Weyl-semi symmetric special Para-Sasakian manifold

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Abstract
In this paper, we investigate the theory of Weyl-semi symmetric special Para-Sasakian. In section 1, we have defined special Para-Sasakian manifold and established a few properties thereof. Section 2 is devoted to the study of Weyl-pseudo symmetric and Weyl-semi symmetric special Para-Sasakian manifold. The results of this paper are believed to be new and unified in nature.

Keywords

AMS Subject Classification
53C25, 53Cxx, 53-XX.

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Article History: Received 04 February 2020; Accepted 09 May 2020

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1. Introduction

Let \( M \) be a connected \( n \)-dimensional Riemannian manifold of class \( C^\infty \) with a positive definite metric \( g \) which admits a unit 1-from \( \eta \) satisfying

\[
\nabla_\beta \eta_\alpha - \nabla_\alpha \eta_\beta = 0
\]  

(1.1)

and

\[
\nabla_\gamma \nabla_\beta \eta_\alpha = -(g_{\gamma \beta} \eta_\alpha + g_{\gamma \alpha} \eta_\beta) + 2 \eta_\gamma \eta_\beta \eta_\alpha
\]  

(1.2)

wherein \( \nabla \) denotes the covariant differentiation with regard to Levi-Civita connection.

If we take

\[
\bar{\zeta}_\alpha = g^{\alpha \beta} \eta_\beta
\]  

(1.3)

\[
\phi^\alpha_\beta = \nabla_\beta \bar{\zeta}_\alpha
\]  

(1.5)

and

\[
\phi^\alpha_\beta = g_{\alpha \gamma} \phi^\gamma_\beta
\]  

(1.6)

Consequently, we obtain

\[
\eta_\alpha \bar{\zeta}_\alpha = 1
\]  

(1.7)

\[
\phi^\alpha_\beta = \phi_\beta \alpha
\]  

(1.8)

\[
\phi^\alpha_\beta \bar{\xi}_\beta = 0
\]  

(1.9)

\[
\phi^\alpha_\beta \eta_\alpha = 0
\]  

(1.10)

\[
\phi_\beta \phi^\alpha_\gamma = \delta^\alpha_\beta - \eta_\beta \bar{\zeta}_\alpha
\]  

(1.11)

\[
g_{\gamma \epsilon} \phi^\gamma_\beta \phi^\epsilon_\beta = g_{\alpha \beta} - \eta_\alpha \bar{\zeta}_\beta
\]  

(1.12)
wherein and the Weyl conformal curvature tensor is defined as

\[ R.C = L_C Q(g, C). \] (2.5)

Definition 2.2. A special Para-Sasakian manifold \( M \) with the properties

\[ C.S = 0 \] (2.6)

is termed as Weyl semi-symmetric special Para-Sasakian manifold.

Remark 2.3. It is noteworthy that a conformally symmetric special Para-Sasakian manifold is Weyl semi-symmetric.

Next, we define the tensor \( C.S \) on \((M, g)\) as follows

\[ S^a \mathcal{C}_b = -\left( \mathcal{C}_b \mathcal{C}^a \right) \mathcal{C}_\eta + \mathcal{C}_b \mathcal{C}^a \mathcal{C}_\eta. \] (2.7)

Equation (2.7) can be written as

\[ S^a \mathcal{C}_b + S_a \mathcal{C}_b = 0. \] (2.8)

Contracting equation (2.8) by \( \xi^a \) and using equation (1.15) yields

\[ \eta \mathcal{C}_\beta + \eta \mathcal{C}_\beta = 0. \] (2.9)

By virtue of equations (1.15), (1.16), (2.2) and (2.3), we obtain

\[ \eta \mathcal{C}_\beta + \eta \mathcal{C}_\beta = 0. \] (2.10)

Contracting equation (2.10) by \( \xi^a \) and using equations (1.15), (2.2), we get

\[ (S,S)_{a\beta} = \frac{k}{n-1} \mathcal{C}_\alpha + (k+1) g_{a\beta}. \] (2.11)

In view of above discussion, we observe the following theorem:

Theorem 2.4. If \( n \)-dimensional special Para-Sasakian manifold is Weyl-semi symmetric then the following condition (2.11) holds good.

Let us consider an \( \eta \)-Einstein special Para-Sasakian manifold, then we can write [2]:

\[ S_{a\beta} = a g_{a\beta} + b \eta_a \eta_\beta, \] (2.12)

wherein \( a \) and \( b \) are smooth functions on \( M \).

