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Fixed point theorems for mappings satisfying implicit relations in multiplicative metric spaces

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Abstract
Fixed point theorems in metric spaces involving implicit relations were introduced by Popa. We modify such a theorem by Berinde so that it applies to pairs of self mappings in multiplicative metric spaces. An illustrative example is given on the use of the theorem.

Keywords
Implicit relations, multiplicative metric spaces, common fixed points.

AMS Subject Classification
47H10 , 54H25.

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Contents

1. Introduction

Researchers have extended the Banach Contraction Principle [2] and generated several fixed point theorems with the form

$$d(Tx,Ty) \leq F\big(d(fx,fy),d(fx,Tx), d(fy,Ty),d(fx,Ty),d(Tx,fy)\big), \quad (1.1)$$

where f and T are self mappings in complete metric spaces. We state the theorems by Imdad and Kumar [9] and Gajić and Rakočević [7] as examples of this form of expressing fixed point theorems.

In 1997, Popa [12] established a class of mappings $F : \mathbb{R}_+^6 \to \mathbb{R}$, which ensures the existence and the uniqueness of a fixed point for a mapping T, if they obey the inequality

$$F[d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)] \leq 0 \quad (1.2)$$

for all $x,y \in X$. When a fixed point theorem is written in the form of (1.2), it is said to satisfy an implicit relation.

Grossman and Katz [8] developed a new type of calculus called multiplicative calculus, where the operations of subtraction and addition were replaced by division and multiplication. Bashirov et al. [3] exploited the efficiency of multiplicative calculus over the Newtonian calculus by showing that it works better than ordinary calculus in addressing some types of problems involving differential equations. Florack and Assen [6] displayed the use of the concept of multiplicative calculus in biomedical image analysis.

Inspired by multiplicative calculus, Özavşar and Çevikel [11] defined and developed the topological properties of the multiplicative metric space. They also obtained some fixed point results in complete multiplicative metric spaces.

Fixed points in multiplicative metric spaces have various applications such as solving multiplicative boundary value problems (Abbas *et al.* [1]) and determining the existence and uniqueness of solutions for a class of nonlinear integral equations (Jiang and Gu [10]).

In this study, we will develop a fixed point theorem satisfying an implicit relation for a pair of mappings in a multiplicative metric space. In doing so, we modify the theorem by Berinde [4] so that it applies to pairs of self mappings in multiplicative metric spaces.

2. Preliminaries

In this section, we recall some definitions and basic results which will be of use in this paper.

In this work, we define $\mathbb{R}_+$ as the set of positive real numbers. We also denote $\mathbb{N}$ as the set of natural numbers. We use the term MMS as an abbreviation of multiplicative metric space.

The following is the definition of a multiplicative metric space.

Definition 2.1. [1] Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$, the following conditions hold:

(m1) $d(x,y) \geq 1$ and $d(x,y) = 1$ if and only if $x = y$;

(m2) $d(x,y) = d(y,x)$;

(m3) $d(x,y) \leq d(x,z) \cdot d(z,y)$.

The pair (X,d) is called a multiplicative metric space.

Examples of multiplicative metric spaces are stated here.

Example 2.2. [11] *Let $d^* : (\mathbb{R}_+)^n \times (\mathbb{R}_+)^n \to [1,\infty)$ be defined as follows*

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in (\mathbb{R}_+)^n$ and $|\cdot|^* : \mathbb{R}_+ \to [1,\infty)$ is defined as

$$|a|^* = \begin{cases} a, & \text{if } a \geq 1 \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}$$

Then $\left((\mathbb{R}_+)^n, d^*\right)$ is a multiplicative metric space.

The following example is modified from Özavşar and Çevikel [11].

Example 2.3. Let $a > 1$ be a fixed number. Then $d_a : \mathbb{R} \times \mathbb{R} \to [1,\infty]$ defined by

$$d_a(x,y) = a^{|x-y|}$$

holds the multiplicative metric conditions.

Özavşar and Çevikel [11] gave the following definitions in multiplicative metric spaces.

Definition 2.4. [11] Let (X,d) be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. Define the following set

$$B_\varepsilon(x) := \{y \in X : d(x,y) < \varepsilon\},$$

which is called the multiplicative open ball of radius ε with center x. Similarly, one can describe the multiplicative closed ball as follows:

$$\bar{B}_\varepsilon(x) := \{y \in X : d(x,y) \leq \varepsilon\}.$$

Definition 2.5. [11] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x)$ there exists a natural number N such that $n \geq N \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by
$x_n \to_* x \, (n \to \infty)$.

