An annotation on the prime graph of an integral domain

Mohiddin Shaw Shaik\textsuperscript{1*}, Nagaraju Dasari\textsuperscript{2} and Neha Gahlot\textsuperscript{3}

Abstract
We introduce the prime graph of the product ring $R_1 \times R_2$ where $R_1$, $R_2$ are integral domains, which is an extension of study on prime graph of an integral domain. We prove that, if $R_1, R_2$ are two integral domains, the graph obtained by removing the isolated vertices from $\text{PG}(R_1 \times R_2)$ is a bipartite graph. We obtain some consequences.

Keywords
Associative ring, Integral Domain, Graph, Prime Graph.

AMS Subject Classification
05C20, 05C25, 13E15, 68R10, 05C99.

1 Department of Mathematics, Narasaraopeta Engineering College, Narasaraopet-522601, Andhra Pradesh, India.
2 Department of Mathematics and Statistics, Manipal University, Jaipur-303007, Rajasthan, India.
*Corresponding author: \textsuperscript{1}mohiddin.shaw26@yahoo.co.in; \textsuperscript{2}dasari.nagaraju@gmail.com; \textsuperscript{3}gahlotnhea1995@gmail.com

Article History: Received \textit{02} February 2020; Accepted \textit{22} May 2020

\section{Introduction}

The prime graph of an associative ring, a concept from algebraic graph theory was introduced by Satyanarayana et.al \cite{11} has shown a new path for the researchers to explore and extend the study in their fields of interest. Satyanarayana et. al \cite{4, 5}, studied prime graphs related to a ring of integers modulo $n$. The complement of a prime graph of a ring was studied by Power and Joshi \cite{2}. These studies motivated us to derive few results in the prime graph of an integral domain which is an extension to the work of Satyanarayana et. al \cite{5}.

Our study is presented in three small sections. Section 1, is a collection of necessary definitions, and results from the literature. Section 2 and 3 contains new findings.

\begin{definition}
\textsuperscript{[7]} An algebraic system with a non-empty set $R$ together with two binary operations addition and multiplication is said to be a ring (or an associative ring) if $(R, +)$ is an abelian group; $(R, \cdot)$ is a semigroup and multiplication is distributive over the addition among the elements of $R$. If in addition $R$ satisfies commutative property with multiplication, then it is called a commutative ring. Further ring containing multiplicative identity is called a ring with unity.
\end{definition}

\begin{definition}
\textsuperscript{[7]} (i) A non-empty subset $I$ of a ring $R$ is called a left ideal if $(I, +)$ is subgroup of $(R, +)$ and for any element $r$ of $R$ and $i$ of $I$, $ri \in I$. It is called right ideal if $ir \in I$ for all elements $r$ of $R$ and $i$ of $I$.

(ii) Let $I,J$ be two ideals of $R$ such that $I \subseteq J$. We say that $I$ is essential (or ideal essential) in $J$ if it satisfies the following condition: $K \subseteq R, K \subseteq J, I \cap K = \{0\} \implies K = \{0\}$.

(iii) Given two distinct ideals $I$ and $J$ of $R$, if $I$ is essential in $J$, then we say that $J$ is proper essential extension of $I$. We use $I \leq_e J$ to represent $I$ is essential in $J$.
\end{definition}

\begin{definition}
\textsuperscript{[7]} (i) A non-zero ideal $I$ of $R$ is said to be uniform if for any other non-zero ideal $J$ of $R$ contained in $I$ imply $I \leq_e J$.

(ii) A non-zero ideal $K$ of $R$ is said to have finite dimension on ideals of $R$ (FDI, in short) if $K$ does not contain an infinite number of non-zero ideals of $R$ whose sum is direct. It is clear that if $R$ has FDI, then every non-zero ideal of $R$ has FDIR.
\end{definition}

\begin{definition}
A commutative ring with unity is said to be an integral domain if for any two element $a$ and $b$, $ab = 0$ implies $a = 0$ or $b = 0$.
\end{definition}
either \( a = 0 \) or \( b = 0 \).

**Theorem 1.6.** [7] Suppose \( H \) is a non-zero ideal of a ring \( R \) and \( H \) has finite dimension on ideals of \( R \). Then there exist ideals \( U_1, U_2, \ldots, U_n \) of \( R \) which are uniform whose sum is direct and essential in \( H \) and further these are unique in number.

**Corollary 1.7.** [7] If \( R \) is a ring with FDI, then there exist uniform ideals \( U_1, U_2, \ldots, U_n \) of \( R \) whose sum is direct and essential in \( R \); and if \( V_1, 1 \leq i \leq k \), possessing the same property as of \( U_j \), \( 1 \leq j \leq n \) mentioned above, then \( k = n \).

