On the solutions of a higher order recursive sequence

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Abstract
In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the recursive sequence
\[ x_{n+1} = \frac{ax_{n-2k-1}}{b + c \prod_{i=0}^{l} x_{n-2i-1}}, \quad n = 0, 1, \ldots, \]
where \(a, b, c\) are positive real numbers, the initial conditions \(x_{-2k-1}, x_{-2k}, \ldots, x_{-1}, x_0\) are real numbers and \(k\) is a nonnegative integer. We show that every admissible solution with \(\prod_{i=0}^{l} x_{n-2(l+1)+i} = \frac{b}{a-b}, \quad i = 1, 2\) is periodic with prime period \(2k+2\). Otherwise, the solution converges to zero if \(a < b\) or converges to a period-(\(2k+2\)) solution if \(a > b\). We finally study some special cases and give illustrative examples.

Keywords
Difference equation, periodic solution, convergence.

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1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [2–7, 9, 10, 13–16, 18–22, 25, 26] and the references therein.

We have discussed in [8] the global behavior of the solutions of the difference equation
\[ x_{n+1} = \frac{B x_{n-2r-1}}{C + D \prod_{i=1}^{l} x_{n-2i}}, \quad n = 0, 1, \ldots, \]
where \(A, B, C\) are nonnegative real numbers and the initial conditions are nonnegative real numbers and \(l, r, k\) are nonnegative integers such that \(l \leq k\) and \(r \leq k\).

In [23], D. Simsek et al. introduced the solution of the difference equation
\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots, \]
where \(x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)\).

Also in [24], D. Simsek et al. introduced the solution of the difference equation
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}}, \quad n = 0, 1, \ldots, \]
with positive initial conditions.

R. Karatas et al. [17] discussed the positive solutions and the attractiveness of the difference equation
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}, \quad n = 0, 1, \ldots, \]
where the initial conditions are nonnegative real numbers. In [11], E.M. Elsayed discussed the solutions of the difference equation

\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, \ldots, \]

where the initial conditions are nonzero real numbers with \( x_{-5} \neq 1, x_{-2}x_{-1} \neq 1 \) and \( x_{-3}x_0 \neq 1 \). Also in [12], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots, \]

where the initial conditions are nonzero positive real numbers.

In [1], the authors obtained the expressions of solutions of the following difference equations

\[ x_{n+1} = \frac{x_{n-2k+1}}{1 + \pm \prod_{i=0}^{k} x_{n-2i+1}}, \quad n = 0, 1, \ldots, \]

with initial conditions \( x_{-j}, \ j = 0, 1, \ldots, 2k - 1 \), where \( k \in \{1, 2, \ldots\} \).

In this paper, we introduce an explicit formula and investigate the global behavior of solutions of the recursive sequence

\[ x_{n+1} = \frac{ax_{n-2k-1}}{b + c \prod_{i=0}^{k} x_{n-2i-1}}, \quad n = 0, 1, \ldots, \quad (1.1) \]

where \( a, b, c \) are positive real numbers, the initial conditions \( x_{-2k-1}, x_{-2k}, \ldots, x_{-1}, x_0 \) are real numbers and \( k \) is a nonnegative integer.

\section{Solution of Equation (1.1)}

In this section, we give an explicit formula for the solution of Equation (1.1) with \( a \neq b \).

Let \( \theta_i = \frac{a - b - c \alpha_i}{a \alpha_i} \), where \( \alpha_i = \prod_{j=0}^{k} x_{n-2(j+1)+i}, \ i = 1, 2 \). We need the following lemma to prove the main result in this section.