Contracting equation (2.12) with \( g^{a\beta} \) and using equation (1.17), we get

\[ \eta a + b = \tau. \] (2.13)

Further, contracting equation (2.12) with \( \xi^\beta \) and using equations (1.7), (1.15) yields

\[ a + b = (1-n). \] (2.14)
Subtracting equation (2.14) from equation (2.13), we get
\[ a = 1 - \frac{\tau}{(1 - n)}. \]  
(2.15)

Inserting this value of \( a \) in equation (2.14), we obtain
\[ b = \frac{\tau}{(1 - n)} - n. \]  
(2.16)

Consequently, we have a theorem:

**Theorem 2.5.** If \( \eta \)-Einstein special Para-Sasakian manifold is Weyl-semi symmetric admits a vector field \( \xi^\alpha \) characterised by the relation (2.12) then the smooth functions are connected by the relations (2.15) and (2.16).

Substituting the values of \( a \) and \( b \) in equation (2.12), we get
\[ S_{\alpha \beta} = (1 - \frac{\tau}{(1 - n)})g_{\alpha \beta} + (\frac{\tau}{(1 - n)} - n)\eta_\alpha \eta_\beta. \]  
(2.17)

Consequently, we have a theorem:

**Theorem 2.6.** If a special Para-Sasakian manifold is an \( \eta \)-Einstein admits a condition \( C.S = 0 \), and a vector field \( \xi^\alpha \) characterised by the relation (2.12) then the Ricci tensor holds the relation (2.17).

In this regard, we have a theorem:

**Theorem 2.7.** For an \( \eta \)-Einstein special Para-Sasakian manifold with the condition \( C.S = 0 \), the following relation
\[ S_{\alpha \beta} \phi^\beta_\gamma = (1 - \frac{\tau}{(1 - n)})\phi_\alpha \gamma \]  
holds good.

\[ \text{Proof.} \] Contracting equation (2.17) with \( \phi^\beta_\gamma \) and using equations (1.6), (1.10) yields
\[ S_{\alpha \beta} \phi^\beta_\gamma = (1 - \frac{\tau}{(1 - n)})\phi_\alpha \gamma. \]  
(2.18)

From equations (1.12) and (2.17), we get
\[ S_{\alpha \beta} = (1 - n)g_{\alpha \beta} - (\frac{\tau}{(1 - n)} - n)g_{\gamma \epsilon} \phi^\gamma_\beta \phi^\epsilon_\alpha. \]  
(2.19)

As a consequence of equations (1.6) and (2.19), we obtain
\[ S_{\alpha \beta} = (1 - n)g_{\alpha \beta} - (\frac{\tau}{(1 - n)} - n)\phi_\alpha \phi_\beta. \]  
(2.20)

By virtue of equations (1.5) and (2.20), we observe that
\[ S_{\alpha \beta} = (1 - n)g_{\alpha \beta} - (\frac{\tau}{(1 - n)} - n)(\nabla_\epsilon \eta_\alpha)(\nabla_\beta \xi^\epsilon). \]  
(2.21)

Contracting equation (2.20) with \( \xi^\beta \) and using equation (1.9) yields
\[ S_{\alpha \beta} \xi^\beta = -(n - 1)\eta_\alpha. \]  
(2.22)

This expression obtained above is similar to the expression (1.15) given by Mileva Prvanovic [2].

In view of above, we have the following theorems:

**Theorem 2.8.** For \( \eta \)-Einstein special Para-Sasakian manifold, the relation \( \tau = -(n - 1) \) holds good.

\[ \text{Proof.} \] Contracting equation (2.22) with \( \eta_\beta \) and using equation (1.7), we obtain
\[ S_{\alpha \beta} = -(n - 1)\eta_\alpha \eta_\beta. \]  
(2.23)

Again contracting equation (2.23) with \( g^{\alpha \beta} \) and using equations (1.3), (1.7) yields
\[ g^{\alpha \beta} S_{\alpha \beta} = -(n - 1). \]  
(2.24)

From equations (1.17) and (2.24), we get
\[ \tau = -(n - 1) \]  
(2.25)

**Theorem 2.9.** If \( \eta \)-Einstein special Para-Sasakian manifold admits \( C.S = 0 \), then the following relation
\[ (S.S)_{\alpha \beta} \phi^\beta_\gamma = (k + n - 1)\phi_\alpha \gamma \]  
holds good.

\[ \text{Proof.} \] Contracting equation (2.23) with \( \phi^\beta_\gamma \) and using the equation (1.10) yields
\[ S_{\alpha \beta} \phi^\beta_\gamma = 0. \]  
(2.26)

Contracting equation (2.11) with \( \phi^\beta_\gamma \) and using equations (2.26), we get
\[ (S.S)_{\alpha \beta} \phi^\beta_\gamma = (k + n - 1)\phi_\alpha \gamma. \]  
(2.27)

**References**


