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Certain fractional integral inequalities using generalized Katugampola fractional integral operator

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Abstract
The purpose of this paper is to obtain some new fractional integral inequalities involving convex functions by applying generalized Katugampola fractional integral operator.

Keywords
Generalized Katugampola fractional integral, convex functions and inequality.

AMS Subject Classification
26A99, 26D10.

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Contents

1. Introduction

In the last years, a number of researchers have found inequalities and application by means results about fractional integrals such as Riemann-Liouville fractional integral operator, Hadamard integral operator, Saigo integral operator, Erdeyi-Kober integral operator, Katugampola fractional integral, see [1–3, 5, 7–12, 15–17, 19, 21]. Fractional inequalities play major role in the development of fractional differential, integral equations and other fields of sciences and technology. Recently, In 2019, V.L. Chinchane and D.B. Pachpatte have studied inequalities using Saigo fractional integral, see [6].

Theorem 1.1. *Let f, h be two positive continuous functions on $[0,\infty)$ and $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing, f is increasing on $[0,\infty)$ and for any convex function ϕ, $\phi(0)=0$, then for $t>0$, $\alpha > max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$, we have*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}[f(t)]}{I_{0,t}^{\alpha,\beta,\eta}[h(t)]} \geq \frac{I_{0,t}^{\alpha,\beta,\eta}[\phi(f(t))]}{I_{0,t}^{\alpha,\beta,\eta}[\phi(h(t))]}. \tag{1.1}$$

and

Theorem 1.2. *Let f, h be two positive continuous functions on $[0,\infty)$ and $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing, f is increasing on $[0,\infty)$ and for any convex function ϕ, $\phi(0)=0$, then we have inequality*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}[f(t)]I_{0,t}^{\psi,\delta,\zeta}[\phi(h(t))]+I_{0,t}^{\psi,\delta,\zeta}[f(t)]I_{0,t}^{\alpha,\beta,\eta}[\phi(h(t))]}{I_{0,t}^{\alpha,\beta,\eta}[h(t)]I_{0,t}^{\psi,\delta,\zeta}[\phi(f(t))]+I_{0,t}^{\psi,\delta,\zeta}[h(t)]I_{0,t}^{\alpha,\beta,\eta}[\phi(f(t))]} \geq 1, \tag{1.2}$$

where for all $t>0$, $\alpha > max\{0,-\beta\}$, $\psi > max\{0,-\delta\}$, $\beta<1$, $\beta-1<\eta<0$, $\delta<1$, $\delta-1<\zeta<0$.

In the literature, some fractional inequalities are obtain by using Generalized Katugampola fractional integral, see [1, 4, 14, 18]. Motivated by above work in this paper we have obtain some new inequalities using generalized Katugampola fractional integral for convex functions.

2. Preliminaries

Here, we devoted to basic concepts of Generalized Katugampola fractional integral,see [1, 13, 20].

Definition 2.1. *Consider the space $X_c^p(a,b)(c\in\mathbb{R}, 1\le p\le\infty)$, of those complex valued Lebesgue measurable functions f on (a,b) for which the norm $||f||_{X_c^p}<\infty$, such that*

$$||f||_{X_c^p}=\left(\int_x^b |x^c f|^p \frac{dx}{x}\right)^{\frac{1}{p}}, (1\le p<\infty)$$

and

$$||f||_{X_c^p}=supess_{x\in(a,b)}[x^c|f|].$$

In particular, when $c=\frac{1}{p}$, the space $X_c^p(a,b)$ coincides with the space $L^p(a,b)$