**Lemma 2.1.** Let \( x_{-2k-1}, x_{-2k}, \ldots, x_{-1} \) and \( x_0 \) be real numbers such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{b}{\sum_{j=0}^{k} \frac{1}{\theta_j^i}} \) for all \( n \in \mathbb{N} \). Then

\[ \prod_{i=0}^{k} x_{-2(i+1)+2i} = \frac{a - b}{(\frac{b}{a})^t \theta_i + c}, \quad 0 \leq t \leq k \quad \text{and} \quad 1 \leq i \leq 2. \]

**Proof.** Let \( \mu(t, i) = \prod_{j=0}^{k} x_{-2(i+1)+2i+j}, \) where \( 0 \leq i \leq k \) and \( 1 \leq i \leq 2 \). It is required to show that

\[ \mu(t, i) = \frac{a - b}{(\frac{b}{a})^t \theta_i + c} \quad \text{for} \quad 0 \leq t \leq k \quad \text{and} \quad 1 \leq i \leq 2. \quad (2.1) \]

The proof is by induction on \( t \) for each \( 1 \leq i \leq 2 \).

When \( t = 0 \), we have for each \( 1 \leq i \leq 2 \),

\[ \mu(0, i) = \prod_{j=0}^{k} x_{-2(i+1)+j} = \alpha_i. \]

On the other hand, the right hand side of Equality (2.1) when \( t = 0 \) for each \( 1 \leq i \leq 2 \) is

\[ \frac{a - b}{\theta_i + c} \quad \text{for each} \quad 1 \leq i \leq 2. \]

Suppose that for a certain \( 0 \leq t \leq k - 1 \), we have

\[ \mu(t, i) = \frac{a - b}{(\frac{b}{a})^t \theta_i + c} \quad \text{for each} \quad 1 \leq i \leq 2. \]

Then for each \( 1 \leq i \leq 2 \),

\[ \mu(t + 1, i) = \prod_{j=0}^{k} x_{-2(t+1)+2(i+1)+j} = \prod_{j=0}^{k} x_{-2(t+1)+2j+i} = \frac{a - b}{(\frac{b}{a})^{t+1} \theta_i + c} \]

This completes the proof.

\[ \square \]

**Theorem 2.2.** Let \( x_{-2k-1}, x_{-2k}, \ldots, x_{-1} \) and \( x_0 \) be real numbers such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{b}{\sum_{j=0}^{k} \frac{1}{\theta_j^i}} \) for all \( n \in \mathbb{N} \). Then the solution \( \{x_n\}_{n=-2k-1}^{\infty} \) of Equation (1.1) can be written as

\[ x_n = \begin{cases} x_{-2k-1} \prod_{j=0}^{k} \left( \frac{\theta_j^{i+1}}{\theta_j^i} + c \right), & n = 1, (2k + 3), \ldots, \\ x_{-2k} \prod_{j=0}^{k} \left( \frac{\theta_j^{i+1}}{\theta_j^i} + c \right), & n = 2, (2k + 4), \ldots, \\ x_{-2k+1} \prod_{j=0}^{k} \left( \frac{\theta_j^{i+1}}{\theta_j^i} + c \right), & n = 3, (2k + 5), \ldots, \\ \vdots \\ x_{-1} \prod_{j=0}^{k} \left( \frac{\theta_j^{i+1}}{\theta_j^i} + c \right), & n = 2k + 1, (4k + 3), \ldots, \\ x_0 \prod_{j=0}^{k} \left( \frac{\theta_j^{i+1}}{\theta_j^i} + c \right), & n = 2(k + 1), (4k + 1), \ldots, \\ \end{cases} \]

where \( \theta_i = \frac{a - b - c \alpha_i}{a \alpha_i} \), \( \alpha_i = \prod_{j=0}^{k} x_{n-2(j+1)+i} \), \( i = 1, 2 \).
Proof. We can write the given solution (2.2) in the form

\[ x_{2(k+1)m+2r+i} = X_{−2(k+1)+2r+i} \prod_{j=0}^{m} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c, \quad (2.3) \]

where \( 0 \leq r \leq k \) and \( 1 \leq i \leq 2 \).

We prove the theorem by induction on \( m \).