**Definition 1.8.** The number \( n \), obtained above, is called the dimension of \( H \), and is denoted by \( \dim H \).

For further developments in this dimension concept in ring theory, we refer [3, 7, 9].

Now we present some Graph theoretic concepts: A graph is a system \( G(V, E, \varphi) \) consist of non-empty set \( V \) of elements called vertices; another set \( E \) of elements called edges and incidence relation \( \varphi \) from \( E \) to \( v_i, v_j \) of \( V \). If in \( G \), both \( |V| \) and \( |E| \) are finite, then \( G \) is called a finite graph. If edge set in graph becomes empty then \( G \) is called an empty graph or a null graph. A simple graph is a graph in which no edge incident to same end vertices and no two edges share the same end vertices. A complete graph is a simple graph in which every vertex is adjacent to every other vertex in the graph. We use \( K_n \) to denote a complete graph with \( n \) vertices. The degree of a vertex \( d(v) \) is the count of number edges incident to it. A component of a graph is a subgraph which is maximally connected. The distance between any two vertices \( u \) and \( v \) of a graph \( G \) is denoted by \( d(u, v) \). In this paper we study only simple graphs. For a graph \( G(V, E, \varphi) \) if there is graph \( G_1 \) with vertex set \( X \) which is a non-empty subset of \( V \) and edge set which are exclusively connecting the vertices of \( X \) is called the subgraph generated by \( X \) or the maximal subgraph with vertex set \( X \).

A star graph is a graph having a fixed vertex \( v \) and edge set containing only edges which are incident with \( v \) and are not forming loop with the fixed vertex. An \( n \)-star graph is a star graph having \( n \) vertices in it.

We refer Herstein [1], and Satyanarayana and Syam Prasad [10] for further readings in ring theory and graph theory.

**Definition 1.9.** [11] A prime graph of a ring \( R \) is a graph \( G(V, E) \) having the vertex set as \( R \) and edge set contains only edges which satisfied either \( xRy = 0 \) or \( yRx = 0 \) for all distinct \( x, y \) from \( V \). It is denoted by \( PG(R) \).

**Example 1.10.** The prime graph of a ring of integers modulo 6 is given in following diagram 1.1.

**Observation 1.11.** [11] (i) Every prime graph of a ring is a simple graph. (ii) The degree of an additive identity element of a ring is always one less than number of elements of the ring. We can find a \( n \)-star graph as a sub graph of it as there always an edge between fixed vertex 0 to any other non-zero vertex of \( V \) together with edge connecting any two non-zero vertices satisfying the property mentioned in the definition. It is always a connected graph with distance from a vertex 0 to any other vertex is 1 and maximum distance from any two vertices 2. (iii) The distance between any two vertices of \( PG(R) \) becomes 2 if and only if when \( xRy \neq 0 \). (iv) The domination number of a prime graphs is 1 as \( \{0\} \) is a dominating set.

For further developments in prime graphs of a ring, we refer [2, 4–6, 9].

**2. \( PG(R) \) where \( R \) is an integral domain**

**Lemma 2.1.** [6] If the ring \( R \) becomes an integral domain, then prime graph of it is a star graph with number of vertices \( |R| \).

**Theorem 2.2.** [6] Given a prime number \( p \), the set of integers modulo \( p, \mathbb{Z}_p \) is a field and hence it is an integral domain. \( PG(\mathbb{Z}_p) \) is a star graph with number of vertices \( p \) and centre ‘0’. Conversely any star graph with \( p \) vertices is isomorphic to the graph \( PG(\mathbb{Z}_p) \).

**Example 2.3.** [6] (Prime graph of \( R \times \mathbb{Z}_2 \)) Suppose \( R \) is an integral domain and \( \mathbb{Z}_2 \) is a ring of integers modulo 2. For \( (a, b), (c, d) \in R \times \mathbb{Z}_2 \), we define addition and multiplication component wise. Then \( R \times \mathbb{Z}_2 \) becomes the product ring, and the zero element of \( R \times \mathbb{Z}_2 \) in \((0, 0), (0, 0) \times (1, 0) \) and \((0, 0) \times (0, 1) \) are two elements in \( R \times \mathbb{Z}_2 \) with \((1, 0) \not= (0, 1) \not= (0, 0) \). So \( R \times \mathbb{Z}_2 \) is not an integral domain.

**Theorem 2.4.** [6] Let \( R \) contains \( n \) elements. Then \( PG(R \times \mathbb{Z}_2) \) contains two particular elements \((0, 0) = a, (say), (0, 1) = b (say) \) such that \( |V(PG(R \times \mathbb{Z}_2))| = 2n \) and \( PG(R \times \mathbb{Z}_2) = \{\text{the } 2n\text{-star graph with } R \times \mathbb{Z}_2 \text{ as vertex set and centre } a\} \cup \{\text{the n-star graph with vertex set } \{(x, 0): 0 \neq x \in R \text{ with centre } b\} \}.

