Transit index of various graph classes

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Abstract
Transit of a vertex $v$ is a graph invariant which was defined as the sum of the length of all shortest paths with $v$ as an internal vertex. In this paper, transit index for various classes of graph like complete graphs, cycles, wheel graph, friendship graph, crown graph, total graph of a path, comet are computed.

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Transit of a vertex, Transit Index.

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1. Introduction

Graph topological indices are widely studied. They find application in many field of science. Chemical graph Theory and Networking are a few to name. In\cite{8}, transit index of a graph was introduced and its correlation with one of the physical property -MON of octane isomers was established. In this paper we compute the transit index for various graph classes and for certain graphs developed from complete graphs.

Throughout $G$ denotes a simple, connected, undirected graph with vertex set $V$ and edge set $E$. For undefined terms we refer\cite{1}.

Preliminaries

Definition 1.1. \cite{8} Let $v \in V$. Then the transit of $v$ denoted by $T(v)$ is “the sum of the lengths of all shortest path with $v$ as an internal vertex” and the transit index of $G$ denoted by $TI(G)$ is

\[ TI(G) = \sum_{v \in V} T(v) \]

Lemma 1.2. \cite{8} $T(v) = 0$ iff $\langle N[v] \rangle$ is a clique.

Theorem 1.3. \cite{8} For a path $P_n$, Transit index is

\[ TI(P_n) = \frac{n(n+1)(n^2 - 3n + 2)}{12} \]

Definition 1.4. Two vertices $v_1$ and $v_2$ of a graph are called transit identical if the shortest paths passing through them are same in number and length.

2. Transit index for various graph classes

2.1 Star

Theorem 2.1. For a star graph $S_n$, $TI(S_n) = (n - 1)(n - 2)$

Proof. In a star graph on $n$ vertices, $n - 1$ vertices are pendant vertices. Hence for them $T(v) = 0$. There are $C(n-1, 2)$ shortest path of length 2 passing through the center vertex. Hence $TI(S_n) = 2.C(n-1, 2) = (n - 1)(n - 2)$.
2.2 Complete Graphs

Theorem 2.2. For the complete graph $K_n$, transit index is zero.

Proof. For every vertex $v$ in a complete graph $K_n$, $|N[v]| = K_n$, a clique. Hence by lemma[1.2], $TI(K_n) = 0$.

Theorem 2.3. For $n \geq 3$, deleting an edge from $K_n$, increases the transit index by $2(n-2)$.

Proof. The deletion of the edge $e = uv$, makes $u$ and $v$ non-adjacent. Hence every other vertex will be an internal vertex of the shortest path between $u$ and $v$ of length 2. Hence $TI(K_n - e) = 2(n-2)$.

Theorem 2.4. Let $G = K_{p,q}$ where $V = V_1 \cup V_2$ is the bipartition with $|V_1| = p$, $|V_2| = q$. Then $TI(G) = pq[p+q-2]$

Proof. When $p = 1$ or $q = 1$, the result is obvious. Let $p, q \geq 2$.
Let $v \in V_1$, then, $T(v) = 2C(q,2)$.
If $v \in V_2$, then $T(v) = 2C(p,2)$.
Hence

$$TI(G) = \sum_{v \in V} T(v) = \sum_{v \in V_1} T(v) + \sum_{v \in V_2} T(v)$$
$$= 2\left[pq\left(q-1\right)\right] + 2\left[pq\left(p-1\right)\right]$$
$$= pq[p+q-2]$$

Theorem[2.4] can be generalised to s-partite graphs as follows.

Theorem 2.5. Let $G$ be the complete s-partite graph [6]. Then $TI(G) = \sum_{i=1}^{s} 2n_i \left[C(n_j,2)\right]$.

Proof. Let $V_1, V_2, \ldots, V_s$ be the partition of the vertex set $V$. Then no two vertices in $V_i$ are adjacent to each other. But every vertex in $V_j$, $j \neq i$ is adjacent to all vertices of $V_i$. The shortest paths passing through $v_i$ are those connecting vertices of the same $V_j$ to itself, of length 2. Hence $T(v_i) = 2 \sum C(n_j,2)$.

$$TI(G) = \sum_{v_i \in V_1} T(v_i) + \sum_{v_j \in V_2} T(v_j) + \cdots + \sum_{v_s \in V_s} T(v_s)$$
$$= \sum_{i=1}^{s} 2n_i \left[C(n_j,2)\right]$$

Corollary 2.6. If $G$ is the cocktail party graph [5], $TI(G) = 4n(n-1)$

Proof. In the theorem [2.5], take $n_i = 2$, $\forall i$ and $s = n$ with $|G| = 2n$.

