On certain geometric properties of generalized polylogarithm function

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Abstract
In this manuscript, we investigate the Hadamard product \( H_f(a, b; z) \) of normalized analytic functions in the unit disc \( \Delta \) and generalized second order polylogarithm function \( G(a, b; z) \), where
\[
G(a, b; z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n, \quad a, b \in \mathbb{C} \setminus \{-1, -2, \ldots\}.
\]

Further, we derive certain characteristics of the function \( H_f(a, b; z) \) and obtain various sufficient conditions for the function \( H_f(a, b; z) \) to be Janowski starlike. Also certain inequalities containing the function \( H_f(a, b; z) \) are obtained.

Keywords
Analytic functions, Convolution, Subordination, Generalized polylogarithm function.

AMS Subject Classification
30C45, 30C80.

1 Introduction

Let \( \mathcal{H} \) signifies the class of analytic functions in the unit disc \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{A}_n \) signifies the class of analytic functions in \( \Delta \) of the form
\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (1.1)
\]
and denote \( \mathcal{A} := \mathcal{A}_1 \).

Let \( f \) and \( g \) be analytic in \( \Delta \), then we say that \( f \) is subordinate to \( g \) in \( \Delta \) (written \( f \prec g \)) if there exists a Schwarz function \( w(z) \), analytic in \( \Delta \) with
\[
w(0) = 0, \quad |w(z)| < 1 \quad (z \in \Delta)
\]
in a way that
\[
f(z) = g(w(z)), \quad (z \in \Delta).
\]

Particularly, if the function \( g \) is univalent in \( \Delta \), then the subordination is similar to
\[
f(0) = g(0) \quad \text{or} \quad f(\Delta) \subset g(\Delta).
\]

Let
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]
be the Maclaurin series, the Hadamard product of \( f \) and \( g \) is defined by the power series
\[
(f \ast g)z = f(z) \ast g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]
The polylogarithm function [4] is defined as the analytic continuation of the Dirichlet series,
\[ L_i(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^i} \quad (z \in \Delta, s \in \mathbb{C}). \]

Ramanujan [1] derived several properties of dilogarithms \( L_2(z) \) and trilogarithms \( L_3(z) \). Ponnusamy and Sabapathy [8] obtained certain geometric properties of generalized polylogarithms and determined the conditions on the parameters for the function to be univalent and starlike.

Let
\[ G(a, b; z) = \sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(n+a)(n+b)} z^n, \quad a, b \in \{-1, -2, \ldots\}, \quad (1.2) \]
be the generalized second order polylogarithm function, which significantly reduces to the Lerch function of order 2, for \( a = b \). For the values \( a = b = 0 \), it reduces to dilogarithm function [2] and to the identity function for \( a = -1 \) and \( b = -1 \).

In the convolution structure \( H_f(a, b; z) = G(a, b; z) \ast f(z) \), for \( a \neq b \), we get the following integral representation
\[ H_f(a, b; z) = \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1-t)^{b-a} f(tz) dt. \quad (1.3) \]

It should be remarked that the convolution structure (1.3) is the generalization of many well known operators.

1. For \( a = -\alpha \) and \( b = 2 - \alpha \), the function (1.3) reduces to the operator
\[ I_\alpha(z) = \frac{(1-\alpha)(3-\alpha)}{2} \int_0^1 t^{-(\alpha+1)}(1-t^\alpha) f(tz) dt \]
with \( 0 \leq \alpha < 1 \).

2. For the limiting case \( b \to \infty \), the function \( H_f(a, b; z) \) represents the Bernardi transform [8, 11]
\[ B_f(a, z) = \frac{a+1}{a} \int_0^z t^{a-1} f(t) dt. \]

Now, it can easily be verified that the function \( H_f(a, b; z) \) satisfies the differential equation
\[ z^2 H'_f(a, b; z) + (a+b+1)z H_f(a, b; z) + ab H_f(a, b; z) = (a+1)(b+1)f(z). \quad (1.4) \]

S. Ponnusamy [9] considered the differential equation (1.4) and discussed certain geometric properties of \( H_f(a, b; z) \) that depend on the parameters \( a \) and \( b \).