When \( m = 0 \), using Lemma (2.1) we get

\[ x_{2r+i} = \frac{ax_{−2(k+1)+2r+i}}{b + c \prod_{j=0}^{r} \left( \frac{\theta_j}{b} \right)}, \quad (2.3) \]

On the other hand, using Formula (2.3), when \( m = 0 \) we get

\[ x_{2r+i} = X_{−2(k+1)+2r+i} \prod_{j=0}^{r} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c. \]

as expected.

Now suppose that Formula (2.3) is true for a given \( m \in \mathbb{N} \). Then

\[ x_{2(k+1)(m+1)+2r+i} = \frac{ax_{2(k+1)m+2r+i}}{b + c \prod_{j=0}^{m} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c} = \frac{ax_{−2(k+1)+2r+i} \prod_{j=0}^{m+1} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c}{b + c \prod_{j=0}^{m+1} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c}. \]

This completes the proof. \( \square \)

### 3. Global behavior of Equation (1.1)

In this section, we investigate the global behavior of Equation (1.1) with \( a \neq b \), using the explicit formula of its solution.

We can write the Solution form (2.3) of Equation (1.1) as

\[ x_{2(k+1)m+2r+i} = X_{−2(k+1)+2r+i} \prod_{j=0}^{m} \beta(j,t,i), \quad m = −1, 0, 1, \ldots, \]

where

\[ \beta(j,t,i) = \frac{1}{b + c \prod_{j=0}^{m} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c} \]

We define the set \( S = \{(t,i) : 0 \leq t \leq k \quad \text{and} \quad 1 \leq i \leq 2 \} \).

**Theorem 3.1.** Let \( \{x_n\}_{n=−2k−1}^{∞} \) be a solution of Equation (1.1) such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{a−b}{c} \) for all \( n \in \mathbb{N} \). If \( \alpha_i = \frac{a−b}{c} \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=−2k−1}^{∞} \) is periodic with period prime \( 2k + 2 \).

**Proof.** Assume that \( \alpha_i = \frac{a−b}{c} \) for all \( i \in \{1, 2\} \). Then \( \theta_i = 0 \) for all \( i \in \{1, 2\} \).

Therefore,

\[ x_{2(k+1)m+2r+i} = X_{−2(k+1)+2r+i} \prod_{j=0}^{m} \beta(j,t,i) = \prod_{j=0}^{m} \beta(j,t,i) \prod_{j=0}^{m} \prod_{i=0}^{2} \left( \frac{\theta_j}{b} \right)^{i+1} \theta_j + c. \]

This completes the proof. \( \square \)

In the following theorem, suppose that \( \alpha_i \neq \frac{a−b}{c} \) for all \( i \in \{1, 2\} \).

**Theorem 3.2.** Let \( \{x_n\}_{n=−2k−1}^{∞} \) be a solution of Equation (1.1) such that for any \( 1 \leq i \leq 2, \alpha_i \neq \frac{a−b}{c} \) for all \( n \in \mathbb{N} \). Then the following statements are true.

1. If \( a < b \), then \( \{x_n\}_{n=−2k−1}^{∞} \) converges to 0.
2. If \( a > b \), then \( \{x_n\}_{n=−2k−1}^{∞} \) converges to a period-\( (2k + 2) \) solution.
Proof. 1. If \( a < b \), then \( \beta(j,t,i) \) converges to \( \frac{a}{b} < 1 \) as \( j \to \infty \), for all \( 0 \leq t \leq k \) and \( 1 \leq i \leq 2 \). So, for every pair \((t,i) \in S\) we have for a given \( 0 < \frac{a}{b} < \varepsilon < 1 \) that, there exists \( j_0(t,i) \in \mathbb{N} \) such that, \( \beta(j,t,i) < \varepsilon \) for all \( j \geq j_0(t,i) \). If we set \( j_0 = \max_{(t,i) \in S} j_0(t,i) \), then for all \((t,i) \in S\) we get

\[
|x_{2(k+1)+2i}| = |x_{2(k+1)+2i} - \prod_{j=0}^{m} \beta(j,t,i) |
\]

\[
= |x_{2(k+1)+2i} - \prod_{j=0}^{j_0-1} \beta(j,t,i) | \prod_{j=j_0}^{m} \beta(j,t,i) |
\]

\[
< |x_{2(k+1)+2i} - \prod_{j=0}^{j_0-1} \beta(j,t,i) | e^{\frac{m-j_0+1}{b}}.
\]