2.3 Cycle

Theorem 2.7. Let $C_n$ be a cycle with $n$ even. Then

(i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$

(ii) $TI(C_{n+1}) = \frac{n(n^2-4)(n+1)}{24}$

Proof. The wheel graph $W_{n+1}$ is the graph obtained from $C_n$, $n \geq 3$ by adding a new vertex and by making it adjacent to all vertices of $C_n$.
Theorem 2.8. \( TI(W_{n+1}) = n(n-1), n > 3 \) and for \( n = 3, TI(W_3+1) = 0 \).

**Proof.** Let \( n > 3 \). In \( W_{n+1} \), the diameter is 2. Hence no shortest path is of length more than 2. The vertices on the outer circle \( C_n \) are transit identical. Let \( v \) be one such vertex. The only shortest path passing through it is between its adjacent vertices. \( \therefore T(v) = 2 \), for \( v \in C_n \).

Consider the center vertex \( c \). To find its transit we consider the contribution of each edge to it. Every edge on \( C_n \) contributes 0 to \( T(c) \). Consider the edges of the type \( e \), as shown in the figure[2], which are the spokes of the wheel. \( e \) will be used only by \( v \) to travel to every vertex other than its adjacent ones. Hence the contribution is \( (n-3) \). \( \because T(c) = n(n-3) \)

i.e. \( TI(W_{n+1}) = 2n + n(n-3) = n(n-1) \)

For \( n = 3 \), we get \( W_{3+1} = K_4 \). \( \therefore \) its transit is zero. \( \square \)

2.5 **Friendship Graph**

The Friendship graph [3], \( F_n \) is constructed by coalescence of \( n \) copies of cycle \( C_3 \) of length 3, with a common vertex.

**Theorem 2.9.** \( TI(F_n) = 4n(n-1), |V| = 2n + 1. \)

**Proof.** In \( F_n \), the diameter is 2. For every vertex \( v \) other than the coalescence vertex, \( \langle N[v] \rangle \) is a clique. Hence \( T(v) = 0 \), by lemma[1.2]. Hence \( TI(F_n) = T(c) \)

The edges of the type \( e' \), as in the figure[3] does not contribute to \( T(c) \). Hence we count the number of times the edges of the type \( e \) is used. The edge \( e \) will be used by the vertex \( v \) to travel to all vertices other than its adjacent ones. Hence contribution of \( e \) is \( 2n + 1 = 2(n-1) \). There are \( 2n \) such edges. \( \therefore T(c) = 4n(n-1) \).

i.e. \( TI(F_n) = 4n(n-1), |V| = 2n + 1. \) \( \square \)

2.6 **Crown Graph**

A crown graph [4] is the unique \( n-1 \) regular graph with \( 2n \) vertices, obtained from the complete bipartite graph \( K_{n,n} \) by deleting a perfect matching. Or it is the graph with vertices as two sets \( \{u_i\} \) and \( \{v_i\} \), with an edge from \( u_i \) to \( v_j \) whenever \( i \neq j \).

![Figure 2. Wheel graph \( W_{n+1} \)](image)

![Figure 3. Friendship Graph \( F_n \)](image)

![Figure 4. Crown graph](image)

**Theorem 2.10.** For the Crown graph \( G \), \( TI(G) = 2n(n^2-1) \).

**Proof.** Let the bipartition be \( V, U \), with \( V = \{v_1, v_2, \ldots, v_n\} \) and \( U = \{u_1, u_2, \ldots, u_n\} \). Consider a vertex of \( V \), say \( v_k \). Note that \( d(u_i, v_1) = 3 \) and \( d(u_i, v_j) = 2, i \neq j \). The shortest path through \( v_k \) are those connecting \( v_i \) to \( v_j \), \( i \neq j \) of length 2 and those connecting \( v_i \) to \( u_i \) of length 3. Hence \( T(v_k) = 2C(n-1, 2) + 3(n-1) = n^2 - 1 \). In this graph every vertex is transit identical. \( \therefore TI(G) = 2n(n^2-1). \) \( \square \)

2.7 **Snake Graph**

The triangular snake graph can be viewed as the graph formed by replacing every edge of \( P_n \) by a triangle, thus adding \( n-1 \) vertices and \( 2(n-1) \) edges.