**Note 2.5.** In the proof of this theorem we arrived at two subgraphs \( H \) and \( K \) of \( PG(R \times \mathbb{Z}_2) \). We can state that \( E(H) \cap E(K) = \emptyset \) and \( a \not\in V(K) \).
Remark 2.6 (6). The graph $PG(R \times \mathbb{Z}_2)$ where $R$ an integral domain, satisfy the following properties:

(i) $|V(G)| = 2n$ where $n = |R|$.
(ii) It contains two particular vertices $a, b \in V(G)$ with $a \neq b$.
(iii) There exists a subgraph $H$ of $G$ such that $H$ is a 2n-star graph (with centre $a$).
(iv) There exists a subgraph $K$ of $G$ such that $K$ is a n-star graph (with centre $b$).
(v) $G = H \cup K$.

Theorem 2.7. [6] Suppose $G$ is a graph satisfying the following conditions:

(i) $|V(G)| = 2p$, where $p$ is a prime number.
(ii) $G$ contains two particular vertices $a^*, b^*$ with $a^* \neq b^*$.
(iii) $H^*$ is a 2p-star graph (with centre $a^*$) which is a subgraph of $G$.
(iv) $K^*$ is a p-star graph of $G$ (with center $b^*$) and $a^* \notin V(K)$.
(v) $G = H^* \cup K^*$. Then $G$ is isomorphic to $PG(Z_{2p} \times \mathbb{Z}_2)$.

Now we obtain the following new results:

Theorem 2.8. If $R$ is an integral domain, then

(i) $R$ is a uniform ideal and (ii) dim($R$) = 1.

Proof. Let $I$ be a non-zero ideal of $R$. We wish to prove that $I$ is essential in $R$. In a contrary way, suppose that $I$ is not essential in $R$. Then there exists a non-zero ideal $J$ of $R$ such that $I \cap J = \{0\}$. Let $0 \neq x \in I$ and $0 \neq y \in J$. Now $xy \in I \cap J = \{0\}$. We proved that $x, y$ are two non-zero elements such $xy = 0$, a contradiction (to the fact that $R$ is an integral domain). This shows that $I$ is essential in $R$. Therefore every non-zero ideal of $R$ is essential in $R$. By Theorem 7[8], we have that $R$ is Uniform and hence dim $R = 1$.

The proof of the following corollary from the fact that every field is an integral domain.

Corollary 2.9. If $R$ is a field, then $R$ is uniform and dim $R = 1$.

We denote the set of all isolated points of graph $G$ by $Iso(G)$.

Theorem 2.10. If $R_1, R_2$ are two integral domains, then $PG(R_1 \times R_2)$ is a bipartite graph.

Proof. Write $R_1^* = \{(a,0)/0 \neq a \in R_1\}$ and $R_2^* = \{(0,b)/0 \neq b \in R_2\}$.
Write $S = (R_1 \times R_2) \cup (R_1^* \times R_2^*)$. We wish to show that (i) $S = Iso(PG(R_1 \times R_2))$ and (ii) subgraph of $PG(R_1 \times R_2)$ generated by $R_1^* \cup R_2^*$ is a complete bipartite graph. It is clear that $S \subseteq (R_1 \times R_2) \cup (PG(R_1 \times R_2))$.

Proof for (i): Let $(a,b) \in S$. If $(a,b) = (0,0)$ then it is isolated. Suppose $(a,b) \neq (0,0)$. We show that $d(a,b) = 0$ where $d(a,b)$ is the degree of the vertex $(a,b)$. Since $(0,0) \notin (R_1^* \times R_2^*)$ we have that $d(a,b) = 0$. In a contrary way, suppose that $d(a,b) \neq 0$. Then there exists $(x,y) \in V(PG(R_1 \times R_2))$ such that $(a,b)$ and $(x,y)$ are adjacent. By the definition of prime graph $(a,b)(x,y) = (0,0)$ that implies $ax = 0$ and by $= 0$ implies that $x = 0$ and $y = 0$. (Since $0 \neq a \in R_1, 0 \neq b \in R_2, R_1$ and $R_2$ are integral domains). implies that $(x,y) = (0,0)$, a contradiction. Hence $d(a,b) = 0$ and so $(a,b)$ is an isolated point. Hence $S \subseteq Iso(PG(R_1 \times R_2))$.

