightarrow v))\).

(iv) \(V_S(u \rightarrow (v \rightarrow w)) \leq V_S((u \rightarrow v) \rightarrow (u \rightarrow w))\).

(v) \(V_S((u \rightarrow v) \rightarrow w) = V_S((u \land v) \rightarrow w)\).

Proof. (i) \(\Rightarrow\) (ii)

Let \(S\) be a VPIF of \(A\).

Then from (ii) definition 3.1, we have
\[
V_S(u \rightarrow w) \geq \text{rmin}\{V_S(u \rightarrow (v \rightarrow w)), V_S(u \rightarrow v)\} \quad (3.1)
\]
for all \(u, v, w \in A\).

Put \(w = v\) and \(v = u\) in (3.1), we get
\[
V_S(u \rightarrow v) \geq \text{rmin}\{V_S(u \rightarrow (u \rightarrow v)), V_S(u \rightarrow u)\}
\]
[From (ii) of Proposition 2.5]
\[
= \text{rmin}\{V_S(u \rightarrow (u \rightarrow v)), V_S(1)\}
\]
[From (ii) of Proposition 2.5]
\[
= V_S(u \rightarrow (u \rightarrow v))
\]
[From Definition 2.2]

Thus, we have
\[
V_S(u \rightarrow v) \geq V_S(u \rightarrow (u \rightarrow v)).
\]

(ii) \(\Rightarrow\) (iii)

Since \(u \rightarrow v \leq u \rightarrow (u \rightarrow v)\), from proposition 2.6, we have
\[
V_S(u \rightarrow v) \leq V_S((u \rightarrow u) \rightarrow (u \rightarrow v))
\]
for all \(u, v \in A\), and from (ii), we get
\[
V_S(u \rightarrow v) = V_S(u \rightarrow (u \rightarrow v)).
\]
Thus, we have,
\[ V_S(u \mapsto w) \geq \text{rmin}\{V_S(u \mapsto v), V_S(u \mapsto v)\}. \]
Hence, \( S \) is a VPIF of \( A \).

**Proposition 3.6.** Let \( S \) be a VF of \( A \). Then \( S \) is a VPIFA if and only if \( V_S(v) \geq \text{rmin}\{V_S((v \mapsto w) \mapsto (u \mapsto v)), V_S(u)\} \) for all \( u,v,w \in A \). 

**Proof.** Let \( S \) be a VPIFA. Then, from (i) of proposition 2.5, we have,
\[ \text{rmin}\{V_S((v \mapsto w) \mapsto (u \mapsto v)), V_S(u)\} = \text{rmin}\{V_S(u \mapsto ((v \mapsto w) \mapsto v))), V_S(u)\} \leq V_S((v \mapsto w) \mapsto v)). \]
From (vii) of proposition 2.5, we have
\[ (v \mapsto w) \mapsto v \leq w \mapsto v \leq (v \mapsto w) \mapsto ((w \mapsto v) \mapsto v). \]

Then, from proposition 2.6, we have
\[ V_S((v \mapsto w) \mapsto v) \leq V_S(w \mapsto v) \leq V_S((v \mapsto w) \mapsto ((w \mapsto v) \mapsto v)) \leq V_S((w \mapsto v) \mapsto v). \]

Thus,
\[ \text{rmin}\{V_S((v \mapsto w) \mapsto v), V_S(w \mapsto v)\} \leq \text{rmin}\{V_S((v \mapsto w) \mapsto v), V_S(w \mapsto v)\} \leq V_S(v). \]

Therefore, we have
\[ V_S(v) \geq \text{rmin}\{V_S((v \mapsto w) \mapsto (u \mapsto v)), V_S(u)\}. \]

Conversely, let \( S \) satisfies
\[ V_S(v) \geq \text{rmin}\{V_S((v \mapsto w) \mapsto (u \mapsto v)), V_S(u)\}. \]
Then we easily prove that,
\[ V_S(u \mapsto w) \geq \text{rmin}\{V_S(u \mapsto (v \mapsto w)), V_S(u \mapsto v)\}. \]
Since \( S \) is VF,
\[ V_S(1) \geq V_S(u). \]
Hence, \( S \) is VPIF of \( A \).

**Proposition 3.7.** Let \( S_1 \) and \( S_2 \) be two VFs, of \( S_1 \subseteq S_2, V_{S_1}(1) = V_{S_2}(1). \) If \( S_1 \) is a VPIF, so is \( S_2 \).

**Proof.** From the proposition 3.5, we only prove that \( V_{S_2}(u \mapsto w) \geq V_{S_2}(u \mapsto (u \mapsto w)) \) for all \( u,w \in A \).

