Distance matrix from adjacency matrix using Hadamard product

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Abstract
Distance matrix of a graph has important applications in the field of hierarchical clustering, phylogenetic analysis, bioinformatics, telecommunication etc. There are nice research works on determinant, characteristic polynomial and eigen values of distance matrices. This paper describes a formula for finding the distance matrix of a simple connected undirected graph from the powers of the adjacency matrix using Hadamard product on matrices.

Keywords
Adjacency matrix, Distance matrix, Binary matrix, Diameter, Hadamard product, m-distance matrix.

AMS Subject Classification
05C12, 05C40, 05C50, 05C62, 05B20.

1. Introduction

The earlier study on distance matrix started by L. Graham, H. O. Pollak and L. Lovasz. Now there are nice research works on determinant, characteristic polynomial [1] and eigen values [2] of distance matrices. But so far, there are not so much significant study on how to find distance matrix from adjacency matrix. Here we define the m-distance matrix and developing a formula for finding this matrix in terms of adjacency matrix. Then distance matrix can be easily calculated from these m-distance matrices.

We are considering a simple, connected, undirected graph \( G = (V, E) \) of order \( n \) with vertex set \( V \) and edge set \( E \) throughout this paper unless otherwise specified.

2. Preliminaries

Let \( A_G \) be the \( n \times n \) adjacency matrix [3] of \( G \). Then \( i^j \) entry of \( A_G(m^th \text{ power of } A_G) \), represent the number of walks of length \( m \) between the vertices \( v_i \) and \( v_j \) of \( G \).

**Definition 2.1.** [4] The distance matrix \( D = (d_{ij}) \), of \( G \) is defined as,

\[
d_{ij} = \begin{cases} 
   d(v_i, v_j), & \text{if } i \neq j \\
   0, & \text{if } i = j 
\end{cases}
\]

where, \( d(v_i, v_j) \) is the distance between the vertices \( v_i \) and \( v_j \).

**Definition 2.2.** [5] The diameter of a graph \( G \) is the maximum distance between any two vertices of \( G \) and it is denote by \( \text{Diam}(G) \).

**Definition 2.3.** (Hadamard product [6]) Consider the vector space \( \mathbb{R}^{m \times n} \) over the field \( \mathbb{R} \). For \( C, F \in \mathbb{R}^{m \times n} \), the Hadamard product \( \circ \) is a binary operation on \( \mathbb{R}^{m \times n} \) defined by,

\[
(C \circ F)_{ij} := (C)_{ij} (F)_{ij}, \quad \forall i, j.
\]

Properties:

(i) \((\mathbb{R}^{m \times n}, \circ)\) and \((B_m^{m \times n}, \circ)\) are commutative Monoids, where \( B_m^{m \times n} \) denote the set of all \( m \times n \) binary matrices of \( \mathbb{R}^{m \times n} \).

(ii) \( C, F, E \in \mathbb{R}^{m \times n}, E \circ (C + F) = E \circ C + E \circ F \).

(iii) \( C, F \in \mathbb{R}^{m \times n}, k \in \mathbb{R}, (kC) \circ F = C \circ (kF) = k(C \circ F) \).

(iv) For \( C, F \in \mathbb{R}^{m \times n} \)

\[
(C \circ F)_{ij} = \begin{cases} 
   1, & \text{if } c_{ij} \text{ and } f_{ij} = 1 \\
   0, & \text{otherwise}
\end{cases}
\]
Since the first two powers of the adjacency matrix $A_G$ of $G$, $(A_G^0 = I_n, A_G^2 = A_G)$ are binary matrices, it will be more convenient that its other powers $A_G^3, A_G^4, \ldots$ are also be treated as binary matrices.

**Definition 2.4.** Let us define a function $\delta : \mathcal{R} \rightarrow B = \{0,1\}$ as follows.

\[
\delta(a) = \begin{cases} 
0, & a \leq 0, a \in \mathcal{R} \\
1, & \text{otherwise.}
\end{cases}
\]

Let $\mathcal{R}^{m \times n}$ denote the vector space of all $m \times n$ real matrices over the real field $\mathcal{R}$ and $B_{m \times n}$ denote the set of all $m \times n$ binary matrices of $\mathcal{R}^{m \times n}$. Define $\delta : \mathcal{R}^{m \times n} \rightarrow B_{m \times n}$ as

\[
\delta(D) = (\delta(D))_{ij} := (\delta(d_{ij})), \quad \forall i,j,D \in \mathcal{R}^{m \times n}.
\]

