\( \beta_\lambda \)-closed spaces

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Abstract

We introduce and study \( \beta_\lambda \)-closed spaces in generalized topological spaces (GTS) as a generalization of \( \beta \)-closed spaces [2] in topological spaces. Several characterizations and mapping properties of such spaces are obtained.

Keywords

Generalized topology, \( \lambda \)-space, \( \lambda \)-\( \beta \)-open, \( \lambda \)-\( \beta \)-regular, \( \lambda \)-\( \beta \)-\( \theta \)-open, \( \beta_\lambda \)-closed, \( \beta_\lambda \)-\( \theta \)-converge, \( \beta_\lambda \)-\( \theta \)-accumulate, \( \beta_\lambda \)-c.a.p.

AMS Subject Classification

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1. Introduction

Concept of generalized topological spaces (GTS) has been introduced by A. Császár [4, 6, 8] in 2002. Since then, several research works have been done to generalize the existing notions of topological spaces to generalized topological spaces. Recently, the concept of covering properties in generalized topological spaces have been studied by some authors [15–18]. On the other hand, \( \beta \)-open sets [1] plays a significant role in the theory of generalized form of open sets in topological spaces. Basu and Ghosh [2] introduced the concept of \( \beta \)-closed spaces in topological spaces and gave several characterizations of \( \beta \)-closed spaces.

In this paper, we have introduced and studied a new kind of covering properties in a generalized topological space \((X, \lambda)\) known as \( \beta_\lambda \)-closed spaces via \( \lambda \)-\( \beta \)-open sets [8].

2. Preliminaries

A collection \( \lambda \) of subsets of \( X \) is called a generalized topology (briefly GT) on \( X \) [6] if and only if \( \emptyset \in \lambda \) and \( G_i \in \lambda \) for \( i \in I \neq \emptyset \) implies \( G = \bigcup_{i \in I} G_i \in \lambda \). A set \( X \) with a GT \( \lambda \) on \( X \) is called a generalized topological space (GTS) and is denoted by \((X, \lambda)\). By a space \( X \) or \((X, \lambda)\), we will always mean a GTS. A space \((X, \lambda)\) is called a \( \lambda \)-space [14] if \( X \in \lambda \). For a space \((X, \lambda)\), the elements of \( \lambda \) are called \( \lambda \)-open sets and the complements of \( \lambda \)-open sets are called \( \lambda \)-closed sets. A GT \( \lambda \) on \( X \) is said to be a quasi-topology [9] if and only if \( A, B \in \lambda \) implies \( A \cap B \in \lambda \). A set \( X \) with a quasi topology \( \lambda \) on \( X \) is called a quasi topological space.

For \( A \subset X \), the \( \lambda \)-closure of \( A \), denoted by \( cA \) is the intersection of all \( \lambda \)-closed sets containing \( A \) and the \( \lambda \)-interior of \( A \), denoted by \( iA \) is the union of all \( \lambda \)-open sets contained in \( A \). It was pointed out in [8] that each of the operations \( iA \) and \( cA \) are monotonic [10] i.e. if \( A \subset B \subset X \), then \( iA \subset iB \) and \( cA \subset cB \), idempotent [10], i.e. if \( A \subset X \), then \( i(iA) = iA \) and \( c(cA) = cA \), i.e. if \( A \subset X \), then \( iA \subset A \), \( cA \) is enlarging [10], i.e., if \( A \subset X \), then \( A \subset cA \). In a space \((X, \lambda)\), for \( A \subset X \), \( x \in iA \) if and only if there exists an \( \lambda \)-open set \( V \) containing \( x \) such that \( V \subset A \) and \( x \in cA \) if and only if \( V \cap A \neq \emptyset \) for every \( \lambda \)-open set \( V \) containing \( x \) [5]. In a space \((X, \lambda)\), \( A \subset X \) is \( \lambda \)-open if and only if \( A = iA \) and is \( \lambda \)-closed if and only if \( A = cA \) [4] and \( cA = X \setminus i(X \setminus A) \).