As \( m \) tends to infinity, the solution \( \{x_n\}_{n=-2k-1}^{\infty} \) converges to 0.

2. If \( a > b \), then \( \beta(j,t,i) \to 1 \) as \( j \to \infty \), 0 \leq t \leq k \) and \( 1 \leq i \leq 2 \). This implies that, for every pair \((t,i) \in S\), there exists \( j_1(t,i) \in \mathbb{N} \) such that, \( \beta(j,t,i) > 0 \) for all \( j \geq j_1(t,i) \). If we set \( j_1 = \max_{(t,i) \in S} j_1(t,i) \), then for all \((t,i) \in S\) we get

\[
x_{2(k+1)+2i} = x_{2(k+1)+2i} - \prod_{j=0}^{m} \beta(j,t,i)
\]

\[
= x_{2(k+1)+2i} - \prod_{j=0}^{j_1-1} \beta(j,t,i) \exp(\sum_{j=j_1}^{m} \ln(\beta(j,t,i))).
\]

We shall test the convergence of the series \( \sum_{j=j_1}^{m} |\ln(\beta(j,t,i))| \).

Since for all \( 0 \leq t \leq k \) and \( 1 \leq i \leq 2 \) we have \( \lim_{j \to \infty} |\ln(\beta(j,t,i))| \) using L'Hospital's rule we obtain

\[
\lim_{j \to \infty} |\ln(\beta(j+1,t,i))/\ln(\beta(j,t,i))| = \left| \frac{b}{a} \right| < 1.
\]

It follows from the ratio test that the series \( \sum_{j=j_1}^{m} |\ln(\beta(j,t,i))| \) is convergent. This ensures that there are \( 2k+2 \) real numbers \( \mu_{ii}, 0 \leq t \leq k \) and \( 1 \leq i \leq 2 \) such that

\[
\lim_{m \to \infty} x_{2(k+1)+2i} = \mu_{ii}, \quad 0 \leq t \leq k \quad \text{and} \quad 1 \leq i \leq 2,
\]

where

\[
\mu_{ii} = x_{2(k+1)+2i} - \prod_{j=0}^{m} \left( \frac{\beta(j+1,k+1)}{\beta(j,k+1)} \mu_{j+1} + c \right).
\]

\( 0 \leq t \leq k \) and \( 1 \leq i \leq 2 \).

\[\square\]

Example (1) Figure (1) shows that if \( a = 1.2, b = 2.5, c = 1 \) \((a < b)\), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of Equation (1.1) with initial conditions \( x_3 = -5, x_4 = 1.8, x_5 = 0.5, x_2 = -2.1, x_1 = 1.1 \) and \( x_0 = 1.4 \) converges to zero.

Example (2) Figure (2) shows that if \( a = 3, b = 1, c = 2.5 \) \((a > b)\), then the solution \( \{x_n\}_{n=-7}^{\infty} \) of Equation (1.1) with initial conditions \( x_7 = -2.1, x_6 = 1.5, x_5 = 0.2, x_4 = -2.1 \)

\[\square\]

4. Special Cases

In this section, we introduce the solutions, show the existence of periodic solutions and discuss the global behavior of some special cases of Equation (1.1).
4.1 Case $a = b$

In this subsection, we study the equation

$$x_{n+1} = \frac{a x_{n-2k-1}}{a + c \prod_{i=0}^{k} x_{n-2i-1}}, \quad n = 0, 1, \ldots, \quad (4.1)$$

where $a, c$ are positive real numbers, the initial conditions $x_{-2k-1}, x_{-2k}, \ldots, x_{-1}, x_0$ are real numbers and $k$ is a nonnegative integer.