Let \( S \) denote the subclass of \( \mathcal{A} \) consisting of univalent functions. For \(-1 \leq B < A \leq 1 \), let \( P[A, B] \) denote the class consisting of normalized analytic functions \( p(z) \) satisfying \( p(0) = 1 \) and \( p'(0) = 0 \) in ways that
\[ p(z) < \frac{1+Az}{1+Bz}, \]
and its subclass \( S[A, B] \) the class of Janowski starlike functions is defined by
\[ S[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz} \right\}. \]

For suitable choices of the parameters \( A \) and \( B \), certain well known subclasses are obtained as special cases of the class \( S[A, B] \). For \( 0 \leq \alpha < 1 \), \( S[1-2\alpha, -1] = S'(\alpha) \) is the familiar class of starlike functions of order \( \alpha \), the class \( S'[1-\alpha, 0] = \left\{ f \in \mathcal{A} : |\frac{zf'(z)}{f(z)} - 1| < (1-\alpha) \right\} \) is denoted by \( S'_1(\alpha) \) and \( S' = S'(0) \) denotes the class of starlike functions.

Recently many authors have investigated the sufficient conditions for functions to belong to \( S[A, B] \) and to various subclasses of \( S[A, B] \). The Janowski starlikeness of Bessel function and Kummer hypergeometric function are studied by Ravichandran et.al. [13]. Tuneski [14] obtained sufficient condition for a function to be Janowski starlike with respect to \( N \)-symmetric points.

Inspired by the aforementioned works, we obtain various sufficient conditions for the univalence and Janowski starlikeness of the function \( H_f(a, b; z) \) and derive certain inequalities involving \( H_f(a, b; z) \).

**Lemma 1.1.** [6] If an analytic function \( f \) has the form \( f(z) = z + a_2z^2 + a_3z^3 + \cdots, (z \in \Delta) \) and satisfies the condition
\[ \frac{zf'(z)}{f(z)} - 1 < 1 \]
then \( f \) is univalent in \( \Delta \).

**Lemma 1.2.** [5] If \( f \in \mathcal{A} \) satisfies \( \left| \frac{f(z)}{z} - 1 \right| < 1, (z \in \Delta) \), then \( f(z) \) is univalent and starlike for \( |z| < 1/2 \).

## 2. Main results

**Theorem 2.1.** Let \( f \in \mathcal{A}, a, b \in \mathbb{C} \setminus \{-1, -2, \ldots\} \) and \(-1 \leq B < A \leq 1 \). If
\[ (1+A)^2(1+B)^2 + (1+A)^2(1+B)^3Re(a+b) + (1+B)^4Re(F(z)) > 0 \]
and
\[ \left\{ \begin{aligned} (1+A)^3(A-B) + (1+A)(1+B)Re(a+b) + (1+B)^2Re(F(z)) \\ 2(A-B)+(1-A^2)+(1-A)(1-B)Re(a+b) + (1-B)^2Re(F(z)) \end{aligned} \right\} \]
\[ - \left\{ \begin{aligned} (1-A^2)+(1-AB)Im(a+b)+(1-B^2)Im(F(z))^2 \end{aligned} \right\} > 0, \]
then \( f \) is starlike of order \( \alpha \) in \( \mathbb{D} \).
where \( F(z) = \left\{ ab + \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)} \right\} \), then
\[ H_f(a,b;z) \in S^+[A,B]. \]