Definition 2.2. *The left and right sided fractional integrals of a function f where $f\in x_c^p(a,b)$, $\alpha>0$ and $\beta,\rho,\eta,k\in\mathbb{R}$, are defined respectively by*

$$ {}^{\rho}\mathscr{I}^{\alpha,\beta}_{a+;\eta,k}f(x)=\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}f(\tau)d\tau, \quad 0\le a<x<b\le\infty, \tag{2.1}$$

and

$$ {}^{\rho}\mathscr{I}^{\alpha,\beta}_{b-;\eta,k}f(x)=\frac{\rho^{1-\beta}x^{k\rho}}{\Gamma(\alpha)}\int_x^b \frac{\tau^{k+\rho-1}}{(\tau^\rho-x^\rho)^{1-\alpha}}f(\tau)d\tau, \quad 0\le a<x<b\le<b\le\infty, \tag{2.2}$$

if the integral exist.
To represent and discuss our new results in this paper we use the left sided fractional integrals, the right sided fractional can be proved similarly, also we consider $a=0$, in (2.1), to obtain

$$ {}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}f(x)=\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}f(\tau)d\tau. \tag{2.3}$$

The above fractional integrals has the following composition (index) formulae

$$\begin{aligned} {}^{\rho}\mathscr{I}^{\alpha_1,\beta_1}_{a+;\eta_1,k_1}\,{}^{\rho}\mathscr{I}^{\alpha_2,\beta_2}_{a+;\eta_2,-\rho\eta_1}f(x)&={}^{\rho}\mathscr{I}^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a+;\eta_2,k_1}f(x),\\ {}^{\rho}\mathscr{I}^{\alpha_1,\beta_1}_{b-;\eta_1,-\rho\eta_2}\,{}^{\rho}\mathscr{I}^{\alpha_2,\beta_2}_{b-;\eta_2,k_2}f(x)&={}^{\rho}\mathscr{I}^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a+;\eta_1,k_2}f(x).\end{aligned} \tag{2.4}$$

For the convenience of establishing our results we define the following function as in [1]: let $x>0,\alpha>0,\rho,k,\beta,\eta\in\mathbb{R}$, then

$$\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta)=\frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)}\rho^{-\beta}x^{k+\rho(\eta+\alpha)}. \tag{2.5}$$

Remark 2.3. *The fractional integral (2.1) contain five well-known fractional integral as its particular cases, see [1, 13, 20]*

1. *Setting $k=0,\eta=0,a=0$ and taking the limit $\rho\to 1$ in (2.1), the integral operator (2.1) reduces to the Riemann-Liouville fractional integral.*
2. *Setting $k=0,\eta=0$ and taking the limit $\rho\to1$ in (2.1), the integral operator (2.1) reduces to the Liouville fractional integral.*
3. *Setting $\beta=\alpha,\eta=0$ and taking the limit $\rho\to0^+$ with L' Hospital rule in (2.1), the integral operator (2.1) reduces to the Hadamard fractional integral.*
4. *Setting $\beta=0,k=-\rho(\alpha+\eta)$ in (2.1), the integral operator (2.1) reduces to the Erdelyi-Kober fractional integral.*
5. *Setting $\beta=\alpha,k=0$ and $\eta=0$ in (2.1), the integral operator (2.1) reduces to the Katugampola fractional integral.*

3. Fractional integral inequalities involving convex functions

In this section, we prove some fractional integral inequalities involving convex function using generalized Katugampola fractional integral,

Theorem 3.1. *Let p, r be two positive continuous functions on $[0,\infty)$ and $p\le r$ on $[0,\infty)$. If $\frac{p}{r}$ is decreasing, p is increasing on $[0,\infty)$ and for any convex function Φ, $\Phi(0)=0$, then for all $\alpha\ge0$, $t>0,\beta,\rho,\eta,k\in\mathbb{R}$ we have,*

$$\frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[r(t)]}\ge\frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\Phi(p(t))]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\Phi(r(t))]}. \tag{3.1}$$

Proof. If the function Φ is convex with $\Phi(0)=0$, then the function $\frac{\Phi(t)}{t}$ is increasing. Since p is increasing, then $\frac{\Phi(p(t))}{r(t)}$ is also increasing. Clearly $\frac{p(t)}{r(t)}$ is decreasing, for all $\tau,\sigma\in[0,\infty)$, and

$$\left(\frac{\Phi(p(\tau))}{p(\tau)}-\frac{\Phi(p(\sigma))}{p(\sigma)}\right)\left(\frac{p(\sigma)}{r(\sigma)}-\frac{p(\tau)}{r(\tau)}\right)\ge0, \tag{3.2}$$

which implies that

$$\begin{aligned}&\frac{\Phi(p(\tau))}{p(\tau)}\frac{p(\sigma)}{r(\sigma)}+\frac{\Phi(p(\sigma))}{p(\sigma)}\frac{p(\tau)}{r(\tau)}\\ &-\frac{\Phi(p(\tau))}{p(\tau)}\frac{p(\tau)}{r(\tau)}-\frac{\Phi(p(\sigma))}{p(\sigma)}\frac{p(\sigma)}{r(\sigma)}\ge0.\end{aligned} \tag{3.3}$$