Theorem 4.1. Let $x_{-2k-1}, x_{-2k}, \ldots, x_{-1}$ and $x_0$ be real numbers such that for any $i \in \{1, 2\}$, $a_i \neq -\frac{a_i}{c(a_i+1)}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-2k-1}^{\infty}$ of Equation (4.1) is

$$x_n = \begin{cases} x_{-2k-1} \frac{a+\alpha_2(k+1)}{a+c \alpha_2((k+1)+1)}, & n = 1, (2k+3), \ldots, \\ x_{-2k} \frac{a+\alpha_2(k+1)}{a+c \alpha_2((k+1)+2)}, & n = 2, (2k+4), \ldots, \\ \vdots \\ x_{-1} \frac{a+\alpha_2(k+1)+k}{a+c \alpha_2((k+1)+2)+k}, & n = 2k+1, (4k+3), \ldots, \\ x_0 \frac{a+\alpha_2(k+1)+k}{a+c \alpha_2((k+1)+2)+k}, & n = 2k+2, (4k+4), \ldots \end{cases} \quad (4.2)$$

where $\alpha_i = \prod_{j=0}^{i} x_{-2(i+1)+i-j+1}$.

For simplicity, let $\gamma(j, t, i) = \frac{a+\alpha_2((k+1)+j)}{a+c \alpha_2((k+1)+j+1)}$. Then we can write the Solution (4.2) as

$$x_{2(k+1)m+2r+i} = x_{-2(k+1)+2r+i} \prod_{j=0}^{m} \gamma(j, t, i), \quad \gamma(j, t, i) = \frac{a+\alpha_2((k+1)+j+t)}{a+c \alpha_2((k+1)+j+t+1)}, \quad 0 \leq t \leq k \text{ and } 1 \leq i \leq 2.$$

Theorem 4.2. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a nontrivial solution of Equation (4.1) such that for any $i \in \{1, 2\}$, $a_i \neq -\frac{a}{c(a+1)}$ for all $n \in \mathbb{N}$. If $\alpha_i = 0$, then $\{x_n\}_{n=-2k-1}^{\infty}$ is periodic with period $2k + 2$.

Proof. Assume that $\alpha_i = 0$ for all $i \in \{1, 2\}$. Then $\gamma(j, t, i) = 1$ for all $0 \leq t \leq k$ and $1 \leq i \leq 2$. Therefore,

$$x_{2(k+1)m+2r+i} = x_{-2(k+1)+2r+i} \prod_{j=0}^{m} \gamma(j, t, i) = x_{-2(k+1)+2r+i}, \quad m = \frac{-1, 0, 1, \ldots}{\text{This completes the proof.}}$$

In the following theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1, 2\}$.

Theorem 4.3. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Equation (4.1) such that for any $i \in \{1, 2\}$, $\alpha_i \neq -\frac{a_i}{c(a_i+1)}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.

Proof. It is clear that $\gamma(j, t, i) \to 1$ as $j \to \infty$, $0 \leq t \leq k$ and $1 \leq i \leq 2$. This implies that, for every pair $(t, i) \in S$ there exists $j_2(t, i) \in \mathbb{N}$ such that $\gamma(j, t, i) > 0$ for all $j \geq j_2(t, i)$. If we set $j_2 = \max_{(t, i) \in S} j_2(t, i)$, then for all $(t, i) \in S$ we get

$$x_{2(k+1)m+2r+i} = x_{-2(k+1)+2r+i} \prod_{j=0}^{m} \gamma(j, t, i)$$

$$= x_{-2(k+1)+2r+i} \prod_{j=0}^{j_2-1} \gamma(j, t, i) \prod_{j=j_2}^{m} \frac{1}{\gamma(j, t, i)}$$