Proof. Let
\[ q(z) = zH_f'(a,b;z) \]
Define the analytic functions
\[ p(z) = \frac{1-(A+B)}{1+A}q(z), \]
then we have
\[ q(z) = \frac{zH_f'(a,b;z)}{H_f(a,b;z)} = \frac{(1-A)+(1+A)p(z)}{(1-B)+(1+B)p(z)}, \]
\[ q'(z) = \frac{2(A-B)p'(z)}{((1-B)+(1+B)p(z))^2}, \]
and
\[ q''(z) = \frac{2(A-B)(1-B)+(1+B)p(z))p''(z) - 4(1+B)(A-B)p'(z)^2}{((1-B)+(1+B)p(z))^3}. \]
From (1.4), we obtain
\[ zq'(z) + q^2(z) + (a+b)q(z) + ab \]
\[ = (a+1)(b+1) \frac{f(z)}{H_f(a,b;z)}. \] (2.4)
Using (2.3) and simplifying (2.4) further, we obtain,
\[ C(z)p^2(z) + D(z)p(z) + 2(A-B)zp'(z) + E(z) = 0, \] (2.5)
where
\[ C(z) = (1-A)^2 + (a+b)(1+A)(1+B) \]
\[ + ab(1+B)^2 + (a+1)(b+1) \frac{f(z)}{H_f(a,b;z)}(1+B)^2, \]
\[ D(z) = 2(1-A)^2 + 2(a+b)(1-AB) \]
\[ + 2ab(1-B)^2 + 2(a+1)(1-B^2) \frac{f(z)}{H_f(a,b;z)}, \]
and
\[ E(z) = (1-A)^2 + (a+b)(1-A)(1-B) \]
\[ + ab(1-B)^2 + (a+1)(b+1) \frac{f(z)}{H_f(a,b;z)}(1-B)^2. \]
Define \( \psi(a,s;z) \) such that,
\[ \psi(a,s;z) = C(z)p^2(z) + D(z)p(z) + 2(A-B)zp'(z) + E(z). \] (2.6)
By letting \( \Omega = \{0\} \), from (2.5), we get
\[ \psi(p(z),zp'(z);z) \in \Omega. \]

Now,
\[ Re \psi(ip,\sigma;z) \]
\[ = Re(C(z)(ip)^2 + D(z)ip + 2(A-B)(z)\sigma + E(z)) \]
\[ \leq -Re C(z)p^2 - Im D(z)p + 2(A-B) - \left( \frac{1+p^2}{2} \right) \]
\[ + Re J(z) \]
\[ = - \left[ Re C(z) + \frac{2(A-B)}{2} \right] p^2 - Im D(z)p - 2(A-B) \]
\[ + Re E(z) \]
\[ = Rp^2 + Sp + T = G(p), \]
where
\[ R = -(Re C(z) + (A-B)), S = -Im D(z), \]
\[ T = -2(A-B) + Re E(z). \]
Now, we observe that
\[ \max_{\rho \in \mathbb{R}} G(p) = \frac{4RT - S^2}{4R}, \quad (R < 0). \]

From (2.1) and (2.2), we have \( Re \psi(ip,\sigma;z) < 0. \)
Therefore by the result [11](pg.35), \( Re p(z) > 0, \)
that is,
\[ - \frac{(1-A)-(1-B)q(z)}{(1+A)-(1+B)q(z)} < \frac{1+z}{1-z}. \]
Hence for an analytic function \( g \) in \( \Delta \) with \( g(0) = 0 \) such that
\[ - \frac{(1-A)-(1-B)q(z)}{(1+A)-(1+B)q(z)} = \frac{1+g(z)}{1-g(z)}. \]
Hence
\[ q(z) < \frac{1+Az}{1+Bz}. \]
In particular
\[ \frac{zH_f'(a,b;z)}{H_f(a,b;z)} \in P[A,B], \]
that is
\[ H_f(a,b;z) \in S^+[A,B]. \]

The following theorem gives the sufficient condition that rely on the parameters \( a \) and \( b \), for the function \( H_f(a,b;z) \) to be in the subclass \( S_1^+[\alpha] \).

