On a subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number

Jenifer Arulmani

Abstract
In this paper we defined a new subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number using the Frasin differential operator. We find some coefficient bounds and solve the linear functional $|a_3 - \mu a_3^2|$. Also we obtained various results proved by several authors as particular cases.

Keywords
Bi-Univalent, Shell-like, Fibonacci Number, Differential operator.

AMS Subject Classification
30C45.

1 Department of Mathematics, Presidency College(Autonomous), Chennai-600005,Tamil Nadu, India.
*Corresponding author: jeniferarulmani06@gmail.com;
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1. Introduction

We denote by $A$ the class of regular functions defined in the open unit disk $\Delta = \{z/|z|<1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$ and the Taylor series expansion,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Consider $S$ to be the class of univalent functions in $A$. For any two analytic functions $f(z)$ and $g(z)$ in $\Delta$. We say that $f(z)$ is subordinate to $g(z)$ [9], (symbolically, $f \prec g$) if there exists a function $\phi(z)$ analytic in $\Delta$ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$f(z) = g(\phi(z)), (|z| < 1).$$

By the Koebe-one quater theorem[4](Theorem.2.3 pg.31) , we know that "The range of every function of the class $S$ contains a disk $\{w : |w| < 1/4\}"$. Hence there exists inverse $f^{-1}$ for every function $f \in S$, defined by

$$f^{-1}(f(z)) = z, (z \in \Delta); \quad \text{and}$$

$$f^{-1}(w) = w, (|w| < r_0(f) : r_0(f) \geq 1/4).$$

Where the inverse of $f$ is given by,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 w^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + ...$$

$$= g(w).$$

A function $f \in A$ is said to be bi-univalent if both $f$ and $f^{-1}$ (its inverse) are univalent in $\Delta$. We denote by $\Sigma$ the class of bi-univalent and analytic functions in $\Delta$ of the form (1.1).

Using the binomial series,

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j;$$

$m \in \mathbb{N} = 1, 2, ...$ and $j \in \mathbb{N}_0 = 0, 1, 2, ...$

Frasin [5] defined the following differential operator for function $f \in A$,

$$D^\phi f(z) = f(z)$$

$$D_{m,\lambda}^m f(z) = (1-\lambda)^m f(z) + (1-(1-\lambda)^m)z f'(z)$$

$$= D_{m,\lambda} f(z), \quad (\lambda > 0; m \in \mathbb{N}).$$

In general,

$$D_{m,\lambda}^n f(z) = D_{m,\lambda} (D_{m,\lambda}^{n-1} f(z)), n \in \mathbb{N}_0$$

$$= z + \sum_{k=2}^{\infty} [1 + (k-1)c_j^m(\lambda)]^n a_k z^k.$$
where, \( c_j^m(\lambda) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \).

**Remarks:**

1. For \( m = 1 \), we get the Al-oboudi differential operator, \( D^p_{1,\lambda} \) [1].
2. For \( m = \lambda = 1 \), we get the Salagean differential operator, \( D^p \) [11].

For \( f \in A \) the class \( SL \) of shell-like functions which is the subclass of the class \( S^* \) of starlike functions was first introduced by Sokol [12] in 1999 as below,

**Definition 1.1.** [12] A function \( f \in A \) having the series expansion (1.1) is said to be in the class \( SL \) of starlike shell-like functions if it satisfies the following conditions:

\[
\frac{zf'(z)}{f(z)} < \bar{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{1 - \sqrt{3}}{2} \approx -0.618 \).

In the year 2011, Dziok et al. [2], introduced the class \( KSL \) of convex functions related to a shell-like curves as follows:

**Definition 1.2.** [2] A function \( f \in A \) of the form (1.1) belongs to the class \( KSL \) of convex shell-like functions if it satisfies the following condition:

\[
1 + \frac{zf'(z)}{f(z)} < \bar{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{1 - \sqrt{3}}{2} \approx -0.618 \).

Again Dziok et al. [3] in the year 2011, defined the following class \( SLM_\alpha \) of \( \alpha \)-convex shell-like functions.