We shall show that

$$\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j, t, i)} = \sum_{j=j_2}^{\infty} \ln \frac{a + c \alpha_2((k+1)+j+t)}{a + c \alpha_2((k+1)+j+t+1)} = \infty$$

by considering the series $\sum_{j=j_2}^{\infty} \frac{c b_1}{a + c \alpha_2((k+1)+j+t)}$. As

$$\lim_{j \to \infty} \frac{b_1}{c \alpha_2((k+1)+j+t)} = \frac{1}{a + c \alpha_2((k+1)+j+t)},$$

using the limit comparison test, we get $\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j, t, i)} = \infty$. Therefore, $\{x_n\}_{n=-2k-1}^{\infty}$ converges to 0.

4.2 Case $a = b = c$

When $a = b = c$, Equation (1.1) reduced to the equation

$$x_{n+1} = \frac{x_{n-2k-1}}{1 + \prod_{i=0}^{k} x_{n-2i-1}}, \quad n = 0, 1, \ldots, \quad (4.3)$$

where the initial conditions $x_{-2k-1}, \ldots, x_{-1}, x_0$ are real numbers and $k$ is a nonnegative integer.

Theorem 4.4. Let $x_{-2k-1}, \ldots, x_{-1}$ and $x_0$ be real numbers such that for any $i \in \{1, 2\}$, $a_i \neq -\frac{a_i}{c(a_i+1)}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-2k-1}^{\infty}$ of Equation (4.3) is

$$x_n = \begin{cases} x_{-2k-1} \frac{1+\alpha_1(k+1)}{1+\alpha_1((k+1)+1)}, & n = 1, (2k+3), \ldots, \\ x_{-2k} \frac{1+\alpha_1(k+1)}{1+\alpha_1((k+1)+2)}, & n = 2, (2k+4), \ldots, \\ \vdots \\ x_{-1} \frac{1+\alpha_1(k+1)+k}{1+\alpha_1((k+1)+2)+k}, & n = 2k+1, (4k+3), \ldots, \\ x_0 \frac{1+\alpha_1(k+1)}{1+\alpha_1((k+1)+2)}, & n = 2k+2, (4k+4), \ldots \end{cases} \quad (4.4)$$

where $\alpha_i = \prod_{j=0}^{i} x_{-2(i+1)+i-j+1}, \quad i = 1, 2$. 

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Theorem 4.5. Let \( \{x_n\}_{n=-2k-1}^\infty \) be a nontrivial solution of Equation (4.3) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{1}{n+1} \) for all \( n \in \mathbb{N} \). If \( \alpha_i = 0 \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=-2k-1}^\infty \) is periodic with prime period \( 2k+2 \).

In the following theorem, suppose that \( \alpha_i \neq 0 \) for all \( i \in \{1, 2\} \).

Theorem 4.6. Let \( \{x_n\}_{n=-2k-1}^\infty \) be a solution of Equation (4.3) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{1}{n+1} \) for all \( n \in \mathbb{N} \). Then \( \{x_n\}_{n=-2k-1}^\infty \) converges to 0.

Example (4) Figure (4) shows that if \( a = c = 1.3, b = 2 \), then the solution \( \{x_n\}_{n=-5}^\infty \) of Equation (4.3) with initial conditions \( x_{-5} = -2, x_{-4} = -1.8, x_{-3} = 1.5, x_{-2} = 2.1, x_{-1} = -1.1 \) and \( x_0 = 1.2 \) converges to zero.

Example (5) Figure (5) shows that if \( a = b = c \), then the solution \( \{x_n\}_{n=-7}^\infty \) of Equation (4.3) with initial conditions \( x_{-7} = 1.1, x_{-6} = -1.6, x_{-5} = -1.2, x_{-4} = 1.5, x_{-3} = 2.1, x_{-2} = 1.7, x_{-1} = -2 \) and \( x_0 = 5.8 \) converges to zero.

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