Lemma 2.6. [11] *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to_* x$ as $n \to \infty$ if and only if $d(x_n, x) \to_* 1$ as $n \to \infty$.*

Lemma 2.7. [11] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

Definition 2.8. [11] Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

Lemma 2.9. [11] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m, x_n) \to_* 1$ as $m, n \to \infty$.*

Definition 2.10. [11] Let (X,d) be a multiplicative metric space. A subset $S \subseteq X$ is called multiplicative closed in (X,d) if S contains all of its multiplicative limit points.

Theorem 2.11. [11] *Let (X,d) be a multiplicative metric space and $S \subseteq X$. Then the set S is multiplicative closed if and only if every multiplicative convergent sequence in S has a multiplicative limit point that belongs to S.*

Theorem 2.12. [11] *Let (X,d) be a multiplicative metric space and $S \subseteq X$. Then (S,d) is complete if and only if S is multiplicative closed.*

We make modifications to the description of the set of implicit relations as defined by Berinde [5]. Let $\mathscr{F}$ be a family of all continuous real functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ and the following conditions:

(F_{1a}) F is non-increasing in the fifth variable and $F(u,v,v,u,uv,1) \leq 1$ for $u, v \geq 1$ implies that $u \leq v^h$ for some $h \in [0,1)$.

(F_2) $F(u,u,1,1,u,u) > 1$, for all $u > 1$.

Example 2.13. The following functions $F \in \mathscr{F}$ satisfy the properties (F_{1a}) and (F_2).

(i) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1/t_2{}^a$ where $a \in [0,1)$;

(ii) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1/(t_3 t_4)^b$, where $t_1 \geq t_4$ and $b \in [0, \frac{1}{2})$;

(iii) $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_3 t_1/(t_5 t_6)^c$, where $c \in [0, \frac{1}{2})$.

We will also make use of the following definitions.

Definition 2.14. Consider the mappings $f, T : X \to X$. A point $x \in X$ is called a *coincidence point* of f and T if $fx = Tx$.

Definition 2.15. The mappings T and f are said to be *coincidentally commuting* if they commute at their coincidence point, that is, $fTx = Tfx$ whenever $fx = Tx$.

We take note of the following theorem by Berinde [4].

Theorem 2.16. [4] *Let (X,d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathscr{F}$ such that for all $x, y \in X$*

$$\begin{aligned}F(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty),\\ d(x,Ty), d(y,Tx)) \le 0.\end{aligned}$$

If F satisfies $(F_{1a})$and (F_2), then:

(p1) T has a unique fixed point $x^\star$ in X;

(p2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$$

converges to $x^\star$, for any $x_0 \in X$.

The aim of this study is to modify Theorem 2.16 so that it applies for pairs of self mappings in multiplicative metric spaces.

3. Main Results

We intend to prove the following theorem:

Theorem 3.1. *Let (X,d) be a multiplicative metric space and $f, T : X \to X$ be self mappings such that $TX \subseteq fX$. Assume that there exists $F \in \mathscr{F}$ satisfying (F_{1a}) such that, for all $x, y \in X$, the following condition holds:*

$$\begin{aligned}F\big(d(Tx,Ty), d(fx,fy), d(fx,Tx), d(fy,Ty),\\ d(fx,Ty), d(fy,Tx)\big) \le 1. \qquad (3.1)\end{aligned}$$

If $TX \subseteq fX$ and fX is a complete subspace of X, then T and f have a coincidence point. Moreover, if T and f are coincidentally commuting and F satisfies also (F_2), then T and f have a unique common fixed point.

Furthermore, for any $x_0 \in X$, the T-f-sequence $fx_{n+1} = Tx_n$ with initial point x_0 converges to the common fixed point.

Proof. Commencing with an arbitrary point $x_0 \in X$, we find Tx_0. From the assumption, we have $TX \subseteq fX$. Hence, there is $x_1 \in X$ such that $fx_1 = Tx_0$. We then find Tx_1.

Proceeding inductively, we generate the sequence $\{fx_n\}$ and $\{Tx_n\}$ by using the relation $fx_{n+1} = Tx_n$.