**Theorem 2.2.** If \( a,b \in C \setminus \{-1,-2,\ldots\} \), with \( 2|a+1||b+1| < (1-\alpha)(\alpha+ab+2) \) then \( H_f(a,b;z) \in S_1^+[\alpha]. \)
Proof. Let
\[ p(z) = \frac{zH'(a,b,z)}{H_f(a,b;z)} - 1. \tag{2.7} \]
From (1.4), we have
\[ zp'(z) + p^2(z) + p(z)(a+b+2) + (a+b+ab+1) \]
\[ - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)} = 0. \]
That is
\[ \psi(p(z),zp'(z);z) = 0, \]
where
\[ \psi(r,s,t) = r(r + (a+b+2)) + s + a + b + ab + 1 \]
\[ - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}. \]
Now, we claim that
\[ |p(z)| < (1 - \alpha), \quad 0 \leq \alpha < 1. \]
From the result [11](Pg.34), it is sufficient to show that,
\[ \psi((1 - \alpha)e^{i\theta},Ke^{i\theta};z) \not\in \Omega, \]
where \( \theta \) is real, \( K \geq (1 - \alpha) \), and \( z \in \Delta \), for \( \Omega = \{0\} \), \( n = 1 \) and \( g(z) = (1 - \alpha)z \).
Now,
\[ \psi(1 - \alpha)e^{i\theta}, Ke^{i\theta};z) = Ce^{i\theta} - D, \tag{2.8} \]
where
\[ C = (1 - \alpha)[(1 - \alpha)e^{i\theta} + (a + b + 2)] + (1 - \alpha) \]
and
\[ D = -(a + b + ab + 1) + \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}. \]
Also
\[ |C| \geq Re C > (1 - \alpha)[(1 - \alpha)e^{i\theta} + (a + b + 2)] + (1 - \alpha) \]
\[ \geq (1 - \alpha)[(1 - \alpha) \cos \theta + (a + b + 2)] + (1 - \alpha) \]
\[ \geq (1 - \alpha)(a + a + b + 2) = \beta, \text{ (say)}, \tag{2.9} \]
and
\[ |D| \leq |(a+1)(b+1) - \frac{(a+1)(b+1)f(z)}{H_f(a,b;z)}| \]
\[ \leq |(a+1)(b+1)| - \frac{|(a+1)(b+1)f(z)|}{H_f(a,b;z)} \]
\[ \leq 2|a+1)(b+1)| \beta, \]
where \( \beta \) is as given in (2.9).
Then we have,
\[ |\psi((1 - \alpha)e^{i\theta},Ke^{i\theta};z)| \leq |e^{i\theta}C - D| \]
\[ \geq |C| - |D| \]
\[ > \beta - \beta = 0. \]
Therefore,
\[ \psi((1 - \alpha)e^{i\theta},Ke^{i\theta};z) \not\in \Omega \]
Thus, \( |p(z)| < (1 - \alpha) \in \Delta \) and hence \( H_f(a,b;z) \in S^*_1(\alpha). \]

The proof of next theorem follows from the definitions of the function \( H_f(a,b;z) \) and the class \( S^*_1(\alpha) \), hence we give the statement alone.

Theorem 2.3. Let \( g : \Delta \to \mathbb{C} \) be defined by
\[ g(z) = \frac{H_f(a,b;z)}{z}, \]
such that
\[ \left| \frac{z^p(z)}{z^{1/2}} \right| < (1 - \alpha), \text{ for } \alpha \in [0,1/2] \text{ and } z \in \Delta, \text{ then } \]
\[ H_f(a,b;z) \in S^*_1(\alpha) \]

3. Univalence and Starlikeness of \( H_f(a,b;z) \)

Theorem 3.1. Let \( a,b \in \mathbb{C} \setminus \{-1,-2,\ldots\} \) such that
\[ |(a+1)(b+1)| \left| \frac{f(z)}{z} - 1 \right| < Re[(a+2)(b+2)] \tag{3.1} \]
then \( H_f(a,b;z) \) is univalent and starlike for \( |z| < 1/2. \)
Proof. Let
\[ p(z) = \frac{H_f(a,b;z)}{z} - 1, \]
then \( p(z) \) is an analytic function with \( p(0) = 0. \)