A subset \( A \) of a topological space is called \( \beta \)-open [1] if \( A \subset cl(int(cl(A))) \). The complement of a \( \beta \)-open set is called \( \beta \)-closed. For a subset \( A \) of a topological space \((X, \tau)\), the \( \beta \)-closure of \( A \), denoted by \( \beta cl(A) \) is the intersection of all \( \beta \)-open sets containing \( A \) and the \( \beta \)-interior of \( A \), denoted by \( \beta int(A) \) is the union of all \( \beta \)-open sets contained in \( A \). A topological space \((X, \tau)\) is said to be \( \beta \)-closed [2] if every cover of \( X \) by \( \beta \)-open sets has a finite subfamily whose \( \beta \)-closures cover \( X \).
A set $A \subseteq X$ is said to be $\lambda$-semi-open (resp. $\lambda$-preopen, $\lambda$-$\alpha$-open, $\lambda$-$\beta$-open) [8] if $A \subseteq \text{ci}A$ (resp. $A \subseteq \text{ci}A$, $A \subseteq \text{ci}A$, $A \subseteq \text{ci}A$). We denote by $\sigma(\lambda)$ (resp. $\pi(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$) the class of all $\lambda$-semi-open sets (resp. $\lambda$-preopen sets, $\lambda$-$\alpha$-open sets, $\lambda$-$\beta$-open sets). From [8], it is clear that $\lambda \subseteq \alpha(\lambda) \subseteq \sigma(\lambda) \subseteq \beta(\lambda)$, $\alpha(\lambda) \subseteq \pi(\lambda) \subseteq \beta(\lambda)$ and each of the collections $\sigma(\lambda)$, $\pi(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$ forms a GTS. The complements of $\lambda$-semi-open sets (resp. $\lambda$-preopen sets, $\lambda$-$\alpha$-open sets, $\lambda$-$\beta$-open sets) are called $\lambda$-semi-closed sets (resp. $\lambda$-preclosed sets, $\lambda$-$\alpha$-closed sets, $\lambda$-$\beta$-closed sets). For $A \subseteq X$, we denote by $scA$ (resp. $pcA$, $acA$, $bcA$) the intersection of all $\lambda$-semi-closed sets (resp. $\lambda$-preclosed sets, $\lambda$-$\alpha$-closed sets, $\lambda$-$\beta$-closed sets) containing $A$ and by $siA$ (resp. $piA$, $aiA$, $biA$) the union of all $\lambda$-semi-open sets (resp. $\lambda$-preopen sets, $\lambda$-$\alpha$-open sets, $\lambda$-$\beta$-open sets) contained in $A$. A subset $A$ of a $\lambda$-space $(X, \lambda)$ is called $\lambda$-compact [16] if any cover of $A$ by $\lambda$-open subsets of $X$ has a finite subcover. A $\lambda$-space $(X, \lambda)$ is weakly $\lambda$-compact [17] if any cover of $X$ by $\lambda$-open sets has finite subfamily, the union of the $\lambda$-closures of whose members covers $X$.

3. $\beta_\lambda$-closed spaces

We first state a lemma which will be used in the sequel. Proofs can be checked easily and therefore omitted.

**Lemma 3.1.** The following hold for a subset $A$ of GTS $X$:

(i) $\beta A = A \cap \text{ci}A$

(ii) $\beta cA = A \cup \text{ci}A$

(iii) $x \in \beta cA$ if $A \cap U \neq \emptyset$ for every $\lambda$-$\beta$-open sets $U$ of $X$ containing $x$

(iv) $\beta c(X \setminus A) = X \setminus \beta A$

(v) $A$ is $\lambda$-$\beta$-closed if and only if $A = \beta cA$.

**Definition 3.2.** A subset $A$ of a space $X$ is said to be $\lambda$-$\beta$-regular if it is both $\lambda$-$\beta$-open and $\lambda$-$\beta$-closed. The family of all $\lambda$-$\beta$-regular sets of a space $X$ is denoted by $\lambda r(X)$ and that of containing a point $x$ of $X$ by $\beta r(X, x)$.