Consider $(fx_n, fx_{n+1}) = (Tx_{n-1}, Tx_n)$. Applying (3.1), we get

$$\begin{aligned}&F\big(d(Tx_{n-1},Tx_n), d(fx_{n-1},fx_n), d(fx_{n-1},Tx_{n-1}),\\ &\quad d(fx_n,Tx_n), d(fx_{n-1},Tx_n), d(fx_n,Tx_{n-1})\big) \le 1.\\ \Rightarrow &F\big(d(fx_n,fx_{n+1}), d(fx_{n-1},fx_n), d(fx_{n-1},fx_n),\\ &\quad d(fx_n,fx_{n+1}), d(fx_{n-1},fx_{n+1}), d(fx_n,fx_n)\big) \le 1.\end{aligned} \qquad (3.2)$$

From (m1) of Definition 2.1, we note that $d(fx_n, fx_n) = 1$.

If we set $u = d(fx_n, fx_{n+1}))$, $v = d(fx_{n-1}, fx_n)$, then (3.2) becomes

$$F\big(u, v, v, u, d(fx_{n-1}, fx_{n+1}), 1\big) \le 1. \qquad (3.3)$$

From (m3) of Definition 2.1, we note that

$$d(fx_{n-1}, fx_{n+1}) \le d(fx_{n-1}, fx_n) \cdot d(fx_n, fx_{n+1}) = uv.$$

From the assumption, F is non-increasing in the fifth variable. Therefore, (3.3) implies that

$$F\big(u, v, v, u, uv, 1\big) \le 1.$$

This implies that there is $h \in [0,1)$ such that $u \le v^h$, meaning

$$d(fx_n, fx_{n+1})) \le [d(fx_{n-1}, fx_n)]^h. \qquad (3.4)$$

Extending (3.4) for large n, we get

$$\begin{aligned}d(fx_n, fx_{n+1}) &\le [d(fx_{n-1}, fx_n)]^h\\ &\le [d(fx_{n-2}, fx_{n-1})]^{h^2}\\ &\le [d(fx_{n-3}, fx_{n-2})]^{h^3}\\ &\;\;\vdots\\ &\le [d(fx_0, fx_1)]^{h^n}\\ \Rightarrow d(fx_n, fx_{n+1}) &\le [d(fx_0, fx_1)]^{h^n}.\end{aligned} \qquad (3.5)$$

Consider $n, m \in \mathbb{N}$ with $n > m$. From (m3) of Definition 2.1, we conclude inductively that

$$\begin{aligned}d(fx_m, fx_n) &\le \prod_{i=m}^{n-1} d(fx_i, fx_{i+1})\\ &\le \prod_{i=m}^{n-1} [d(fx_0, fx_1)]^{h^i}, \text{ from (3.5)}\\ &\le \prod_{i=m}^{\infty} [d(fx_0, fx_1)]^{h^i}, \text{ by (m1)}.\end{aligned}$$

We utilize the fact that the logarithmic is a continuous and increasing function. This leads to

$$\begin{aligned}\log[d(fx_m, fx_n)] &\le \log\left[\prod_{i=m}^{\infty} [d(fx_0, fx_1)]^{h^i}\right]\\ &= \sum_{i=m}^{\infty} h^i \log[d(fx_0, fx_1)]\\ &= \log[d(fx_0, fx_1)] \sum_{i=m}^{\infty} h^i\\ &= \log[d(fx_0, fx_1)] \frac{h^m}{1-h},\\ &\quad \{sum\ of\ a\ Geometric\ Progression\}.\end{aligned}$$

As $m, n \to \infty$ we get

$$\lim_{m,n\to\infty} \log[d(fx_m, fx_n)] = 0. \qquad (3.6)$$

We use the fact that the exponential function is continuous. We apply the exponential function on both sides of (3.6) and get

$$\lim_{m,n\to\infty} d(fx_m, fx_n) = 1.$$

This makes $\{fx_n\}$ a multiplicative Cauchy sequence by Lemma 2.9. From the assumption, we know that fX is complete. This means that there is $z \in fX$ such that

$$\lim_{n\to\infty} fx_n = z. \tag{3.7}$$

From the construction of the sequence $\{fx_n\}$, we have $fx_n = Tx_{n-1}$. Taking $n \to \infty$ we have

$$\lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} Tx_n. \tag{3.8}$$

As $z \in fX$, there is $w \in X$ such that $fw = z$. We claim that w is a coincidence point of f and T.

