On a class of $b$-$\gamma$-open sets in a topological space

C. Sivashanmugaraja 1*

Abstract
In this paper, we analyze the properties of $b$-$\gamma$-open sets in a topological space. Further, the concept of $b$-$\gamma$-boundary, $b$-$\gamma$-exterior, $b$-$\gamma$-limit point, $b$-$\gamma$-neighborhood, locally $b$-$\gamma$-closed and $b$-$\gamma$-generalized closed sets are introduced and investigated.

Keywords
$b$-$\gamma$-open sets, $b$-$\gamma$-boundary, $b$-$\gamma$-exterior, $b$-$\gamma$-limit point, $b$-$\gamma$-neighborhood, $b$-$\gamma$-generalized closed.

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1 Department of Mathematics, Periyar Government Arts College, Cuddalore-607001, Tamil Nadu, India.
*Corresponding author: csrajamaths@yahoo.co.in
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Contents
1 Introduction ........................................... 977
2 Preliminaries ........................................ 977
3 $b$-$\gamma$-open and $b$-$\gamma$-closed sets .................. 977
4 $b$-$\gamma$-boundary and $b$-$\gamma$-exterior ................. 978
5 $b$-$\gamma$-g-open and $b$-$\gamma$-g-closed sets ............... 980
6 Conclusion .......................................... 981
References ............................................. 981

1. Introduction

Kasahara [2] introduced the notion of an operation $\gamma$ in 1979. The notion of $\gamma$-open sets were introduced and investigated by Ogata [4] in 1991. Ibrahim [3] introduced the concept of $b$-$\gamma$-open set by using the operation $\gamma$. Further, he continued studying the weak forms of $\gamma$-open sets in his work. Andrijevic [1] introduced the notion of $b\gamma$-open sets. The aim of this paper is to analyze some properties of $b\gamma$-open sets in a topological space. Further the concepts of $b\gamma$-boundary, $b\gamma$-exterior, $b\gamma$-limit point, $b\gamma$-neighborhood, $b\gamma$-generalized closed set and locally $b\gamma$-closed spaces are introduced. Also, the relationship among these sets are discussed.

2. Preliminaries

Throughout this paper, $(X, \tau)$ or $X$ always mean topological space.

Definition 2.1. [4] Let $(X, \tau)$ be a space and $\gamma$ be an operation on $\tau$. A $\subseteq X$ is called $\gamma$-open if $\forall x \in A, \exists$ an open set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$. Then the collection of all $\gamma$-open sets in $X$ are denoted by $\tau_\gamma$. Evidently $\tau_\gamma \subseteq \tau$. A subset $A$ of $X$ is called $\gamma$-closed $\iff$ its complement is $\gamma$-open.

Definition 2.2. [4] Let $(X, \tau)$ be a space and $\gamma$ be an operation on $\tau$. Then $X$ is said to be $\gamma$-regular, if $\forall x \in X$ and $\forall$ open neighborhood $V$ of $x$, $\exists$ an open neighborhood $U$ of $x$, such that $\gamma(U) \subseteq V$. A space $X$ is $\gamma$-regular space $\iff$ $\tau = \tau_\gamma$.

Definition 2.3. [3] Let $(X, \tau)$ be a space. $A \subseteq X$ is said to be $b\gamma$-open if $A \subseteq \tau_\gamma$-$\text{int}(\text{cl}(A)) \cup \text{cl}(\tau_\gamma$-$\text{int}(A))$.

Definition 2.4. [1] Let $(X, \tau)$ be a space. $A \subseteq X$ is said to be $b\gamma$-open if $A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$.

Definition 2.5. [3] Let $(X, \tau)$ be a space with an operation $\gamma$ on the topology $\tau$. Then the intersection of two $b\gamma$-open sets may not be $b\gamma$-open.

Definition 2.6. [3] Let $(X, \tau)$ be a space with an operation $\gamma$ on the topology $\tau$. Then if $\{A_i : i \in \Delta\}$ is a collection of $b\gamma$-open sets of a space $(X, \tau)$, then $\bigcup_{i \in \Delta} A_i$ is a $b\gamma$-open set.

3. $b\gamma$-open and $b\gamma$-closed sets

Remark 3.1. Let $(X, \tau)$ be a space and $B$ is a subset of $X$. Then $B$ is said to be $b\gamma$-closed $\iff B^c$ is $b\gamma$-open.

Further, the set of all $b\gamma$-open sets and $b\gamma$-closed sets of $(X, \tau)$ are denoted by $b\gamma O(X)$ and $b\gamma C(X)$ respectively.
Definition 3.2. Let \((X, \tau)\) be a space and \(A \subseteq X\). Then the \(b\gamma\)-closure of \(A\) (briefly, \(b\gamma\text{cl}(A)\)) is given by \(b\gamma\text{cl}(A) = \bigcap\{B : A \subseteq B\) and \(B \in b\gamma\text{C}(X)\}\).

Definition 3.3. Let \((X, \tau)\) be a space and \(A \subseteq X\). Then the \(b\gamma\)-interior of \(A\) (briefly, \(b\gamma\text{int}(A)\)) is given by \(b\gamma\text{int}(A) = \bigcup\{B : A \supseteq B\) and \(B \in b\gamma\text{O}(X)\}\).

Theorem 3.4. Let \((X, \tau)\) be a space with an operation \(\gamma\) on the topology \(\tau\). Then the below statements hold:

(i) Each \(\gamma\)-open set of \((X, \tau)\) is \(b\gamma\)-open set in \((X, \tau)\);

(ii) Each \(b\gamma\)-open set of \((X, \tau)\) is \(b\gamma\)-open set in \((X, \tau)\).