**Lemma 3.3.** For a subset $A$ of a space $X$, $A \in \beta(\lambda)$ if and only if $\beta cA \subseteq \beta r(X)$.

**Proof.** First suppose, $A \in \beta(\lambda)$. Then $A \subseteq \text{ci}A$ and therefore, $\beta c(A \cap \text{ci}A) \subseteq \beta c(\text{ci}A) = \text{ci}(\beta cA)$ i.e. $\beta cA$ is $\lambda$-$\beta$-open. Since $\beta cA$ is $\lambda$-$\beta$-open and $\lambda$-$\beta$-closed, $\beta cA \subseteq \beta r(X)$. Next suppose, $\beta cA \subseteq \beta r(X)$. Then $A \subseteq \beta cA \subseteq \text{ci}(\beta cA) \subseteq \text{ci}(\text{ci}A) = \text{ci}A$. Hence $A \in \beta(\lambda)$.

**Definition 3.4.** A point $x \in X$ is said to be in the $\lambda$-$\beta$-closure of $A$, denoted by $\beta \theta c A$, if $A \cap \beta cV \neq \emptyset$ for every $\lambda$-$\beta$-open set $V$ of $X$ containing $x$.

If $\beta \theta c A = A$, then $A$ is said to be $\lambda$-$\beta$-closed. The complement of a $\lambda$-$\beta$-closed set is said to be $\lambda$-$\beta$-open.

**Lemma 3.5.** For a subset $A$ of a space $X$,

$\beta \theta c A = \cap \{ V : A \cap V \subseteq \text{ci}V \}$

$\cap \{ V : A \subseteq V \text{and} V \subseteq \beta r(X) \}$

**Proof.** We give proof of the first equality. Other is quite similar. Suppose that, $x \notin \beta \theta c A$. Then there exists, $\lambda$-$\beta$-open set $V$ containing $x$ such that $\beta cV \cap A = \emptyset$. Therefore by Lemma 3.3, $X \setminus \beta cV$ is $\lambda$-$\beta$-regular and so $\lambda$-$\beta$-closed set containing $A$ such that $x \notin X \setminus \beta cV$. Hence, $x \notin \cap \{ V : A \subseteq \text{ci}V \text{and} V \subseteq \beta r(X) \}$. Conversely, suppose that, $x \notin \beta \theta c A$. Then there exist, a $\lambda$-$\beta$-closed set $V$ containing $A$ and $x \notin V$. Also, there exists $\gamma \subseteq \beta(\lambda)$ such that $x \in U \subseteq \beta cU \subseteq V \subseteq X$. Then we have, $\beta cU \cap A \subseteq \beta cU \cap V = \emptyset$ and so $x \notin \beta \theta c A$.

**Lemma 3.6.** Let $A$ and $B$ be any subset of a space $X$. Then the following properties hold:

(i) $x \in \beta \theta c A$ if and only if $A \cap V \neq \emptyset$ for every $V \in \beta r(X, x)$.

(ii) If $A \subseteq B$ then $\beta \theta c A \subseteq \beta \theta c B$.

(iii) $\beta \theta c(A \cap \beta \theta c A) = \beta \theta c A$.

(iv) intersection of an arbitrary family of $\lambda$-$\beta$-closed sets in $X$ is $\lambda$-$\beta$-closed in $X$.

(v) $A$ is $\lambda$-$\beta$-open if and only if for each $x \in A$, there exists $V \in \beta r(X, x)$, such that $x \in V \subseteq C A$.

(vi) If $A \subseteq \beta(\lambda)$ then $\beta cA = \beta \theta c A$.

(vii) If $A \subseteq \beta r(X)$ then $A$ is $\lambda$-$\beta$-closed.

(viii) $A \subseteq \beta r(X)$ if and only if $A$ is $\lambda$-$\beta$-open and $\lambda$-$\beta$-closed.