Then $\delta(A_G^m) \in B_{n \times n}, \forall m \in \{0,1,2,\ldots\}$ and it is the equivalent binary matrix representation of $A_G^m$. Let it be denoted by $A_G^{(m)}$.

\[
(A_G^{(m)})_{ij} = \begin{cases} 
1, & \text{if there exist a walk of length } m \text{ form } \nu_i \text{ to } \nu_j \text{ in } G \\
0, & \text{otherwise.}
\end{cases}
\]

Also, for $C,F \in B_{m \times n}$, $\delta(C \circ F) = \delta(C) \circ \delta(F)$.

**Definition 2.5.** Consider the graph $G = (V,E)$ with $n$ vertices. Then the $m$-distance matrix $D_m$ of $G$ is an $n \times n$ symmetric binary matrix defined by,

\[
(D_m)_{ij} := \begin{cases} 
1, & \text{if } d(\nu_i,\nu_j) = m \\
0, & \text{otherwise.}
\end{cases}
\]

**Properties**

(i) $D_m \in B_{n \times n}, \forall m = 0,1,2,\ldots$

(ii) $D_0 = I_n$

(iii) $\text{Diam}(G) = \max_m \{m : D_m \neq 0\}$

(iv) $D_i \circ D_j = \begin{cases} 
0, & \text{if } i \neq j \\
D_i, & \text{if } i = j, \text{ for } 0 \leq i, j \leq d.
\end{cases}$

**Remark 2.6.** By the property (iv), we get $\{D_0,D_1,\ldots,D_d\}$ as an orthogonal subset of $B_{n \times n} \subset \mathcal{R}^{m \times n}$ with respect to the Hadamard product.

### 3. Main Results

**Theorem 3.1.** The set $\{D_0,D_1,\ldots,D_d\}$ is a Linearly independent subset of the vector space $\mathcal{R}^{n \times n}$.

**Proof.** If $c_0D_0 + c_1D_1 + \ldots + c_dD_d = 0$ for some $c_0,c_1,\ldots,c_d \in \mathcal{R}$.

Then by taking Hadamard product $\circ$ by $D_j(0 \leq j \leq d)$ on both side, we have

\[
D_j \circ (c_0D_0 + c_1D_1 + \ldots + c_dD_d) = 0
\]

By property (iv), we have

\[
0 + 0 + \cdots + c_jD_j \circ D_j + \cdots + 0 = 0,
\]

Since, $D_j \neq 0$, then

\[
c_jD_j = 0 \Rightarrow c_j = 0, \quad \forall j.
\]

Therefore, $\{D_0,D_1,\ldots,D_d\}$ is a Linearly independent subset of $\mathcal{R}^{n \times n}$. \hfill \Box

**Theorem 3.2.** Let $D$ be the distance matrix of $G = (V,E)$, then $D = \sum_{t=1}^{d} tD_t$, where $d = \text{Diam}(G)$.

**Proof.** Let $D_t$ be the $t$-distance binary matrix of $G$. Then $tD_t$ is a matrix whose non negative entry $t$ represent the distance between the corresponding vertices of $G$. Since $G$ is connected, there should be a finite distance varies from $1$ to $d$ between any two vertices of $G$. so,

\[
D = \sum_{t=1}^{d} tD_t.
\]

**Remark 3.3.** If we include the zero distance in the above sum, then $D = \sum_{t=0}^{d} tD_t$.

**Remark 4.** Let $W$ denote the subspace of all symmetric matrices of $\mathcal{R}^{n \times n}$ such that all its diagonal entries $0$. Consider the $n \times n$ binary matrices $B_{u,t}(u < t), u,t \in N$ in $W$ gives as below

\[
(B_{u,t})_{ij} = \begin{cases} 
1, & \text{if } (i,j) = (u,t) \text{ or } (i,j) = (t,u) \\
0, & \text{otherwise.}
\end{cases}
\]

Then $B_W = \{B_{u,t} : 1 \leq u < t \leq n\}$ will be the standard basis of $W$ and dimension of $W$ is $\frac{n(n-1)}{2}$. Since, $D_m \in W$, it can be represented as a linear combination of elements of $B_W$.