**Definition 1.3.** [3] A function \( f \in A \) of the form (1.1) belongs to the class \( SLM_\alpha \) of \( \alpha \)-convex shell-like functions if it satisfies the following condition:

\[
(1 - \alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) < \bar{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

where \( \tau = \frac{1 - \sqrt{3}}{2} \approx -0.618 \).

We note that \( SLM_0 \equiv SL \) and \( SLM_1 \equiv KSL \). We consider \( \tau = \frac{1 - \sqrt{3}}{2} \approx -0.618 \) and \( \bar{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \) throughout this paper. The function \( \bar{p}(z) \) does not belong to the class \( S \). Since \( \bar{p}(z) \) is univalent in the disc \( |z| < \tau^2 \simeq 0.38 \). We can observe the following from \( \bar{p}(z) \) [6]: \( \bar{p}(0) = \bar{p}(\frac{1}{2\tau}) = 1 \); \( \bar{p} \) takes the unit circle to a curve described by \((10x - \sqrt{5})y^2 = (\sqrt{5}x - 1)^2\), which is translated and revolved trisectrix of Maclaurin. The curve \( \bar{p}(re^\theta) \) is a closed curve without any loops for \( 0 < r < r_0 = \tau^2 \simeq 0.38 \). For \( r_0 < r < 1 \), it has a loop, and for \( r = 1 \) it has a vertical asymptote. In the year 2016, Raina and Sokol [10] proved the following,

\[
\bar{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
\]

\[
= 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1})\tau^n z^n
\]

where \( u_n = \frac{(1 - \tau)\tau - \tau^n}{\sqrt{5}} \), such that

\[
u_n = u_{n-2} + u_{n-1} \text{ for } n = 2, 3, ...
\]

By simple calculation we can decompose all the higher powers \( \tau^n \) as a linear combination of \( \tau \) and 1. The resulting recurrence relationships yield Fibonacci number \( u_n \),

\[
\tau^n = u_n \tau + u_{n-1}.
\]

Thus \( \bar{p}(z) \) is related to Fibonacci number. So we can rewrite \( \bar{p}(z) \) as,

\[
\bar{p}(z) = 1 + \sum_{n=1}^{\infty} \bar{p}_n \tau^n z^n
\]

where \( \bar{p}_n = (u_{n-1} + u_{n+1}) \). Now using (1.2) in (1.3) we have,

\[
\bar{p}(z) = 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + ...
\]

Motivated by the works of earlier authors we define a new subclass of bi-univalent functions related to shell-like curves connected to Fibonacci number using Frasin differential operator.

**Definition 1.4.** A function \( f(z) \in \sum \) given by (1.1) is said to be in the class \( \alpha - SLM_\sum(n,m,\lambda,\bar{p}(z)) \) if the following conditions are satisfied,

\[
(1 - \alpha)\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)} + \alpha\left(\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)}\right)' < \bar{p}(z)
\]

\[
(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \alpha \leq 1, z \in \Delta)
\]

and

\[
(1 - \alpha)\frac{D^{n+1}_{m,\lambda} g(w)}{D^n_{m,\lambda} g(w)} + \alpha\left(\frac{D^{n+1}_{m,\lambda} g(w)}{D^n_{m,\lambda} g(w)}\right)' < \bar{p}(w)
\]

\[
(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \alpha \leq 1, w \in \Delta)
\]

**Remarks**

1. \( \alpha - SLM_\sum(1,n,\lambda,\bar{p}(z)) = SLM_{\alpha,\sum}(n,\bar{p}(z)) \), the class of bi-univalent functions defined by Gurmeet Singh et al. [7].
2. \( \alpha - SLM_\sum(1,0,1,\bar{p}(z)) = SLM_{\alpha,\sum}(\bar{p}(z)) \), the class of bi-univalent functions defined by Guney et al. [6].