**Theorem 2.11.** If \( G \) is the triangular snake graph of a path on \( 2n-1 \) vertices, \( TI(G) = TI(P_n) + \frac{4(n-2)(n-1)n+1}{4} \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) denote the vertices of the path \( P_n \). The newly added vertices are named as \( u_1, u_2, \ldots, u_{n-1} \). For every \( u_i \), \( \langle N[u_i] \rangle \) is a clique. Hence \( T(u_i) = 0, \forall i \), by lemma[1.2]. Also \( \langle N[v_1] \rangle, \langle N[v_n] \rangle \) are cliques. \( \therefore T(v_1) = T(v_n) = 0. \)

Hence we need to compute only the transit of \( v_i \) for \( 1 < i < n \). The transit of these vertices are due to path connecting \( v_i \) among themselves, path connecting \( u_i \) among themselves and paths connecting \( v_i \) to \( u_i \), i.e. \( TI(G) = TI(P_n) + I, \)
where I denote the increase in transit of \( v_i \) due to the addition of \( u_i \).

Consider \( v_k \). The increase in its transit is due to:
1. Paths connecting \( u_i \) to \( u_j \), \( i < k, j > k \)
2. Paths connecting \( u_i \) to \( v_j \), \( i < k, j > k \)
3. Paths connecting \( v_i \) to \( u_j \), \( i < k, j > k \)

It can be seen that the increase in all the three cases are the same and equal to:

\[
A = (2 + 3 + \ldots + n - k + 1) + (3 + 4 + \ldots + n - k + 2) + \ldots (k + (k + 1) + \ldots + (n - 1))
\]

Hence increase in transit of \( v_k \) is \( 3A \).

If we take \( a = 2 + 3 + \ldots + n - k + 1 \), then
\[
A = a + (a + n - k) + (a + 2(n - k)) + \ldots
\]

Hence increase in transit of \( v_k \) is
\[
I = \sum_{k=1}^{n} \frac{3}{2} (n + 1)(n + 1 - k^2 - n)
\]

Hence the theorem. \( \Box \)

### Remark 2.13
Applying the recursive formula for a path, \( \text{TI}(P_{n+1}) = \text{TI}(P_n) + \frac{n(n-1)}{3} \), the transit of a comet \( G \) of Theorem 2.12 can be expressed as, \( \text{TI}(G) = m\text{TI}(P_{n-1}) - (m-1)\text{TI}(P_n) \).

### 3. Transit index for some graphs derived from Complete graph

#### Theorem 3.1
Let \( G \) be the graph obtained by attaching a pendant edge to one of the vertices of a complete graph, i.e., \( |V(G)| = |V(K_n)| + 1 \) and \( |E(G)| = |E(K_n)| + 1 \). Then \( \text{TI}(G) = 2(n-1) \)

**Proof.** Let the new vertex be \( v \) and the vertex to which it is attached be \( u \). Then for every vertex in \( G \) other than \( u \), \( N[v_i] \) is a clique. Hence transit is zero. There are \( n - 1 \) paths of length 2 connecting \( v \) to vertices of \( K_n - \{u\} \), passing through \( v \).

\[ \therefore \text{TI}(G) = 2(n-1) \] \( \Box \)

#### Theorem 3.2
Let \( G \) be the graph formed by attaching a pendant edge to every vertex of \( K_n \). Then \( \text{TI}(G) = 5n(n-1) \)

**Proof.** Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of \( K_n \) and \( u_1, u_2, \ldots, u_n \), be the vertices attached to \( v_1, v_2, \ldots, v_n \) respectively. Since \( u_i \) are pendant vertices \( T(u_i) = 0 \), \( \forall i \). The shortest path passing through \( v_i \) are either \( u_i, v_j \) paths or \( u_i, u_j \) paths of length 2 and 3 respectively. Hence \( T(v_i) = 2(n-1) + 3(n-1) \).