\[
D_m = \sum_{r,k=1}^{n-1} \sum_{r} b_{r,k,m}B_{r,k},
\]

where $b_{r,k,m} \in \mathcal{R}$. Also, the distance matrix $D \in W$, so

\[
D = \sum_{u=1}^{n-1} \sum_{u+t=1}^{n} b_{u,t}B_{u,t}.
\]

By Theorem 3.2, $D = \sum_{m=1}^{d} mD_m$, so

\[
D = \sum_{m=1}^{d} m \left( \sum_{r=1}^{n-1} \sum_{k=r+1}^{n} b_{r,k,m}B_{n,k} \right)
\]

(3.1)

\[
\sum_{r=1}^{n-1} \sum_{k=r+1}^{n} \left( \sum_{m=1}^{d} m b_{r,k,m}B_{r,k} \right)
\]

(3.2)

Since $B_W$ is basis, (3.1) & (3.2) gives,

\[
b_{u,t} = \sum_{m=1}^{d} mb_{r,k,m}.
\]

This gives the relationship between scalars $b_{u,t}$ and $b_{r,k,m}$ which are in the linear combination of $D$ and $D_m$ respectively, with respect to the basis $B_W$. 

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Lemma 3.5. Let $A_G$ be the adjacency matrix of the graph $G = (V, E)$. Let $d = \text{Diam}(G)$. Then for $0 \leq k, m \leq d$,

$$(A_G^{(k)} \circ A_G^{(m)})_{ij} = \begin{cases} 1, & \text{if there exist a walk of length } m \text{ and} \\ & \text{a walk of length } k \text{ between } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since, $(A_G^{(k)} \circ A_G^{(m)})$ is a binary matrix, its entries are either 1 or 0. If $(A_G^{(k)} \circ A_G^{(m)})_{ij} = (A_G^{(k)})_{ij} \circ (A_G^{(m)})_{ij} = 1$, then $(A_G^{(k)})_{ij} = 1$ and $(A_G^{(m)})_{ij} = 1$.\[\therefore (A_G^{(k)} \circ A_G^{(m)})_{ij} = 1 \Rightarrow \text{a walk of length } k \text{ and a walk of length } m \text{ between } v_i \text{ and } v_j.\] Similarly, $(A_G^{(k)} \circ A_G^{(m)})_{ij} = 0 \Rightarrow \text{either } (A_G^{(k)})_{ij} = 0 \text{ or } (A_G^{(m)})_{ij} = 0,$ which means a walk of length $k$ or a walk of length $m$ between $v_i$ and $v_j$ does not exist. Therefore \[ (A_G^{(k)} \circ A_G^{(m)})_{ij} = 0, \] when there does not exist a walk of length $k$ or a walk of length $m$ between $v_i$ and $v_j$. By Theorem 3.2, to express the distance matrix $D$ in terms of adjacency matrix $A_G$, it is enough to express the $m$-distance matrix $D_m$ in terms of the adjacency matrix $A_G$. The following result gives relation between $D_m$ and $A_G$. \[ \square \]

Theorem 3.6. Let $A_G$ be the adjacency matrix of $G = (V, E)$. Let $d = \text{Diam}(G)$, Then for $1 \leq m \leq d$,

$$D_m = A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right),$$

where $D_m$ is the $m$-distance matrix of $G$.

Proof. By Lemma 3.5,

$$(A_G^{(m)} \circ A_G^{(s)})_{ij} = \begin{cases} 1, & \text{if there exist a walk of length } m \text{ and} \\ & \text{a walk of length } s \text{ between } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)}$ need not be a binary matrix. But $\delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right)$ will be the equivalent binary matrix of it.

$$\left( \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right) \right)_{ij} = \delta \left( \sum_{s=0}^{m-1} (A_G^{(m)})_{ij} \circ (A_G^{(s)})_{ij} \right) = \begin{cases} 1, & \text{if there exist a walk of length } m \text{ between } v_i \text{ and } v_j \\ & \text{that can also be joined by a walk of length } < m \\ 0, & \text{otherwise.} \end{cases}$$

Then the subtraction of $\delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right)$ from $A_G^{(m)}$ removes all the 1’s in $A_G^{(m)}$ that corresponds to the $m$-length walks joining any two vertices of $G$ which can also be joined by a walk of length fewer than $m$. The remaining 1’s in $A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right)$ corresponds to all the $m$-length paths joining vertices of $G$. These paths will be the shortest $m$-length paths joining the vertices because those vertices cannot be joined by a path of length fewer than $m$. Thus the 1’s of $A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right)$ corresponds to all the $m$-distance paths joining the vertices of $G$. In other words,

$$D_m = A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right).$$

\[ \square \]