We consider \( \mathbb{P} \) to be the class of Caratheodory functions. i.e., for \( p \in \mathbb{P}, \Re\{p(z)\} > 0 \), \( p(z) \) is analytic in \( \Delta \) and have the series expansion

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \Delta.
\]

**Lemma 1.5.** If \( p(z) \in \mathbb{P} \), then \( |p_n| \leq 2 \) for each \( n = 1, 2, ... \).
2. Coefficient estimate for the functions in the class $\alpha - \text{SLM}_\Sigma(n, m, \lambda, \bar{p}(z))$

**Theorem 2.1.** If $f(z)$ is in the class $\alpha - \text{SLM}_\Sigma(n, m, \lambda, \bar{p}(z))$, then:

$$|a_2| \leq \frac{|\tau|}{\sqrt{(c_j^m(\lambda))(\tau_\zeta + \psi)}}$$

and

$$|a_3| \leq \frac{|\tau|(|\psi - \tau(1 + 3\alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}|}{2\Psi[\tau_\zeta + \psi]}$$

where

$$\varsigma = 2(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n - (1 + 3\alpha)(1 + c_j^m(\lambda))^{2n},$$

$$\psi = (1 + \alpha)^2c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}(1 - 3\tau)$$

and

$$\Psi = (1 + 2\alpha)(c_j^m(\lambda))^2(1 + 2\varsigma m_j^m(\lambda))^n.$$

**Proof.** Since $f(z) \in \alpha - \text{SLM}_\Sigma(n, m, \lambda, \bar{p}(z))$, from the definition 1.4, we have

$$(1 - \alpha)\frac{D_{m, \lambda}^{n+1}f(z)}{D_{m, \lambda}^n f(z)} + \alpha \frac{(D_{m, \lambda}^{n+1}f(z))'}{(D_{m, \lambda}^n f(z))'} = \bar{p}(r(z)) \quad (2.1)$$

and

$$(1 - \alpha)\frac{D_{m, \lambda}^{n+1}g(w)}{D_{m, \lambda}^n g(w)} + \alpha \frac{(D_{m, \lambda}^{n+1}g(w))'}{(D_{m, \lambda}^n g(w))'} = \bar{p}(s(w)) \quad (2.2)$$

where $r(z)$ and $s(w)$ are analytic functions in $\Delta$ with $r(0) = s(0) = 0$ and $|r(z)| < 1$ and $|s(w)| < 1$.

Now define the function,

$$h(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + r_1z + r_2z^2 + ...$$

Then,

$$\bar{p}(r(z)) = 1 + \frac{r_1}{2}\tau z + \frac{1}{2}(r_2 - \frac{r_1^2}{2} + \frac{3r_1^2}{4})\tau z^2 + ... \quad (2.3)$$

Similarly we define the function,

$$k(w) = \frac{1 + s(z)}{1 - s(z)} = 1 + s_1z + s_2z^2 + ...$$

Then,

$$\bar{p}(s(w)) = 1 + \frac{s_1}{2}\tau w + \frac{1}{2}(s_2 - \frac{s_1^2}{2} + \frac{3s_1^2}{4})\tau w^2 + ... \quad (2.4)$$

and by considering the LHS of (2.1), we have

$$(1 - \alpha)\frac{D_{m, \lambda}^{n+1}f(z)}{D_{m, \lambda}^n f(z)} + \alpha \frac{(D_{m, \lambda}^{n+1}f(z))'}{(D_{m, \lambda}^n f(z))'} = 1 + (1 + \alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^n a_2 z + [2(1 + 2\alpha)c_j^m(\lambda)(1 + 2c_j^m(\lambda))^n a_2 z + (1 + 3\alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}a_2^2]z^2 + ...$$

and

$$(1 - \alpha)\frac{D_{m, \lambda}^{n+1}g(w)}{D_{m, \lambda}^n g(w)} + \alpha \frac{(D_{m, \lambda}^{n+1}g(w))'}{(D_{m, \lambda}^n g(w))'} = 1 - (1 + \alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^n a_2 w + [2(1 + 2\alpha)c_j^m(\lambda)(1 + 2c_j^m(\lambda))^n a_2 w - a_3] + (1 + 3\alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}a_2^2 w^2 + ...$$

Using (2.3), (2.4), and the above two equations in (2.1) and (2.2) and equating the coefficients of $z, z^2, w$ and $w^2$ we have the following equations,

$$c_j^m(\lambda)(1 + c_j^m(\lambda))^n (1 + \alpha)a_2 = \frac{r_1}{2}\tau \quad (2.5)$$

$$2(1 + 2\alpha)c_j^m(\lambda)(1 + 2c_j^m(\lambda))^n a_3 - (1 + 3\alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}a_2^2 = \frac{r_2^2}{2} + \frac{3r_1^2}{4}\tau^2 \quad (2.6)$$