Multiplying equation (3.3) by $r(\tau)r(\sigma)$, we have

$$\begin{aligned}&\frac{\Phi(p(\tau))}{p(\tau)}p(\sigma)r(\tau)+\frac{\Phi(p(\sigma))}{p(\sigma)}p(\tau)r(\sigma)-\\ &\frac{\Phi(p(\tau))}{p(\tau)}p(\tau)r(\sigma)-\frac{\Phi(p(\sigma))}{p(\sigma)}p(\sigma)r(\tau)\ge0.\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{\rho^{1-\beta}t^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}$, $\tau\in(0,t)$ which is positive, and integrating obtained result with respect to τ from 0 to t, we have

$$p(\sigma)^{\rho}\,\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(x)}r(t)\right]+\frac{\Phi(p(\sigma))}{p(\sigma)}r(\sigma)^{\rho}\,\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[p(t)\right] -r(\sigma)^{\rho}\,\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}p(t)\right]-\frac{\Phi(p(\sigma))}{p(\sigma)}p(\sigma)^{\rho}\,\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right] \geq 0. \tag{3.5}$$

Multiplying both sides of (3.5) by $\frac{\rho^{1-\beta}t^{k}}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(t^{\rho}-\sigma^{\rho})^{1-\alpha}},\sigma\in(0,t)$ which is positive, and integrating obtained result with respect to σ from 0 to t, we have

$$\begin{aligned}&[p(t)]\,{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]+\\&{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[p(t)\right]\\&\geq{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}p(t)\right]+\\&{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}p(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right].\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}&{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[p(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)r(t)}\right]\\&\geq{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(f(t))}{p(t)}p(t)\right],\end{aligned} \tag{3.7}$$

$$\frac{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[p(t)\right]}{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right]}\geq\frac{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}p(t)\right]}{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]}. \tag{3.8}$$

Since $p\leq r$ on $[0,\infty)$ and function $\frac{\Phi(t)}{t}$ is increasing, then for $\tau,\sigma\in[0,\infty)$, we have

$$\frac{\Phi(p(\tau))}{p(\tau)}\leq\frac{\Phi(r(\tau))}{r(\tau)}. \tag{3.9}$$

Multiplying (3.9) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}r(\tau),\tau\in(0,t)$ which is positive, and integrating equation (3.9) on both side with respective τ from 0 to t, we get

$$\begin{aligned}&\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{a}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\Phi(p(\tau))}{p(\tau)}r(\tau)d\tau\leq\\&\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{a}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\Phi(r(\tau))}{r(\tau)}r(\tau)d\tau,\end{aligned} \tag{3.10}$$

which implies that,

$${}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]\leq{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(r(t))}{r(t)}r(t)\right]. \tag{3.11}$$

Hence, from (3.8) and (3.11) we obtain required inequality (3.1). □

Theorem 3.2. *Let p, r be two positive continuous functions on $[0,\infty)$ and $p\leq r$ on $[0,\infty)$. If $\frac{p}{r}$ is decreasing, p is increasing on $[0,\infty)$ and for any convex function Φ, $\Phi(0)=0$, then for all $k\geq0$, $t>0$, $\alpha,\theta\geq0$, $t>0,\beta,\pi,\rho,\eta,k\in\mathbb{R}$ we have,*

$$\frac{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[p(t)]\,{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}[\phi(r(t))]+{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}[p(t)]\,{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[\Phi(r(t))]}{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[r(t)]\,{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}[\Phi(p(t))]+{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}[r(t)]\,{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[\Phi(p(t))]}\geq1, \tag{3.12}$$