Proof of (ii): To show that the subgraph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph we show the following four conditions. (i) $R_1^* \cap R_2^* = \emptyset$. (ii) there is no edge between two vertices belonging to $R_1^*$. (iii) There is no edge between two vertices belonging to $R_2^*$. (iv) $(a,b) \in R_1^*, (c,d) \in R_2^*$ implies there is an edge between $(a,b)$ and $(c,d)$. $R_1^* \cap R_2^* = \{(a,0)/0 \neq a \in R_1\} \cap \{(0,b)/0 \neq b \in R_2\} = \emptyset$. Let $(a,0)(0,b) = (0,0)$ and so $uv = 0$. That implies $u = 0$ or $v = 0$ (since $R_1$ is an integral domain) and hence $(0,0) \notin R_1^*$, a contradiction. So we verified that there is no edge between any two vertices in $R_1^*$. A similar valid argument shows that there is no edge between any two vertices of $R_2^*$. Let $(a,0) \in R_1^*$ and $(0,b) \in R_2^*$. Then $(a,0) \neq (0,0) \neq (0,b)$ and $(a,0)(0,b) = (0,0)$ and so there is an edge between $(a,0)$(0,b). Hence one can conclude that the graph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph.

Proof of (iii) By Part(iii), we have that $R_1^* \cup R_2^* = R_1 \times R_2$ Iso ($PG(R_1 \times R_2)$). So vertex set of the subgraph generated by $R_1^* \cup R_2^* = V(PG(R_1 \times R_2))$ Iso ($PG(R_1 \times R_2)$). By part (ii), the subgraph generated by $(R_1 \times R_2)$ is a complete bipartite graph. This shows that $PG(R_1 \times R_2)$ Iso ($PG(R_1 \times R_2)$)

3. An application to $Z_p$, ring of integers modulo a prime number $p$

Let $p, q$ be two prime numbers. Then $Z_{pq}$ are two integral domains.

Lemma 3.1. $PG(Z_p \times Z_q)$Iso ($PG(Z_p \times Z_q)$) forms a complete bipartite graph $K_{(p-1)(q-1)}$.

Proof. Write $R_1 = Z_p$ and $R_2 = Z_q$. Then the proof follows from Theorem 2.9.

Theorem 3.2. Suppose that $p, q$ are prime numbers. Then the subgraph $PG(Z_p \times Z_q)$Iso ($PG(Z_p \times Z_q)$) is complete bipartite graph $K_{(p-1)(q-1)}$. Conversely any complete bipartite graph $K_{(p-1)(q-1)}$ (where $p, q$ are primes) is isomorphic to a subgraph of $PG(R_1 \times R_2)$ that is generated by $R_1^* \cup R_2^*$ where $R_1 = Z_p$ and $R_2 = Z_q$. 

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Proof. Write $R_1 = Z_p$ and $R_2 = Z_q$. Then the first part is Lemma 3.1.

Converse: Consider the complete bipartite graph $(K_{(p-1)(q-1)})$ with $p, q$ are prime. Suppose the set of vertices of $(K_{(p-1)(q-1)})$ are divided into the partition $\{x_1, x_2, \ldots, x_\frac{p-1}{2}\}$ and $\{y_1, y_2, \ldots, y_\frac{q-1}{2}\}$. Now $V(K_{(p-1)(q-1)}) = \{x_1, x_2, \ldots, x_\frac{p-1}{2}\} \cup \{y_1, y_2, \ldots, y_\frac{q-1}{2}\}$. Write $R_1 = Z_p$, the integral domain of integer modulo $p$, and $R_2 = Z_q$ the integral domain of integer modulo $q$. Now $R_1^* = \{(i, 0) | 1 \leq i < p -1\}$ and $R_2^* = \{(0, j) | 1 \leq j < q -1\}$. Define $f: R_1^* \cup R_2^* \rightarrow V(K_{(p-1)(q-1)})$ by $f((i, 0)) = x_i$ for all $1 \leq i \leq p -1$ and $f((0, j)) = y_j$ for all $1 \leq j \leq q -1$. Also $f(((i, 0)(j, 0))) = f((i, 0))f((j, 0))$. We proved that $K_{(p-1)(q-1)}$ is isomorphic to the subgraph PG$(Z_p \times Z_q)$. (Is PG$(Z_p \times Z_q)$ of PG$(Z_p \times Z_q)$.

Example 3.3. $Z_p \times Z_q$

Figure 2. PG$(Z_6 \times Z_3)$

Observation 3.4. PG$(Z_6 \times Z_3)$ is not a complete graph because there is no edge between $(1, 0)$ and $(2, 0)$. PG$(Z_6 \times Z_3)$ is not bipartite graph because it contains a triangle $\{(2, 0), (0, 2), (3, 0)\}$.

Note 3.5. Example 3.3. shows that Theorem 3.2 fails if $p$ is not a prime number. So our main result 3.2 of this section is not true if both $p, q$ are not prime numbers.

References


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