\[ \therefore \text{TI}(G) = 5n(n-1) \] \( \Box \)

#### Theorem 3.3
Let \( G \) be the graph formed by merging a vertex of \( K_n \) and \( K_m \), i.e., \( |V(G)| = m+n-1 \) and \( |E(G)| = |E(K_n)| + |E(K_m)| \). Then \( \text{TI}(G) = 2(n-1)(m-1) \)

**Proof.** Let \( v \) be coalescence vertex. For every vertex \( u \) of \( G \) other than \( v \), \( T(u) = 0 \), as \( N[u] \) is a clique. The shortest paths
Let \( K_n \) with \( m-1 \) vertices of \( K_m \), each of length 2. Hence
\[
TI(G) = T(v) = 2(n-1)(m-1)
\]

**Theorem 3.4.** Let \( G \) be the graph formed by merging a vertex of \( K_n \) with a vertex of \( C_m \).
Then \( TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12} \), if \( m \) is even and
\( TI(G) = TI(C_m) + \frac{(n-1)(m-1)(m+1)(m+3)}{12} \), if \( m \) is odd.

**Proof.** Let us denote the coalescence vertex by \( v \)

**Case 1 [m even]**

Clearly, \( TI(G) = TI(C_m) + TI(K_n) + I \), where I denote the

\[
\begin{align*}
\text{Figure 6. } K_n \text{ and } C_m \text{ merged at } v, \text{ m even.}
\end{align*}
\]

increment in transit due to merging of graphs. The transit for vertices in \( K_n \) remains zero, except for \( v \). The vertex at the distance \( \frac{m}{2} \) from \( v \) on \( C_m \) has no increment. Let \( v_k \) denote the kth vertex on \( P_1, v_1 \) being \( v \). For \( 1 < k < \frac{m}{2} \), the increment for \( v_k \) is due to the shortest paths from vertices on its right to vertices of \( K_n \) including \( v \). This can be computed as
\[
= \left[ (\frac{m}{2} + 1) + (\frac{m}{2} + 2) + \ldots + (\frac{m}{2} + 1) \right] (n-1)
\]
\[
= \left[ (\frac{m}{2} + 1)(\frac{m}{2} + 2) - k^2 \right] \frac{(n-1)}{2}
\]
Now do similar positions, \( T(v_k), T(v_{m-k+2}) \) are transit identical.

Hence we have \( I = \)
\[
= 2 \sum_{1}^{\frac{m}{2}} \left[ \left( \frac{m}{2} + 1 \right) \left( \frac{m}{2} + 2 \right) - k^2 \right] \frac{(n-1)}{2}
\]
\[
= \frac{1}{12} \frac{(n-1)(m+4)(m+2)m}{2}, \text{ on simplification.}
\]
\[
\therefore TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}
\]

**Case 2 [m odd]**

Let \( v_k \) denote the kth vertex on \( P_1, v_1 = v \). For \( 1 < k < \frac{m-1}{2} \),

\[
\begin{align*}
\text{Figure 7. } K_n \text{ and } C_m \text{ merged at } v, \text{ m odd.}
\end{align*}
\]

the increment for \( v_k \) is due to the shortest paths from vertices on its right to vertices of \( K_n \) including \( v \). This can be computed as
\[
I = (k + 1) + (k + 2) + \ldots + \frac{m+1}{2}
\]
\[
= \frac{(m+1)(m+3)}{4} - k - \frac{k^2}{2}
\]
In this case also \( T(v_k) = T(v_{m-k+2}) \)

Hence \( TI(G) = 2 \sum_{1}^{\frac{m+1}{2}} \left[ \left( \frac{m+1}{2} \right)(m+3) - k - \frac{k^2}{2} \right] \)
\[
= \frac{(m+1)(m+3)}{12} - \frac{1}{2} \frac{k}{2} - \frac{k^2}{2}
\]
\[
\therefore TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}
\]

4. Conclusion

In this paper, transit index for various graph classes and for graphs obtained from complete graphs are computed. In future, authors are planning to extend the study to sub-division graphs, graph products and various graphs of importance in chemical graph theory and communication networks.

**References**