**Proof.** We give only proof of (iv). Others proofs are obvious. (iv) Let $A$ be a $\lambda$-$\beta$-closed for each $\alpha \subseteq A$. Then for each $\alpha \subseteq A$, we have $A \subseteq \beta \theta c A$. Therefore, $\beta \theta c(\cap A \subseteq A) \subseteq \cap A \subseteq A \subseteq \beta \theta c(\cap A \subseteq A)$. Hence, $\beta \theta c(\cap A \subseteq A) = \cap A \subseteq A$. Therefore, $\cap A \subseteq A$ is $\lambda$-$\beta$-closed.

**Remark 3.7.** If $A$ be a $\lambda$-$\beta$-regular set in a GTS $(X, \lambda)$, then $A$ is $\lambda$-$\beta$-open and $\lambda$-$\beta$-open.

We now introduce $\beta_\lambda$-closed subset $A$ of a $\lambda$-space $(X, \lambda)$. As a special case, we obtain $\beta_\lambda$-closed spaces when $A = X$. Several characterizations in terms of filter bases and generalized complete accumulation point are obtained.

**Definition 3.8.** A subset $A$ of a $\lambda$-space is called $\beta_\lambda$-closed in $X$ if any cover of $A$ by $\lambda$-$\beta$-open subsets of $X$ has a finite subfamily, the union of $\lambda$-$\beta$-closures of whose members covers $A$.

A $\lambda$-space $(X, \lambda)$ is called $\beta_\lambda$-closed if any cover of $X$ by $\lambda$-$\beta$-open sets has a finite subfamily, the union of $\lambda$-$\beta$-closures of whose members covers $X$.

**Remark 3.9.** We observe that the concept of $\beta_\lambda$-closed subset of a $\lambda$-space generalizes the concept of $\beta$-closed subset [3] of a topological space. Also, if $A$ is a subset of a topological space $(X, \tau)$ and $\lambda = \tau$, then the concepts of $\beta_\lambda$-closedness and $\beta$-closedness are equivalent.

**Theorem 3.10.** Every $\beta_\lambda$-closed space $(X, \lambda)$ is weakly $\lambda$-compact [17].

**Proof.** Proof follows from the fact that every $\lambda$-$\beta$-open set is $\lambda$-$\beta$-open in a space $(X, \lambda)$. 

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Theorem 3.11. For a $\lambda$-space $X$, the following are equivalent:
(i) $A$ is $\beta_\lambda$-closed;
(ii) every cover of $A$ by $\lambda$-$\beta$-regular sets has a subcover;
(iii) for each family $\{\{U_\alpha \in B_\beta(X) : \alpha \in I_0\} \cap A = \emptyset\}$, there exists a finite subset $I_0$ of $I$ such that $\{\{U_\alpha \in B_\beta(X) : \alpha \in I_0\} \cap A = \emptyset\}$;
(iv) every cover of $A$ by $\lambda$-$\beta$-$\theta$-open has a finite subcover.

Proof. Straightforward.

Proposition 3.12. For a subset $A$ of a $\lambda$-space $X$, the following are equivalent:
(i) $A$ is $\beta_\lambda$-closed in $X$.
(ii) for any family $\mathcal{W} = \{U_\alpha : \alpha \in \Lambda\}$ of $\lambda$-$\beta$-closed subsets of $X$ such that $\{\{U_\alpha : \alpha \in \Lambda\} \cap A = \emptyset\}$, there exist a finite subset $\Lambda_0$ of $\Lambda$ such that $\{\{U_\alpha : \alpha \in \Lambda_0\} \cap A = \emptyset\}$.

Proof. Straightforward.

Definition 3.13. A filter base $F$ on a $\lambda$-space $X$ is said to be $\beta_\lambda$-$\theta$-converge to a point $x \in X$, if for each $\lambda$-$\beta$-$\theta$-open subset $V$ of $X$ containing $x$, there exists $F' \in F$ such that $F' \subset \beta cV$.

A filter base $F$ is said to be $\beta_\lambda$-$\theta$-accumulate at $x \in X$, if $F \cap \beta cV \neq \emptyset$ for every $F \in F$ and for every $\lambda$-$\beta$-$\theta$-open subset $V$ of $X$ containing $x$.