Proof. If function Φ is convex with $\Phi(0)=0$, then $\frac{\Phi(t)}{t}$ is increasing. Since p is increasing, then $\frac{\Phi(p(t))}{p(t)}$ is also increasing. Clearly $\frac{p(t)}{r(t)}$ is decreasing, for all $\tau,\sigma\in[0,t)\ t>0$. Multiplying equation (3.5) by $\frac{\rho^{1-\pi}t^{k}}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(t^{\rho}-\sigma^{\rho})^{1-\theta}},\sigma\in(0,t),t\in\mathbb{R}$, which remains positive. Now integrating obtained result with respect to σ from 0 to t, we have

$$\begin{aligned}&{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[p(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\frac{\Phi(f(t))}{p(t)}r(t)\right]\\&+{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[p(t)\right]\\&\geq{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[r(t)\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[\Phi(p(t))\right]\\&+{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}\left[\Phi(p(t))\right]{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[r(t)\right].\end{aligned} \tag{3.13}$$

Since $p\leq r$ on $[0,\infty)$ and as function $\frac{\Phi(t)}{t}$ is increasing, for $\tau,\sigma\in[0,t)\ t>0$, we have

$$\frac{\Phi(p(\tau))}{p(\tau)}\leq\frac{\Phi(r(\tau))}{r(\tau)}. \tag{3.14}$$

Multiplying both sides of (3.14) by $\frac{\rho^{1-\pi}t^{k}}{\Gamma(\theta)}\frac{\tau^{\rho(\eta+1)-1}}{(t^{\rho}-\tau^{\rho})^{1-\theta}}$, $\tau\in(0,t),t\in\mathbb{R}$ positive, and integrating obtained result with respect to τ from 0 to t, we have

$${}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[\frac{\Phi(p(t))}{p(t)}r(t)\right]\leq{}^{\rho}\mathscr{I}_{\eta,k}^{\theta,\pi}\left[\Phi(p(t))\right]. \tag{3.15}$$

Hence, using (3.13) and (3.15), we obtain required inequality (3.12). □

Now, we prove our main result.

Theorem 3.3. *Let p, r and q be three positive continuous functions on $[0,\infty)$ and $p\leq r$ on $[0,\infty)$. If $\frac{p}{r}$ is decreasing, p and q are increasing functions on $[0,\infty)$, and for any convex function ϕ such that $\phi(0)=0$, then for all $\alpha\geq0$, $t>0,\beta,\rho,\eta,k\in\mathbb{R}$ we have,*

$$\frac{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[p(t)]}{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[r(t)]}\geq\frac{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[\phi(p(t))q(t)]}{{}^{\rho}\mathscr{I}_{\eta,k}^{\alpha,\beta}[\phi(r(t))q(t)]}, \tag{3.16}$$

Proof: Since $p \leq r$ on $[0,\infty)$ and function $\frac{\phi(t)}{t}$ is increasing, then for $\tau,\sigma \in [0,t)$, $t>0$, we have

$$\frac{\phi(p(\tau))}{p(\tau)} \leq \frac{\phi(r(\tau))}{r(\tau)}. \tag{3.17}$$

Multiplying both sides of (3.17) by $\frac{\sigma^{1-\beta}t^k}{\Gamma(\alpha)}\frac{\tau^{\sigma(\eta+1)-1}}{(t^\sigma-\tau^\sigma)^{1-\alpha}}r(\tau)$ positive, and integrating obtained result with respect to τ from 0 to t, we have

$${}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\frac{\phi(p(t))}{p(t)}r(t)q(t)\right] \leq {}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\phi(r(t))q(t)\right]. \tag{3.18}$$