Let $x = x_n$ and $y = w$ in (3.1). Then, we have

$$\begin{aligned}
&F\big(d(Tx_n,Tw), d(fx_n,fw), d(fx_n,Tx_n), d(fw,Tw),\\
&\qquad d(fx_n,Tw), d(fw,Tx_n)\big) \le 1\\
\Rightarrow &F\big(d(Tx_n,Tw), d(fx_n,z), d(fx_n,Tx_n), d(z,Tw),\\
&\qquad d(fx_n,Tw), d(z,Tx_n)\big) \le 1.
\end{aligned}$$

From the assumption, F is continuous. Hence, on taking $n \to \infty$ and appying (3.8), we get

$$\begin{aligned}
&F\big(d(z,Tw), d(z,z), d(z,z), d(z,Tw),\\
&\qquad d(z,Tw), d(z,z)\big) \le 1.\\
\Rightarrow &F\big(d(z,Tw), 1, 1, d(z,Tw), d(z,Tw), 1\big) \le 1.
\end{aligned} \tag{3.9}$$

We use the property (F_{1a}) with $u = d(z,Tw), v = 1$ and deduce that, for some $h \in [0,1)$, we have

$$\begin{aligned}
&u \le v^h\\
\Rightarrow &d(z,Tw) \le 1\\
\Rightarrow &d(z,Tw) = 1, \text{ by (m1) of Definition 2.1},\\
\Rightarrow &z = Tw, \text{ by (m1).}
\end{aligned}$$

Thus w is a coincidence point for f and T because

$$Tw = z = fw. \tag{3.10}$$

If f and T are coincidentally commuting at point w, then we have

$$fTw = Tfw \Rightarrow fz = Tz. \tag{3.11}$$

We claim that z is a common fixed point of f and T. Let us use (3.1) with $x = w,\ y = z$.

$$\begin{aligned}
&F\big(d(Tw,Tz), d(fw,fz), d(fw,Tw), d(fz,Tz),\\
&\qquad d(fw,Tz), d(fz,Tw)\big) \le 1,\\
\Rightarrow &F\big(d(z,fz), d(z,fz), d(z,z), d(fz,fz),\\
&\qquad d(z,fz), d(fz,z)\big) \le 1,\\
&\qquad\qquad \text{from (3.11) and (3.10)}\\
\Rightarrow &F\big(d(z,fz), d(z,fz), 1, 1, d(z,fz), d(z,fz)\big) \le 1,
\end{aligned} \tag{3.12}$$

Let us assume $d(z,fz) > 1$. Applying property (F_2) , we get

$$F\big(d(z,fz), d(z,fz), 1, 1, d(z,fz), d(z,fz)\big) > 1,$$

which is a contradiction. Hence

$$\begin{aligned}
&d(z,fz) \le 1\\
\Rightarrow &d(z,fz) = 1, \text{ according to (m1) in Definition 2.1},\\
\Rightarrow &fz = z = Tz, \text{ by (m1)and (3.11)}
\end{aligned}$$

making z a common fixed point of f and T.

Now we will show that z is unique. Suppose z' is also a common fixed point of f and T. Let us set $x = z, y = z'$ in (3.1). We get

$$\begin{aligned}
&F\big(d(Tz,Tz'), d(fz,fz'), d(fz,Tz), d(fz',Tz'),\\
&\qquad d(fz,Tz'), d(fz',Tz)\big) \le 1\\
\Rightarrow &F\big(d(z,z'), d(z,z'), d(z,z), d(z',z'),\\
&\qquad d(z,z'), d(z',z)\big) \le 1\\
\Rightarrow &F\big(d(z,z'), d(z,z'), 1, 1,\\
&\qquad d(z,z'), d(z,z')\big) \le 1.
\end{aligned}$$

Let us assume $u = d(z,z') > 1$. Using property (F_2), we get

$$F\big(d(z,z'), d(z,z'), 1, 1, d(z,z'), d(z',z)\big) > 1,$$

which is a contradiction.

Hence, we have

$$\begin{aligned}
&d(z,z') \le 1\\
\Rightarrow &d(z,z') = 1, \text{ by (m1) in Definition 2.1},\\
\Rightarrow &z = z', \text{ by (m1)}
\end{aligned}$$

making z the unique common fixed point of f and T. □

If we set $f = I$, the identity mapping, we get the following corollary, which is Theorem 2.16 by Berinde [4], modified to multiplicative metric spaces.