Let \( t = u \mapsto (u \mapsto w) \), then
\[ u \mapsto (u \mapsto (t \mapsto w)) = t \mapsto (u \mapsto (u \mapsto w)) = t \mapsto t = 1. \]
If \( S_1 \) is a VPIF, and from (iii) of the proposition 3.5, then
\[ V_{S_1}(u \mapsto (t \mapsto w)) = V_{S_1}(u \mapsto (u \mapsto (t \mapsto w))) = V_{S_1}(1). \]
That is
\[ V_{S_1}(t \mapsto (u \mapsto w)) = V_{S_1}(1) = V_{S_2}(1). \]
From \( S_1 \subseteq S_2 \), we get
\[ V_{S_2}(t \mapsto (u \mapsto w)) \geq V_{S_1}(t \mapsto (u \mapsto w)) = V_{S_2}(1), \]
from (i) of definition 2.3, we have,
\[ V_{S_2}(t \mapsto (u \mapsto w)) = V_{S_2}(1). \]
Since \( S_2 \) is a VF,
\[ V_{S_2}(u \mapsto w) \geq \text{rmin}\{V_{S_2}(t \mapsto (u \mapsto w)), V_{S_2}(t)\} \]
Thus
\[ V_{S_2}(u \mapsto w) \geq \text{rmin}V_{S_2}(1), V_{S_2}(t) = V_{S_2}(u \mapsto (u \mapsto w)). \]
Hence, \( S_2 \) is a VPIF.

**Proposition 3.8.** Every VBF is a VPIF, the converse may not be true.

**Proof.** Let \( S \) be a VBF. Then
\[ V_S(u \mapsto w) \geq \text{rmin}\{V_S((u \lor u^c) \mapsto (u \mapsto w)), V_S(u \lor u^c)\} = \text{rmin}\{V_S((u \lor u^c) \mapsto (u \mapsto w)), V_S(1)\} \]
Since
\[ (u \lor u^c) \mapsto (u \mapsto w) = (u \mapsto (u \mapsto w)) \lor (u^c \mapsto (u \mapsto w)) = u \mapsto (u \mapsto w), \]
and from the proposition 2.5, we have
\[ V_S((u \lor u^c) \mapsto (u \mapsto w)) = V_S(u \mapsto (u \mapsto w)). \]
Thus, we have
\[ V_S(u \mapsto w) \geq V_S(u \mapsto (u \mapsto w)). \]
We consider proposition 3.5, we get \( S \) is a VPIF.
We prove converse is not true from the following example.

Example 3.9. We consider the example 3.2. S is a VPIF, but S is not a VBF, since $V_S(q \lor q) = V_S(q) \neq V_S(1)$.

Proposition 3.10. Let $S$ be a VPIF of $A$. $S$ is a VBF if and only if

$V_S((u \rightarrow v) \rightarrow v) = V_S((v \rightarrow u) \rightarrow u)$ for all $u, v \in A$.

Proof. Let $S$ be a VPIF of $A$. We know that

$u = 1 \rightarrow u \leq (v \rightarrow u) \rightarrow u$

and

$v \leq (v \rightarrow u) \rightarrow u$,

it follows that

$((v \rightarrow u) \rightarrow u)^- \leq u^- \leq u \rightarrow v$

and

$(u \rightarrow v) \rightarrow v \leq ((v \rightarrow u) \rightarrow u) \rightarrow v$

$\leq ((v \rightarrow u) \rightarrow u)^- \rightarrow ((v \rightarrow u) \rightarrow u)$.

Then, we have

$V_S((v \rightarrow u) \rightarrow u)^- \rightarrow ((v \rightarrow u) \rightarrow u) \geq V_S((u \rightarrow v) \rightarrow v)$.

Since $S$ is a VBF, from $V_S(u) = V_S(u^- \rightarrow u)$, and (i) of proposition 2.6, we get

$V_S((v \rightarrow u) \rightarrow u) = V_S(((v \rightarrow u) \rightarrow u)^- \rightarrow ((v \rightarrow u) \rightarrow u))$

Thus, we have

$V_S((v \rightarrow u) \rightarrow u) \geq V_S((u \rightarrow v) \rightarrow v) \tag{3.2}$

Same method to prove

$V_S((v \rightarrow u) \rightarrow u) \leq V_S((u \rightarrow v) \rightarrow v) \tag{3.3}$

From (3.2) and (3.3), we get

$V_S((u \rightarrow v) \rightarrow v) = V_S((v \rightarrow u) \rightarrow u)$

Conversely, if $S$ be a VPIF of $A$, and satisfies $V_S((u \rightarrow v) \rightarrow v) = V_S((v \rightarrow u) \rightarrow u)$.

Replace $v$ by $u^-$, we have

$V_S((u \rightarrow u^-) \rightarrow u^-) = V_S((u^- \rightarrow u) \rightarrow u)$.

Then, we get

$V_S(u \lor u^-) = V_S((u \rightarrow u^-) \rightarrow u^-) \tag{3.4}$

To Prove: $S$ is a VBF.

It is enough to prove $V_S((u \rightarrow u^-) \rightarrow u^-) = V_S(1)$. Since $S$ is a VPIF, from (v) of proposition 3.5, we have

$V_S((u \rightarrow u^-) \rightarrow u^-) = V_S((u \rightarrow u^-) \rightarrow (u \rightarrow 0))$

$V_S((u \rightarrow (u^- \rightarrow 0))$

$V_S((u \rightarrow u^-) \rightarrow V_S(1) \tag{3.5}$

From (3.4) and (3.5), we get

$V_S(u \lor u^-) = V_S(1)$.

Thus, $S$ is a VBF.

4. Conclusion

In the present paper, we have introduced the notion of a VPIF of $BL$-algebra, and investigate some related properties. Moreover, we have obtained some necessary and sufficient condition between VPIF and BF of $BL$-algebra.

References