Using the differential equation (1.4), we obtain the equation,
\[ z^2 p''(z) + zp'(z)(a + b + 3) + p(z)(a + 1)(b + 1) \]
\[ + (a + 1)(b + 1) - (a + 1)(b + 1) \frac{f(z)}{z} = 0. \tag{3.2} \]
If we consider
\[ \psi(r,s,t) = r + (a + b + 3)s + (a + 1)(b + 1)t \]
\[ -(a + 1)(b + 1) \left( \frac{f(z) - 1}{z} \right) \text{ and } \Omega = \{0\}, \]
then (3.2) implies
\[ \psi(p(z),zp'(z),z^2 p''(z);z) \in \Omega \text{ for all } z \in \Delta. \]
Further, for any \( \theta \in \mathbb{R} \), \( K \geq M \), \( \text{Re}(Le^{-i\theta}) \geq 0 \) and
\[
M = \frac{|(a+1)(b+1)(f(z) - 1)|}{\text{Re}(a+2)(b+2)}, \quad \text{we have}
\]
\[
|\psi(Me^{i\theta}, Ke^{i\theta}, L; z)|
= |L + (a+b+3)Ke^{i\theta} + (a+1)(b+1)Me^{i\theta} - (a+1)(b+1)K(z)|
\geq \text{Re}(a+b+3)M + \text{Re}(a+1)(b+1)M
- |(a+1)(b+1)K(z)|
= \text{Re}[(a+b+3 + (a+1)(b+1)]M
- |(a+1)(b+1)K(z)|
= \text{Re}[(a+2)(b+2)M - |(a+1)(b+1)K(z)|
= 0.
\]

That is, \( \psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega \).

Now by applying the result [11] (Pg. 34), we get
\[
|p(z)| < M, \quad |z| < 1/2.
\]

The assertion of the theorem follows by using (3.1) and applying Lemma 1.2.

**Theorem 3.2.** For \( a, b \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\} \), if \( H_f(a, b; z) \) satisfies any one of the following conditions for all \( z \in \Delta \):
\[
\frac{z^2H'_f(a, b; z)}{H_f(a, b; z)} \left( \frac{[H_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} \right) < 1,
\]  
then \( H_f(a, b; z) \) is univalent.

**Proof.** Let
\[
\frac{z^2H'_f(a, b; z)}{H_f(a, b; z)} - 1 = h(z), \tag{3.7}
\]
then \( h(z) \) is analytic in \( \Delta \) and \( h(0) = 0 \).

Further differentiating (3.7), we obtain
\[
\frac{[zH_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} = \frac{zh'(z)}{1 + h(z)}. \tag{3.8}
\]

Hence, from (3.7) and (3.8), we have
\[
K_1(z) = \frac{z^2H'_f(a, b; z)}{H_f(a, b; z)} \left( \frac{[H_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} \right) - 1 = \frac{zh'(z)}{1 + h(z)}. \tag{3.9}
\]
\[
K_2(z) = \frac{[H_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} - 1 = \frac{zh'(z)}{1 + h(z)}. \tag{3.10}
\]
\[
K_3(z) = \left[ \frac{[H_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} - 1 \right] = \frac{zh'(z)}{1 + h(z)}.
\]
\[
K_4(z) = \left[ \frac{[H_f(a, b; z)]'' - 2zH'_f(a, b; z)}{H_f(a, b; z)} - 1 \right] = \frac{zh'(z)}{h(z)}.
\]

Now, suppose that there exist \( z_0 \in \Delta \) such that
\[
\max_{|z| < |z_0|} |h(z)| = |h(z_0)| = 1,
\]
then by Jack’s Lemma [3], we have
\[
z_0h'(z_0) = \delta h(z_0),
\]
where \( \delta \in \mathbb{R} \) and \( \delta \geq 1 \). Therefore by letting \( h(z_0) = e^{i\theta} \) in each equation of (3.9), we obtain that
\[
|K_1(z_0)| = |z_0h'(z_0)| = |\delta h(z_0)| = |\delta e^{i\theta}| \geq 1 \tag{3.11}
\]
\[
|K_2(z_0)| = \left| \frac{z_0h'(z_0)}{(1 + h(z_0))^2} \right| = \frac{\delta h(z_0)}{(1 + h(z_0))^2}
\geq \frac{\delta}{1 + e^{i\theta}} \geq 1/4 \tag{3.12}
\]
\[
|K_3(z_0)| = \left| \frac{z_0h'(z_0)}{h(z_0) + 1 + h(z_0)} \right| = \frac{\delta h(z_0)}{h(z_0) + 1 + h(z_0)} \geq 1/2 \tag{3.13}
\]
\[
|K_4(z_0)| = \left| \frac{z_0h'(z_0)}{h(z_0)} \right| = \frac{\delta h(z_0)}{h(z_0)} \geq 1
\]
which contradicts our assumption (3.3) to (3.6) respectively and hence \( |h(z)| < 1 \) for all \( z \in \Delta \). Therefore, we obtain
\[
\frac{z^2H'_f(a, b; z)}{H_f(a, b; z)} - 1 = |h(z)| < 1
\]
which implies \( H_f(a, b; z) \) is univalent, by Lemma 1.1.