Corollary 3.2. *Let (X,d) be a complete multiplicative metric space and $T : X \to X$ be a self-mapping. Assume that there exists $F \in \mathscr{F}$ satisfying (F_{1a}) and (F_2) such that, for all $x,y \in X$, the following condition holds:*

$$\begin{aligned}
&F\big(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty),\\
&\qquad d(x,Ty), d(y,Tx)\big) \le 1.
\end{aligned}$$

Then the mapping T has a unique fixed point in X.

Furthermore, for any $x_0 \in X$, the sequence $\{Tx_n\}$, where $x_{n+1} = Tx_n$ with initial point x_0, converges to the fixed point.

Example 3.3. Consider the function $F : \mathbb{R}_+{}^6 \to \mathbb{R}_+$ defined as

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = \frac{t_1}{(t_2)^a \cdot (t_3)^b \cdot (t_4)^c}, \quad \text{where } 0 < a+b+c \leq 1. \tag{3.13}$$

We will show that F has the (F_{1a}) property. Note that F is non-increasing in the fifth variable. Also note that,

$$\begin{aligned} &F(u,v,v,u,uv,1) \leq 1 \\ &\Rightarrow \frac{u}{v^a v^b u^c} \leq 1 \\ &\Rightarrow u \leq v^{\frac{a+b}{1-c}}. \end{aligned} \tag{3.14}$$

Because $0 \leq a+b+c < 1$, we have $0 \leq h = \dfrac{a+b}{1-c} < 1$. Hence (3.14) becomes $u \leq v^h$ with $h \in [0,1)$, obeying the (F_{1a}) property for all $u, v \geq 1$.

We now show that F has the F_2 property which states that for all $u > 1$ we have $F(u,u,1,1,u,u) > 1$.

Because $0 \leq a+b+c < 1$ we have $1-a > b+c \geq 0$. Hence for $u > 1$ we have

$$F(u,u,1,1,u,u) = u^{1-a} > u^0 = 1.$$

The function F described in (3.13) leads to the following theorem which is a modification of Reich's Fixed Point Theorem [13] so that it applies to two self mappings in multiplicative metric spaces.

Theorem 3.4. *Let (X,d) be a multiplicative metric space and $f,T : X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that, for all $x,y \in X$, the following condition holds:*

$$d(Tx,Ty) \leq [d(fx,fy)]^a \cdot [d(fx,Tx)]^b \cdot [d(fy,Ty)]^c, \tag{3.15}$$

where $0 \leq a+b+c < 1$. If $TX \subseteq fX$ and fX is a complete subspace of X, then T and f have a coincidence point. Moreover, if T and f are coincidentally commuting, then T and f have a unique common fixed point.

Furthermore, for any $x_0 \in X$, the T-f-sequence $fx_{n+1} = Tx_n$ with initial point x_0, converges to the common fixed point.

Proof. We provide the proof to the theorem using conventional methods. Commencing with an arbitrary point $x_0 \in X$, we find Tx_0. From the assumption, we have $TX \subseteq fX$. Hence, there is $x_1 \in X$ such that $fx_1 = Tx_0$. We then find Tx_1.

Proceeding inductively, we generate the sequence $\{fx_n\}$ and $\{Tx_n\}$ by using the relation

$$fx_{n+1} = Tx_n.$$

If we set $x = x_{n-1}, y = x_n$ in (3.15), we get

$$\begin{aligned} d(Tx_{n-1},Tx_n) &= d(fx_n,fx_{n+1}) \\ &\leq [d(fx_{n-1},fx_n)]^a \cdot [d(fx_{n-1},Tx_{n-1})]^b \cdot \\ &\qquad [d(fx_n,Tx_n)]^c \\ &= [d(fx_{n-1},fx_n)]^a \cdot [d(fx_{n-1},fx_n)]^b \cdot \\ &\qquad [d(fx_n,fx_{n+1})]^c \\ \Rightarrow d(fx_n,fx_{n+1}) &\leq [d(fx_{n-1},fx_n)]^{\frac{a+b}{1-c}}. \end{aligned} \tag{3.16}$$

From the assumption we have $0 \leq a+b+c < 1$. This implies $h = \dfrac{a+b}{1-c} < 1$. Hence, for all $n \geq 1$, we have

$$d(fx_n,fx_{n+1}) \leq [d(fx_{n-1},fx_n)]^h, \; 0 \leq h < 1. \tag{3.17}$$

The proof continues as after (3.4). □

4. Example

Now, we illustrate an example on the use of the theorem.