Definition 3.14. A net $\{x_\lambda\}_{\lambda \in \mathcal{D}}$ on a $\lambda$-space $X$, where $D$ is a directed set, is said to be $\beta_\lambda$-$\theta$-converge to a point $x \in X$, if for each $\lambda$-$\beta$-$\theta$-open subset $V$ of $X$ containing $x$, there exists $n_0 \in D$ such that $\forall n > n_0, x_\lambda \in \beta cV$.

A net $\{x_\lambda\}_{\lambda \in \mathcal{D}}$ on a $\lambda$-space $X$, where $D$ is a directed set, is said to be $\beta_\lambda$-$\theta$-accumulate at $x \in X$, if for each $\lambda$-$\beta$-$\theta$-open subset $V$ of $X$ containing $x$ and $\forall n_0 \in D$ there exist $n > n_0$ such that $x_\lambda \in \beta cV$.

Definition 3.15. A point $x$ in a space $X$ is called $\beta_\lambda$-$\theta$-complete accumulation point (or $\beta_\lambda$-$\theta$-c.a.p. for short) of a subset $S$ of $X$, if $|S| = |S \cap V|$ for each $V \in B_\beta(X, x)$, where $|S|$ denotes the cardinality of the set $S$.

Theorem 3.16. The following conditions are equivalent for a $\lambda$-space $X$:
(i) $X$ is $\beta_\lambda$-closed;
(ii) every infinite subset of $X$ has a $\beta_\lambda$-$\theta$-c.a.p. in $X$;
(iii) each net with a well ordered directed set as its domain $\beta_\lambda$-$\theta$-accumulate to a point in $X$.

Proof. (i) $\Rightarrow$ (ii): Let $I$ be an infinite subset in a $\beta_\lambda$-closed space and $A = \{x \in X : x$ is not a $\beta_\lambda$-$\theta$-c.a.p. of $I\}$. Then for each $x \in A$, there exists a $B_\lambda \in B_\beta(X, x)$ such that $|B_\lambda \cap I| < |I|$. If $A$ is the whole space, then it follows from the Theorem 3.11, that the cover $\{B_\lambda : x \in A\}$ has a finite subcover, say $\{B_{i_1}, B_{i_2}, ..., B_{i_k}\}$. Then, $I \subset \cap B_{i_1} : i = 1, 2, ..., k$ and $|I| = \max\{|B_{i_j} \cap I| : i = 1, 2, ..., k\}$, a contradiction. Hence, $A$ has a $\beta_\lambda$-$\theta$-c.a.p. in $X$.

(ii) $\Rightarrow$ (i): Suppose $X$ is not $\beta_\lambda$-closed. Then by the Theorem 3.11, there exists a cover $\mathcal{W}$ of $X$ by $\lambda$-$\beta$-regular sets having no finite subcover. We consider $\mathcal{W} = \min\{|\mathcal{W}^* : \mathcal{W}^* \subset \mathcal{W} \text{ and } \mathcal{W}^* \text{ is a cover of } X\}$, where $|.|$ denotes the cardinality. Let $\mathcal{W}_0 \subset \mathcal{W}$ be a cover of $X$ for which $|\mathcal{W}_0| = \mathcal{W}$. Clearly $\mathcal{W} \geq \mathcal{W}_0$ and then by the well ordering of $\mathcal{W}_0$, by some minimal well ordering $<$. We have $\{U : U \in \mathcal{W}_0 \text{ and } U \prec U_0\} \subset \{U : U \in \mathcal{W}_0\}$ for each $U_0 \in \mathcal{W}_0$. It is clear that, $X$ can not have any subcover with cardinality less than $\mathcal{W}$. Therefore, $Z = \{U \in \mathcal{W}_0 \text{ and } U_0 \prec U\} \subset \{U \in \mathcal{W}_0 : U \prec U^*\}$. By the minimality of $\mathcal{W}^*, |Z| < \mathcal{W}$ and so $|\mathcal{W} \cap U^*| < \mathcal{W}$. Since, for each $U_1, U_2 \in \mathcal{W}_0$ with $U_1 \neq U_2$, we have $x_{U_1} \neq x_{U_2}$, then $|Z| = \mathcal{W} > \mathcal{W}_0$. Therefore, the infinite set $W$ has no $\beta_\lambda$-$\theta$-c.a.p. in $X$ a contradiction. Hence, $X$ is $\beta_\lambda$-closed.