Using (2.5) and (2.6),

$$c_j^m(\lambda)(1 + c_j^m(\lambda))^n (1 + \alpha)a_2 = \frac{s_1}{2}\tau \quad (2.7)$$

and

$$2(1 + 2\alpha)c_j^m(\lambda)(1 + 2c_j^m(\lambda))^n a_3 - (1 + 3\alpha)c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}a_2^2 = (r_2^2 - \frac{s_1^2}{2})\tau + \frac{3s_1^2}{4}\tau^2 \quad (2.8)$$

from (2.5) and (2.6),

$$r_1 = -s_1, \quad (2.9)$$

also

$$2|c_j^m(\lambda)|^2 [1 + c_j^m(\lambda)]^{2n}(1 + \alpha)^2a_2^2 = \frac{1}{4}(r_1^2 + s_1^2)\tau^2 \quad (2.10)$$

Adding (2.6) and (2.8), we get

$$a_2^2 [4c_j^m(\lambda)(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n - 2c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}(1 + 3\alpha)](r_2 + s_2)\tau - \frac{1}{4}(r_1^4 + s_1^4)\tau + \frac{3}{4}(r_1^2 + s_1^2)\tau^2. \quad (2.11)$$

Using (2.10) in the above equation we get

$$a_2^2 = \frac{(r_2 + s_2)\tau^2}{4|c_j^m(\lambda)|\tau_\zeta + \psi} \quad (2.12)$$

where

$$\varsigma = 2(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n - (1 + 3\alpha)(1 + c_j^m(\lambda))^{2n}$$

and

$$\psi = (1 + \alpha)^2c_j^m(\lambda)(1 + c_j^m(\lambda))^{2n}(1 - 3\tau).$$
By using Lemma 1.5 and triangular inequality we get the required inequality for \( |a_2| \).
To estimate \( |a_1| \) first we subtract (2.8) from (2.6) and then by using (2.9), we get
\[
4e^m(\lambda)(1 + 2\alpha)(1 + 2e^m(\lambda))^n(a_3 - a_3^2) = (r_2 - s_2)\frac{\tau}{2}.
\]
(2.13)
Now by using (2.12) in the above equation we get the coefficient bound for \( |a_3| \).

For \( m = 1 \) in theorem 2.1 we get the following corollary,

**Corollary 2.2.** If \( f(z) \in SLM_{w, \lambda}^\alpha(n, \bar{\omega}(z)) \), then
\[
|a_2| \leq \frac{|\tau|}{\sqrt{3}}
\]
and
\[
|a_3| \leq \frac{|\tau|(1 + \lambda)^3(1 + \alpha)^2 (1 - 3\tau) - \tau(1 + 3\alpha)\lambda}{2\xi(1 + 2\alpha)(1 + 2\lambda)^n}
\]
where \( \xi = \lambda(1 + 2\alpha)(1 + \lambda)^n - (1 + 3\alpha)(1 + \lambda)^2n + (1 + \alpha)^2\lambda(1 + \lambda)^2(1 - 3\tau) \) which agrees with the results of Gurmeet Singh et al. Corollary 1.

For \( m = \lambda = 1 \) in theorem 2.1 gives the following corollary,

**Corollary 2.3.** If \( f(z) \in SLM_{\alpha, \lambda}^w(n, \bar{\omega}(z)) \), then
\[
|a_2| \leq \frac{1}{\sqrt{4(1 + \alpha)^2 + 2(1 + 2\alpha)3^\alpha - \eta^n}}
\]
and
\[
|a_3| \leq \frac{|\tau|^3[(1 + \alpha)^2 - \eta^2]}{2(1 + 2\alpha)^3\tau[(1 + \alpha)^2 + 2(1 + 2\alpha)^3\tau - \eta^2\tau]}
\]
where \( \eta = 3\alpha^2 + 9\alpha + 4 \) which agrees with the results of Gurmeet Singh et al. Corollary 1.