\[\square\]
Let \( c > 0, d \geq 0 \) such that \( c + 2d \leq 1 \). If \( H_f(a, b; z) \) satisfies
\[
\text{Re} \left\{ \frac{(zH_f(a, b; z))^n}{(H_f(a, b; z))^2} - \frac{2zH_f(a, b; z)}{H_f(a, b; z)} \right\} < \frac{c + d}{(1 + c)(1 - d)}, \quad z \in \Delta
\]
then \( H_f(a, b; z) \) is univalent in \( \Delta \).

**Proof.** Let
\[
\frac{z^2H_f'(a, b; z)}{(H_f(a, b; z))^2} = \frac{1 + ah(z)}{1 - bh(z)} (z \in \Delta). \tag{3.14}
\]
then \( h(z) \) is analytic in \( \Delta \) and \( h(0) = 0 \).

Differentiating (3.14), we get
\[
\frac{(zH_f(a, b; z))^n}{(H_f(a, b; z))^2} - \frac{2zH_f(a, b; z)}{H_f(a, b; z)} = \frac{c + d|z|h'(z)}{(1 + ch(z))(1 - dh(z))} = \mathcal{T}(z), \text{ (say)}. \tag{3.15}
\]
If there exist \( z_0 \in \Delta \) such that
\[
\max_{|z| < |z_0|} |h(z)| = |h(z_0)| = 1.
\]
Then from Lemma due to Jack [3] we have \( z_0 h'(z_0) = \delta h(z_0) \) and
\[
\text{Re} \left\{ 1 + \frac{z_0 h'(z_0)}{h'(z_0)} \right\} > \delta
\]
Now letting \( h(z_0) = e^{i\theta} , (\theta \in [0, 2\pi]) \) in (3.15), we have
\[
\text{Re} \{ \mathcal{T}(z_0) \} = \delta (c + d) \text{Re} \left\{ \frac{h(z_0)}{(1 + ch(z_0))(1 - dh(z_0))} \right\}
\]
\[
= \delta \text{Re} \left\{ \frac{1}{1 - dh(z_0)} - \frac{1}{1 + ch(z_0)} \right\}
\]
\[
= \delta \text{Re} \left\{ \frac{1}{1 - e^{i\theta} - 2cd \cos \theta} - \frac{1}{1 + e^{-i\theta} + 2ce^{-i\theta}} \right\}
\]
\[
= \delta \left\{ \frac{1}{2 + \frac{d^2-1}{1-\cos \theta}} - \frac{1}{2 + \frac{c^2-1}{1+\cos \theta}} \right\}
\]
where \( \theta \neq \cos^{-1}(-1/c) \) and \( \theta \neq \cos^{-1}(1/d) \) we have
\[
\text{Re} \{ \mathcal{T}(z_0) \} > \frac{c + d}{(1 + c)(1 - d)}. \]
This contradicts the inequality given in the hypothesis and therefore \( |h(z)| < 1 \) for all \( z \in \Delta \).

Thus, we have
\[
\frac{z^2H_f'(a, b; z)}{(H_f(a, b; z))^2} - 1 = \left| \frac{(c + d)h(z)}{1 - dh(z)} \right| < \frac{c + d}{1 - d} \leq 1, \quad z \in \Delta.
\]
In view of Lemma 1.1, it implies that \( H_f(a, b; z) \) is univalent \( \square \)