(iii) $\Rightarrow$ (ii): Let $I$ be an infinite subset of $X$. Then by Zorn’s lemma, $I$ can be assumed to be a net with a well ordered directed set as its domain. So, it has a $\beta_\lambda$-$\theta$-adherent point say, $x$ and then clearly $x$ is an $\beta_\lambda$-$\theta$-c.a.p. of $I$.

(i) $\Rightarrow$ (ii): Let $\{x_\lambda\}_{\lambda \in \mathcal{D}}$ be a net with well ordered directed set $D$ and it has no $\beta_\lambda$-$\theta$-adherent point in $X$. Then, for each $x \in X$, there exists $U_1 \in B_\beta(X, x)$ and $\lambda_1 \in D$ such that $x_\lambda \notin U_1 \forall \lambda \geq \lambda_1$. Now, since $X$ is $\beta_\lambda$-closed, the cover $\{U_1 : x \in U_1\}$ has a finite subcover $\{U_{i_1}, U_{i_2}, ..., U_{i_k}\}$ (say). Suppose $\{\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_k}\}$ be the corresponding elements in $D$ and since it is finite, by the well orderedness of $D$, there exists a largest element say, $\lambda_{i_k}$ in $D$. Therefore, we have $x_{i_k} \in \cap_{\lambda \geq \lambda_{i_k}}(X \setminus U_{i_k}) = X \setminus \cup_{\lambda \geq \lambda_{i_k}}U_{i_k} = \emptyset$ for $\lambda > \lambda_{i_k}$ a contradiction. Hence, the net $\{x_\lambda\}_{\lambda \in \mathcal{D}}$ has a $\beta_\lambda$-$\theta$-c.a.p. in $X$.

The proof of the following proposition is straightforward and is omitted.

Proposition 3.17. Let $F$ be a filter base on a $\lambda$-space $X$.
(i) If $F$ is $\beta_\lambda$-$\theta$-converge to $x$, then $F$ $\beta_\lambda$-$\theta$-accumulates at $x$.
(ii) If $F$ is a maximal filter base, then $F$ $\beta_\lambda$-$\theta$-converges if and only if $\beta_\lambda$-$\theta$-accumulates at $x$.

Theorem 3.18. For a subset $A$ of a $\lambda$-space $X$, the following are equivalent:
(i) $A$ is $\beta_\lambda$-closed in $X$;
(ii) every maximal filter base on $X$, each of whose members meet $A$, $\beta_\lambda$-$\theta$-converges to some point of $A$;
(iii) every filter base on $X$, each of whose members meet $A$, $\beta_\lambda$-$\theta$-accumulate to a point of $A$.

Proof. (i) $\Rightarrow$ (ii): Let $F$ be a maximal filter base on $X$, each of whose members meet $A$, such that $F$ does not $\beta_\lambda$-$\theta$-converge to any point of $A$. Now, since $F$ is maximal, by above Proposition 3.17, $F$ does not $\beta_\lambda$-$\theta$-accumulates at any point of $A$. So, for each $x \in A$, there exists $F_x \in F$ and $\lambda$-$\beta$-open set $U_x$ of $X$ containing $x$ and $F_x \cap \beta cU_x = \emptyset$. 

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But, $A$ being $β_2$-closed in $X$, there exists $x_1, x_2, ..., x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} βcU_{x_i}$. Again, since $\mathcal{F}$ is filter base on $X$, there exists $F \in \mathcal{F}$ such that $F \cap \bigcap_{i=1}^{n} F_{x_i} \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$. Therefore, $F \cap βcU_{x_i} = \emptyset$ for each $i \in \{1, 2, ..., n\}$ i.e. $\emptyset = (\bigcup_{i=1}^{n} βcU_{x_i}) \cap F \supset A \cap F$ - a contradiction.