On the other hand, since the fact that the function ϕ is convex with $\phi(0)=0$. Then the function $\frac{\phi(t)}{t}$ is increasing. Since p is increasing, $\frac{\phi(p(t))}{p(t)}$ is also increasing. Clearly we can say that $\frac{p(t)}{r(t)}$ is decreasing, for all $\tau,\sigma \in [0,t)$ $t>0$

$$\left(\frac{\phi(p(\tau))}{p(\tau)}q(\tau)-\frac{\phi(p(\sigma))}{p(\sigma)}q(\sigma)\right)(p(\sigma)r(\tau)-p(\tau)r(\sigma)) \geq 0, \tag{3.19}$$

which implies that

$$\begin{aligned}&\frac{\phi(p(\tau))q(\tau)}{p(\tau)}p(\sigma)r(\tau)+\frac{\phi(p(\sigma))q(\sigma)}{p(\sigma)}p(\tau)r(\sigma)\\&-\frac{\phi(p(\tau))q(\tau)}{p(\tau)}p(\tau)r(\sigma)-\frac{\phi(p(\sigma))q(\sigma)}{p(\sigma)}p(\sigma)r(\tau) \geq 0.\end{aligned} \tag{3.20}$$

Hence, we can write

$$\begin{aligned}&p(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\frac{\phi(p(t))}{p(t)}r(t)q(t)\right]\\&+\frac{\phi(p(\sigma))}{p(\sigma)}r(\sigma)q(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p(t)]\\&-r(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(p(t))q(t)]\\&-\frac{\phi(p(\sigma))}{p(\sigma)}p(\sigma)q(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[r(t)] \geq 0,\end{aligned} \tag{3.21}$$

with the same argument as before, we have

$$\frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[r(t)]} \geq \frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(p(t))q(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\frac{\phi(p(t))}{p(t)}r(t)q(t)\right]}. \tag{3.22}$$

Hence, using equation (3.18) and (3.22), we obtain (3.16). Now, we give generalization of Theorem 3.3.

Theorem 3.4. *Let p, r and q be three positive continuous functions on $[0,\infty)$ and $p \leq r$ on $[0,\infty)$. If $\frac{p}{r}$ is decreasing, p and q are increasing functions on $[0,\infty)$, and for any convex function ϕ such that $\phi(0)=0$, then for all $k \geq 0$, $t>0$, $\alpha,\theta \geq 0$, $t>0,\beta,\pi,\rho,\eta,k \in \mathbb{R}$ we have,*

$$\frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p(t)]\,{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[\phi(r(t))q(t)]+{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[p(t)]\,{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(r(t))q(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[r(t)]\,{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[\phi(p(t))q(t)]+{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[r(t)]\,{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(p(t))q(t)]} \geq 1. \tag{3.23}$$

Proof:- Multiplying equation (3.21) by $\frac{\rho^{1-\pi}t^k}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(t^\rho-\sigma^\rho)^{1-\theta}}, \sigma \in (0,t), t \in \mathbb{R}$, which remains positive . Then integrate the resulting identity with respect to σ from 0 to t, we have

$$\begin{aligned}&{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[p(t)]\,{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\frac{\phi(p(t))q(t)}{p(t)}r(t)\right]+{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}\left[\frac{\phi(p(t))q(t)}{p(t)}r(t)\right]\times\\&{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p(t)] \geq {}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[r(t)]\,{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[\phi(p(t))q(t)]\\&+{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(p(t))q(t)]\,{}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[r(t)],\end{aligned} \tag{3.24}$$

and since $p \leq r$ on $[0,\infty)$ and use the fact that $\frac{\phi(t)q(t)}{t}$ is increasing, we obtain

$${}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}\left[\frac{\phi(p(t))q(t)}{p(t)}r(t)\right] \leq {}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[\phi(r(t))q(t)], \tag{3.25}$$

and

$${}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}\left[\frac{\phi(p(t))q(t)}{p(t)}r(t)\right] \leq {}^{\rho}\mathscr{I}^{\theta,\pi}_{\eta,k}[\phi(r(t))q(t)]. \tag{3.26}$$

Hence, from equation (3.24), (3.25) and (3.26), we obtain (3.23).