On substituting \( m = \lambda = 1 \) and \( n = 0 \) in theorem 2.1 gives the following corollary,

**Corollary 2.4.** If \( f(z) \in SL_{\alpha, \lambda}^w(\bar{\omega}(z)) \), then
\[
|a_2| \leq \frac{|\tau|}{\sqrt{(1 + \alpha)^2 - (1 + \alpha)(2 + 3\alpha)\tau}}
\]
and
\[
|a_3| \leq \frac{|\tau|[(1 + \alpha)^2 - (3\alpha^2 + 9\alpha + 4)\tau]}{2(1 + 2\alpha)(1 + \alpha)^2 - (1 + \alpha)(2 + 3\alpha)\tau]}
\]
which agrees with the results of Guney et al. Corollary 1.

On substituting \( m = 1 \) and \( n = 0 \) in theorem 2.1 gives the following corollary,

**Corollary 2.5.** If \( f(z) \in SL_{\lambda}(\bar{\omega}(z)) \), then
\[
|a_2| \leq \frac{|\tau|}{\sqrt{1 - 2\tau}}
\]
and
\[
|a_3| \leq \frac{|\tau|(1 - 4\tau)}{2(1 - 2\tau)}
\]
which agrees with the results of Guney et al. Corollary 1.

Also for \( m = \lambda = 1 \) and \( n = 0 \) in theorem 2.1 gives the following corollary,

**Corollary 2.6.** If \( f(z) \in KSL_{\lambda}(\bar{\omega}(z)) \), then
\[
|a_2| \leq \frac{|\tau|}{\sqrt{4 - 10\tau}}
\]
and
\[
|a_3| \leq \frac{|\tau|(1 - 4\tau)}{3(2 - 5\tau)}
\]
which agrees with the results of Guney et al. Corollary 2.

**3. Fekete-Szego Inequality for the function class** \( \alpha - SLM_{\lambda}(n, m, \lambda, \bar{\omega}(z)) \)

**Theorem 3.1.** If \( f(z) \in \text{the class } \alpha - SLM_{\lambda}(n, m, \lambda, \bar{\omega}(z)) \) then,
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|\tau|}{|1 - \mu^2|}, & |\mu - 1| \leq \frac{\tau}{\lambda} \\
\frac{|\tau|}{|1 - \mu^2|}, & |\mu - 1| \geq \frac{\tau}{\lambda}
\end{array} \right.
\]
(3.1)
where
\[
Y = c_j^m(\lambda)(\tau[2(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n - (1 + 3\alpha)(1 + c_j^m(\lambda))2n] + c_j^m(\lambda)(1 + c_j^m(\lambda))2n(1 + \alpha)^2(1 - 3\tau))
\]
and
\[
X = 2|\tau|c_j^m(\lambda)(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n
\]

**Proof.** From (2.12) and (2.13), we have
\[
a_3 - \mu a_2^2 = \frac{\tau(c_2 - d_2)}{8c_j^m(\lambda)(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n} + (c_2 + d_2)\chi(\mu)
\]
(3.2)
where
\[
\chi(\mu) = \frac{(1 - \mu)^2}{4c_j^m(\lambda)(\tau\zeta + \phi)}.
\]
with \( \zeta = 2(1 + 2\alpha)(1 + 2c_j^m(\lambda))^n - (1 + 3\alpha)(1 + c_j^m(\lambda))2n \) and \( \phi = c_j^m(\lambda)(1 + c_j^m(\lambda))2n(1 + \alpha)^2(1 - 3\tau) \).
The above equation can be expressed as,
\[
\begin{align*}
  a_3 - \mu a_2^2 &= |\chi(\mu)| + \frac{\tau}{8\epsilon_f^{\alpha}(\lambda)(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))} |\zeta_2| \\
  &\quad + |\chi(\mu)| - \frac{\tau}{8\epsilon_f^{\alpha}(\lambda)(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))} |\zeta_2| \\
  &\quad + |\chi(\mu)| - \frac{\tau}{8\epsilon_f^{\alpha}(\lambda)(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))} |\zeta_2|.
\end{align*}
\]
(3.3)

Taking modulus on the above equation, we obtain,
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\zeta_1|}{58}, & 0 \leq |\chi(\mu)| \leq \frac{|\zeta_1|}{58} \\
\frac{4|\chi(\mu)|}{58}, & |\chi(\mu)| > \frac{|\zeta_1|}{58}.
\end{cases}
\]
(3.4)

where \( \delta = c_f^{\alpha}(\lambda)(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))^{n} \). Using the above equation we can get the desired bound for the Fekete-Szego problem.