(ii) $\Rightarrow$ (iii): Let $A$ be a filter base on $X$, each of whose members meet $A$. Then $\mathcal{F}^A = \{F \cap A : F \in \mathcal{F}\}$ is a filter base on $X$. Therefore, $\mathcal{F}^A$ is contained in a maximal filter base $\mathcal{A}$ on $X$, each of whose members meet $A$. Hence, $\mathcal{A}_{β_2}$-converges to some $y$ of $A$ and so by Proposition 3.17 (ii), $\mathcal{A}_{β_2}$-accumulates at $y$. But since $\mathcal{F}^A \subset \mathcal{A}$, so $\mathcal{F}^A_{β_2}$-accumulates at $y$. Hence, $\mathcal{F} \beta_2$-accumulates at $y$.

(iii) $\Rightarrow$ (i): If possible, suppose that, $A$ is not $β_2$-closed. Then by Proposition 3.12, there exist a cover $\mathcal{U} = \{U_α : α \in A\}$ of $A$ by $λ$-$β$-open subsets of $X$ such that for any finite subset $A_0$ of $A$ such that $β(X \setminus U_α) : α \in A_0 \} \cap A \neq \emptyset$. For each finite subset $A_0$ of $A$, let $F_{A_0} = \{\{β(U_α) : α \in A_0\} \cap A$. Then $\mathcal{F} = \{F_{A_0} : A_0$ is a finite subset of $A\}$ is a filter base on $X$, each of whose members meet $A$. Therefore by (iii), $\mathcal{F} \beta_2$-accumulates at some point $α$ of $A$. Since $\mathcal{U}$ is a cover of $A$, there exists $α_0 \in A$ such that $x \in U_{α_0}$, but $\mathcal{F} \beta_2$-accumulates at $x$ and $U_{α_0}$ being a $λ$-$β$-open set, $F \cap βcU_{α_0} \neq \emptyset$ for every $F \in \mathcal{F}$. Let $F = β(x \setminus U_{α_0})$, then $F \in \mathcal{F}$. Thus $β(X \setminus U_{α_0}) \cap A \cap βcU_{α_0} = \emptyset$ - a contradiction. Hence, $A$ is $β_2$-closed in $X$.

**Theorem 3.19.** Let $X$ be a $λ$-space. If $A$ is $λ$-$β$-$θ$-closed subset of $X$ and $B$ is a $β_2$-closed in $X$, then $A \cap B$ is $β_2$-closed in $X$.

**Proof.** Let $\mathcal{U} = \{U_α : α \in A\}$ be a cover of $A \cap B$ by $λ$-$β$-open subsets of $X$. Then $\mathcal{U} \cup \{X \setminus A\}$ is a cover of $B$. Now, since $X \setminus A$ is $λ$-$β$-open, for each $x \notin A$, there exists a $λ$-$β$-open set $U_x$ such that $x \in U_x \subset βcU_x \subset \{X \setminus A\}$. Then $\mathcal{U} \cup \{U_x : x \in X \setminus A\}$ is a cover of $B$ by $λ$-$β$-open subsets of $X$. Since $B$ is $β_2$-closed in $X$, so there exists $α_1, α_2, ..., α_m \in A$ and $x_1, x_2, ..., x_m \in X \setminus A$ such that $B \subset \bigcup_{i=1}^{m} βcU_{x_i} \cup \bigcup_{i=1}^{n} βcU_{α_i}$. But, $βcU_x \subset X \setminus A$ and so $A \cap B \subset \bigcup_{i=1}^{n} βcU_{α_i}$. Hence, $A \cap B$ is $β_2$-closed in $X$.