Example 4.1. Consider the multiplicative metric space (X,d) where $X = [1,2] \in \mathbb{R}$ and for all $x,y \in X$, $d(x,y) = |\frac{x}{y}|^*$. Define the mappings $f,T : X \to X$ as $fx = x^{(1/2)}, Tx = x^{(1/3)}$. Note that we have $[1,\ 2^{(1/3)}] = TX \subset fX = [1,\ 2^{(1/2)}]$. We also have fX closed as the theorem requires.

Consider the function $F \in \mathscr{F}$ defined as

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = \frac{t_1}{(t_2)^{2/5}\,(t_5)^{1/5}\,(t_6)^{1/5}}. \tag{4.1}$$

We show that F has the (F_{1a}) property. Note that F is non-increasing in the fifth variable. Also note that,

$$\begin{aligned} &F(u,v,v,u,uv,1) \leq 1 \\ &\Rightarrow \frac{u}{v^{2/5}(uv)^{1/5}1^{1/5}} \leq 1 \\ &\Rightarrow u \leq v^{3/4}. \end{aligned}$$

Then we show that F has the F_2 property which states that for all $u > 1$ we have $F(u,u,1,1,u,u) > 1$.

$$\begin{aligned} F(u,u,1,1,u,u) &= \frac{u}{(u)^{2/5}\,u^{1/5}u^{1/5}} \\ &= u^{1/5} \\ &> 1. \end{aligned}$$

Here we show that function (4.1) obeys the condition (3.1). Without loss of generality, assume $x,y \in X$, $x \geq y$. Note that in this case

$$d(x,y) = |\frac{x}{y}|^* = \frac{x}{y}.$$

$$\begin{aligned}\Gamma = & F\big(d(Tx,Ty), d(fx,fy), d(fx,Tx),\\ & \qquad d(fy,Ty), d(fx,Ty), d(fy,Tx)\big)\\ = & \frac{d(Tx,Ty)}{(d(fx,fy))^{2/5}(d(fx,Ty))^{1/5}(d(Tx,fy))^{1/5}}\\ = & \frac{d(x^{\frac{1}{3}},y^{\frac{1}{3}})}{(d(x^{\frac{1}{2}},y^{\frac{1}{2}}))^{2/5}(d(x^{\frac{1}{2}},y^{\frac{1}{3}}))^{1/5}(d(x^{\frac{1}{3}},y^{\frac{1}{2}}))^{1/5}}.\end{aligned}$$

We consider two cases:
Case 1: If $x^{1/3} \geq y^{1/2}$, we have

$$\begin{aligned}\Gamma = & \frac{\frac{x^{1/3}}{y^{1/3}}}{(\frac{x^{1/2}}{y^{1/2}})^{2/5}\ (\frac{x^{1/2}}{y^{1/3}})^{1/5}\ (\frac{x^{1/3}}{y^{1/2}})^{1/5}}\\ = & (\frac{x}{y})^{-1/30},\\ \leq & 1,\ \text{because}\ \frac{x}{y} \geq 1.\end{aligned}$$

Case 2: If $x^{1/3} < y^{1/2}$, we have

$$\begin{aligned}\Gamma = & \frac{\frac{x^{1/3}}{y^{1/3}}}{(\frac{x^{1/2}}{y^{1/2}})^{2/5}\ (\frac{x^{1/2}}{y^{1/3}})^{1/5}\ (\frac{y^{1/2}}{x^{1/3}})^{1/5}}\\ < & \frac{\frac{x^{1/3}}{y^{1/3}}}{(\frac{x^{1/2}}{y^{1/2}})^{2/5}\ (\frac{x^{1/2}}{y^{1/3}})^{1/5}\ (\frac{x^{1/3}}{y^{1/2}})^{1/5}}\\ = & (\frac{x}{y})^{-1/30},\\ \leq & 1.\end{aligned}$$

From the theorem, f and T have a unique fixed point at $z = 1$ which can be found by the Picard iteration $fx_{n+1} = Tx_n$ using x_0 as an arbitrary point in X.

5. Conclusion

In this paper, an approach has been developed for existence and uniqueness of fixed point for the maps involving implicit relations in multiplicative metric spaces. Illustrative examples are aslo given to support the theorems.

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