By varying the parameters in Theorem 3.1 we get the following corollary.

When we consider \( m = 1 \) in Theorem 3.1 we get the following corollary, which is proved by Gurney et al.\[6\] in Theorem 11.

**Corollary 3.2.** If \( f(z) \in SL^\lambda_{\alpha,\Sigma}(\bar{\beta}(z)) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\tau}{2|\lambda(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))^{n}|}, & |\mu - 1| \leq \frac{2}{21|\tau|} \\
\frac{2}{21|\tau|} |\lambda(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))^{n}|, & |\mu - 1| > \frac{2}{21|\tau|}.
\end{cases}
\]
(3.5)

where
\[
X = \lambda |\tau[2(1 + 2\alpha)(1 + 2\epsilon_f^{\alpha}(\lambda))^{n} - (1 + 3\alpha)(1 + \lambda)^{2n}] + \lambda(1 + \lambda)^{2n}(1 + \lambda)^{2n}\). \]

If we consider \( m = \lambda = 1 \) in Theorem 3.1 we get the following corollary, which is proved by Gurney et al.\[8\] in Corollary 12.

**Corollary 3.3.** If \( f(z) \in SL^\lambda_{\alpha,\Sigma}(\bar{\beta}(z)) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\tau}{2(1 + 2\alpha)|\lambda(1 + 2\epsilon_f^{\alpha}(\lambda))^{n}|}, & |\mu - 1| \leq \frac{2}{21|\tau|} \\
\frac{2}{21|\tau|} |\lambda(1 + 2\alpha)|\lambda(1 + 2\epsilon_f^{\alpha}(\lambda))^{n}|, & |\mu - 1| > \frac{2}{21|\tau|}.
\end{cases}
\]
(3.6)

where
\[
Z = (2(1 + 2\alpha)^3 - (6\alpha^2 + 9\alpha + 4)^n + (1 + \alpha)^2 4^n).
\]

If we consider \( m = \lambda = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Gurney et al.\[6\] in Theorem 11.

**Corollary 3.4.** If \( f(z) \in SL^\lambda_{\alpha,\Sigma}(\bar{\beta}(z)) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{p}{2|\tau(1 + 2\alpha)|}, & |\mu - 1| \leq \frac{p}{2|\tau(1 + 2\alpha)|} \\
\frac{p}{2|\tau(1 + 2\alpha)|}, & |\mu - 1| > \frac{p}{2|\tau(1 + 2\alpha)|}.
\end{cases}
\]
(3.7)

where \( P = (1 + \alpha)(1 + \alpha) - (2 + 3\alpha)^2 \).

If we consider \( m = \lambda = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Gurney et al.\[6\] in Corollary 4.

**Corollary 3.5.** If \( f(z) \in SL^\lambda_{\alpha,\Sigma}(\bar{\beta}(z)) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\zeta_1|}{21|\tau|}, & |\mu - 1| \leq \frac{3}{21|\tau|} \\
\frac{21|\tau|}{3(2 - 5\tau)}, & |\mu - 1| > \frac{3}{21|\tau|}.
\end{cases}
\]

If we consider \( m = \lambda = \alpha = 1 \) and \( n = 0 \) in Theorem 3.1 we get the following corollary, which is proved by Gurney et al.\[6\] in Corollary 5.

**Corollary 3.6.** If \( f(z) \in KSL^\lambda_{\alpha,\Sigma}(\bar{\beta}(z)) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{2}{3|\tau|}, & |\mu - 1| \leq \frac{2}{3|\tau|} \\
\frac{2}{3|\tau|}, & |\mu - 1| > \frac{2}{3|\tau|}.
\end{cases}
\]

References

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