4. Other fractional integral inequalities

In [9], authors have proved the inequalities using Riemann-Liouville fractional integral. Now, we prove the similar results using generalized Katugampola fractional integral

Theorem 4.1. *Let p, q be two positive and continuous functions on $[0,\infty)$ such that p is decreasing and q is increasing on $[0,\infty)$, then for all $\alpha \geq 0$, $t>0,\beta,\rho,\eta,k \in \mathbb{R}$ we have,*

$$\frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^l(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^m(t)]} \geq \frac{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^n p^l(t)]}{{}^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^n p^m(t)]}. \tag{4.1}$$

Proof. Consider $\sigma,\tau \in (0,t)$, we have

$$(q^n(\sigma)-q^n(\tau))\left(p^l(\tau)p^m(\sigma)-p^m(\tau)p^l(\sigma)\right) \geq 0,$$

which implies that

$$\begin{aligned}&q^n(\sigma)p^l(\tau)p^m(\sigma)+q^m(\tau)p^m(\tau)p^l(\sigma)\\&\geq q^m(\sigma)p^m(\tau)p^l(\sigma)+q^m(\tau)p^m(\sigma)p^l(\tau),\end{aligned} \tag{4.2}$$

Multiplying both sides of (4.2) by $\frac{\rho^{1-\beta}t^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}$ which is positive, and integrating obtained result with respect to τ from 0 to t, we have

$$\begin{aligned}&q^n(\sigma)p^m(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^l(t)]+p^l(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^n p^m(t)]\\&\geq g^r(\sigma)f^p(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[f^q(t)]+p^m(\sigma)^{\rho}\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^n p^l(t)].\end{aligned} \tag{4.3}$$

Now, multiplying both side of (4.3) by $\frac{\rho^{1-\beta}t^k}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(t^\rho-\sigma^\rho)^{1-\alpha}}$ which is positive from (2.4). Now integrating obtained result with respect to σ from 0 to t, we have

$$\begin{aligned}&{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^np^m(t)]{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^l(t)]\\&\geq{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^np^l(t)]{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^m(t)],\end{aligned}\tag{4.4}$$

which gives the inequality 4.1. □

Theorem 4.2. *Let p, q be two positive and continuous functions on $[0,\infty)$ such that p is decreasing and q is increasing on $[0,\infty)$. then for all $k\geq 0$, $t>0$, $\alpha,\theta\geq 0$, $t>0,\beta,\pi,\rho,\eta,k\in\mathbb{R}$, we have,*

$$\frac{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^l(t)]{}^\rho\mathscr{I}^{\theta,\pi}_{\eta,k}[q^np^m(t)]+{}^\rho\mathscr{I}^{\theta,\pi}_{\eta,k}[p^l(t)]{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^np^m(t)]}{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^m(t)]{}^\rho\mathscr{I}^{\theta,\pi}_{\eta,k}[q^np^m(t)]+{}^\rho\mathscr{I}^{\theta,\pi}_{\eta,k}[p^m(t)]{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}q^np^m(t)]}\geq 1.\tag{4.5}$$

Proof. Multiplying equation (4.3) by $\frac{\rho^{1-\pi}t^k}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(t^\rho-\sigma^\rho)^{1-\theta}}$ ($\sigma\in(0,t)$, $t>0$), which remains positive. Then integrate the resulting identity with respect to σ from 0 to t, we obtain the result 4.5. □

Theorem 4.3. *Let p, q be two positive and continuous functions on $[0,\infty)$ such that p is decreasing and q is increasing on $[0,\infty)$, Such that*

$$(p^n(\tau)q^n(\sigma)-p^n(\sigma)q^n(\tau))\left(p^{l-n}(\tau)-p^{l-n}(\tau)\right)\geq 0,$$

then for all $\alpha\geq 0$, $t>0,\beta,\sigma,\eta,k\in\mathbb{R}$ we have,

$$\frac{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^{n+l}(t)]}{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[p^{n+m}(t)]}\geq\frac{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^np^l(t)]}{{}^\rho\mathscr{I}^{\alpha,\beta}_{\eta,k}[q^np^m(t)]},\quad n>0.\tag{4.6}$$

Proof. Consider $\tau,\sigma\in(0,t)$, we get

$$(p^n(\tau)q^n(\sigma)-p^n(\sigma)q^n(\tau))\left(p^m(\sigma)p^l(\tau)-p^m(\tau)p^l(\sigma)\right)\geq 0.$$

and using the same arguments as in Theorem [4.1]. we obtain the result.

□

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