**Theorem 3.20.** For a $λ$-space $X$, the following are equivalent:

(i) $X$ is $β_2$-closed;

(ii) every proper $λ$-$β$-$θ$-closed set is $β_2$-closed relative to $X$;

(iii) every proper $λ$-$β$-$θ$-regular set is $β_2$-closed relative to $X$.

**Proof.** (i) $\Rightarrow$ (ii): Let $\{U_α : α \in I\}$ be a cover of proper $λ$-$β$-$θ$-closed set $F$ by $λ$-$β$-regular sets of $X$. Then $X \setminus F$ is $λ$-$β$-open, so for each $x \in (X \setminus F)$, there exists a $V_x \subset βR(X, x)$ such that $x \in V_x \subset (X \setminus F)$. Hence, the family $\{V_x : x \in X \setminus F\} \cup \{U_α : α \in I\}$ is a cover of $X$ by $λ$-$β$-regular sets of $X$. Now, since $X$ is $β_2$-closed, there is a finite subset $I_0$ of $I$ such that $F \subset \bigcup\{U_α : α \in I_0\}$. Hence by Theorem 3.11, $F$ is $β_2$-closed relative to $X$.

(i) $\Rightarrow$ (iii): Follow from Theorem 3.6 (vii).

(iii) $\Rightarrow$ (i): Let $F$ be a $λ$-$β$-regular set. Then, $X = F \cup (X \setminus F)$ and since $F$ and $X \setminus F$ are both $λ$-$β$-regular, $X$ is $β_2$-closed.

### 4. Mapping Properties

**Definition 4.1.** A function $f : (X, λ) \to (Y, λ')$ is called $(λ, λ')$-continuous [6] if the inverse image of each $λ'$-open set is $λ$-open.

**Definition 4.2.** A function $f : (X, λ) \to (Y, λ')$ is called $β(λ, λ')$-irresolute if the inverse image of each $λ'$-open set is $λ$-open.

Proof. The following lemma is quite straightforward and thus omitted.

**Lemma 4.3.** Let $f : (X, λ) \to (Y, λ')$ be a function. Then the following are equivalent:

(i) $f$ is $β(λ, λ')$-irresolute;

(ii) for every $x \in X$ and for every $λ'$-open set $V$ containing $f(x)$, there exists a $λ$-open set $U$ containing $x$ such that $f(U) \subset V$;

(iii) $f(βcA) \subset βc_{λ'}f(A)$ for every subset $A$ of $X$ (where $c_{λ'}A$ denotes $λ'$-closure of $A$ in $(X, λ)$);

(iv) $βc_{λ'}f^{-1}(B) \subset f^{-1}(βc_B)$ for every subset $B$ of $Y$.

**Theorem 4.4.** Let $f : (X, λ) \to (Y, λ')$ be a $β(λ, λ')$-irresolute function, where $(X, λ)$ is a $λ$-space and $(Y, λ')$ is a $λ'$-space. If $A$ is $β_2$-closed in $X$, then $f(A)$ is $β_2$-closed in $Y$.

**Proof.** Let $\mathcal{U} = \{U_α : α \in A\}$ be a cover of $f(A)$ by $λ'$-$β$-open subsets of $Y$. Since $f$ is $β(λ, λ')$-irresolute, $\mathcal{V} = \{f^{-1}(U_α) : α \in A\}$ is a cover of $A$ by $λ$-$β$-open subsets of $X$. But, since $A$ is $β_2$-closed in $X$, there exists $α_1, α_2, ..., α_m \in A$ such that $A \subset \bigcup_{i=1}^{m} βc_{λ'}f^{-1}(U_{α_i})$. This implies $f(A) \subset \bigcup_{i=1}^{m} f(βc_{λ'}f^{-1}(U_{α_i}))$. Again, since $f$ is $β(λ, λ')$-irresolute, by above Lemma 4.3, it follows that $f(βc_{λ'}f^{-1}(U_{α_i})) \subset βc_{λ'}f(f^{-1}(U_{α_i})) \subset βc_{λ'}U_{α_i}$. Therefore, $f(A)$ is $β_2$-closed in $Y$.

### References