Theorem 3.7. Let $A_G$ be the adjacency matrix of $G = (V, E)$ with diameter $d$. Then,

$$D = \sum_{m=1}^{d} \left( A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right) \right).$$

Proof. By Theorem 3.2, $D = \sum_{m=1}^{d} mD_m$. Also, by Theorem 3.6,

$$D_m = A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right).$$

Combining above two, we have

$$D = \sum_{m=1}^{d} m \left( A_G^{(m)} - \delta \left( \sum_{s=0}^{m-1} A_G^{(m)} \circ A_G^{(s)} \right) \right).$$

\[ \square \]

This is the formula for finding the distance matrix $D$ of a graph $G$ from the powers of its adjacency matrix using Hadamard product.

4. Illustration

Consider the following graph $G$. Then the adjacency ma-
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The adjacency matrix \( A_G \) of \( G \) is

\[
A_G = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & 0 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 & 1 & 1 \\
4 & 0 & 0 & 1 & 0 & 0 \\
5 & 1 & 1 & 1 & 0 & 0 \\
6 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Here \( d = \text{Diam}(G) = 3 \), \( A_G^{(0)} = I \), \( A_G^{(1)} = \delta(A_G) = A_G \)

\[
A_G^2 = \begin{bmatrix}
3 & 2 & 1 & 1 & 2 & 1 \\
2 & 2 & 1 & 0 & 1 & 1 \\
1 & 1 & 3 & 0 & 1 & 2 \\
1 & 0 & 0 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 4 & 2 \\
1 & 1 & 2 & 0 & 2 & 3
\end{bmatrix}, \delta(A_G^2) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

\[
A_G^{(2)} \circ A_G^{(1)} = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A_G^3 = \begin{bmatrix}
4 & 3 & 6 & 1 & 7 & 7 \\
3 & 2 & 3 & 1 & 6 & 5 \\
6 & 3 & 2 & 3 & 7 & 3 \\
1 & 1 & 3 & 0 & 1 & 2 \\
7 & 6 & 7 & 1 & 6 & 7 \\
7 & 5 & 3 & 2 & 7 & 4
\end{bmatrix}, \delta(A_G^3) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
A_G^{(3)} \circ A_G^{(1)} = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
A_G^{(3)} \circ A_G^{(2)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A_G^{(1)} \circ I = 0, \quad D_0 = A_G^{(0)} = I \\
D_1 = A_G^{(1)} - \delta(A_G^{(1)} \circ A_G^{(0)}) = A_G^{(1)} - 0 = A_G^{(1)}
\]

\[
\left(A_G^{(2)} \circ A_G^{(0)} + A_G^{(2)} \circ A_G^{(1)}\right) = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

\[
D_2 = A_G^{(2)} - \delta\left(A_G^{(2)} \circ A_G^{(0)} + A_G^{(2)} \circ A_G^{(1)}\right) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\left(A_G^{(3)} \circ I + A_G^{(3)} \circ A_G^{(1)} + A_G^{(3)} \circ A_G^{(2)}\right) = \begin{bmatrix}
2 & 1 & 2 & 1 & 2 & 2 \\
1 & 2 & 1 & 0 & 2 & 2 \\
2 & 1 & 2 & 1 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 0 \\
2 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 0 & 2 & 2
\end{bmatrix}
\]

\[
D_3 = A_G^{(3)} - \delta\left(A_G^{(3)} \circ I + A_G^{(3)} \circ A_G^{(1)} + A_G^{(3)} \circ A_G^{(2)}\right) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

Now

\[
D = \sum_{m=1}^{3} mD_m = 1.D_1 + 2.D_2 + 3.D_3
\]

Generally it is difficult to find the distance matrix of a large order simple undirected graph. Here we provide a formula for finding the distance matrix from the adjacency matrix of a simple, connected , undirected graph of any finite order . A computer program can be easily written in any of the programming languages for computing distance matrix of the graph using this formula that shall give instant result. So far we considered only simple connected undirected graphs. We can extend this formula for weighted undirected graphs as well as digraphs.

### 5. Conclusion

### References