Proof. (i) Let \(B\) be a \(\gamma\)-open set. Then \(\gamma(B) = \tau\gamma\text{int}(B)\). Since \(\tau\gamma\text{int}(B) \subseteq \text{cl}(\tau\gamma\text{int}(B)) \subseteq \text{cl}(\tau\gamma\text{int}(B)) \cup \tau\gamma\text{int}(\text{cl}(B))\).

Therefore, \(B\) is \(b\gamma\)-open.

(ii) Evident. \(\square\)

Remark 3.5. The converse of the above Theorem 3.4 may not be true as shown in the below examples.

Example 3.6. Let \(X = \{a, b, c\}\) and \(\tau X = \{X, \phi, \{a, c\}\}\). Define an operation \(\gamma\) on \(\tau X\) by \(\gamma(B) = B\). Here, the set \(\{b, c\}\) is \(\gamma\)-open but it is not \(b\gamma\)-open.

Example 3.7. Let \(X = \{a, b, c\}\) and \(\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\). Define an operation \(\gamma\) on \(\sigma\) by

\[
\gamma(B) = \begin{cases} 
B, & \text{if } B = \{b\} \\
X, & \text{if } B \neq \{b\}.
\end{cases}
\]

Then the set \(\{a\}\) is \(\gamma\)-open but not \(b\gamma\)-open.

Remark 3.8. The notion of \(b\)-open and \(b\gamma\)-open sets are independent. A space \(X\) is \(\gamma\)-regular space if and only if the sets \(b\)-open and \(b\gamma\)-open are equal.

Definition 3.9. In the above Example 3.7, the set of all \(\gamma\)-open sets \(\tau\gamma = \{X, \phi, \{b\}\}\). Here, \(b\gamma\)-open and \(b\)-open sets are not equal. Again, suppose we define \(\gamma\) on \(\tau\) by \(\gamma(B) = B\), then the sets \(b\gamma\)-open and \(b\)-open are equal.

Proposition 3.10. Let \(B\) and \(C\) be two subsets of a space \((X, \tau)\) with an operation \(\gamma\) on the topology \(\tau\). Then the below statements hold:

(i) \(b\gamma\text{cl}(\emptyset) = \emptyset\) and \(b\gamma\text{cl}(X) = X\);

(ii) \(B\) is a \(b\gamma\)-closed \(\iff\) \(b\gamma\text{cl}(B) = B\);

(iii) \(b\gamma\text{cl}(B)\) is a \(b\gamma\)-closed set of \((X, \tau)\) and \(B \subseteq b\gamma\text{cl}(B)\);

(iv) If \(B \subseteq C\), then \(b\gamma\text{cl}(B) \subseteq b\gamma\text{cl}(C)\);

(v) \(b\gamma\text{cl}(B) \cup b\gamma\text{cl}(C) \subseteq b\gamma\text{cl}(B \cup C)\);

(vi) \(b\gamma\text{cl}(B \cap C) \subseteq b\gamma\text{cl}(B) \cap b\gamma\text{cl}(C)\).

Proof. Evident. \(\square\)

Proposition 3.11. Let \(B\) and \(C\) be two subsets of a space \((X, \tau)\) with an operation \(\gamma\) on the topology \(\tau\). Then the below statements hold:

(i) \(b\gamma\text{int}(\emptyset) = \emptyset\) and \(b\gamma\text{int}(X) = X\);

(ii) \(B\) is a \(b\gamma\)-open \(\iff\) \(b\gamma\text{int}(B) = B\);

(iii) \(b\gamma\text{int}(B)\) is a \(b\gamma\)-open set of \((X, \tau)\) and \(b\gamma\text{int}(B) \subseteq B\);

(iv) If \(B \subseteq C\), then \(b\gamma\text{int}(B) \subseteq b\gamma\text{int}(C)\);

(v) \(b\gamma\text{int}(B) \cup b\gamma\text{int}(C) \subseteq b\gamma\text{int}(B \cup C)\);

(vi) \(b\gamma\text{int}(B \cap C) \subseteq b\gamma\text{int}(B) \cap b\gamma\text{int}(C)\).

Proof. Evident. \(\square\)

Proposition 3.12. Let \(B \subseteq X\) with an operation \(\gamma\) on the topology \(\tau\). Then the below statements hold:

(i) \(\text{int}(B) \subseteq \text{int}(B) \subseteq \text{int}(B) \subseteq \text{cl}(B) \subseteq \text{cl}(B)\);

(ii) \(\text{int}(B) \subseteq \text{int}(B) \subseteq \text{int}(B) \subseteq \text{cl}(B) \subseteq \text{cl}(B)\).

Proposition 3.13. Let \(B \subseteq X\) with an operation \(\gamma\) on the topology \(\tau\). Then the below statements are equivalent:

(i) \(B\) is a \(b\gamma\)-open set in \((X, \tau)\);

(ii) \(X \setminus B\) is a \(b\gamma\)-closed set in \((X, \tau)\);

(iii) \(b\gamma\text{cl}(X \setminus B) = X \setminus B\).

Proof. Evident. \(\square\)

Proposition 3.14. Let \(B \subseteq X\) with an operation \(\gamma\) on the topology \(\tau\). Then the below statements are equivalent:

(i) \(B\) is a \(b\gamma\)-closed set in \((X, \tau)\);

(ii) \(X \setminus B\) is a \(b\gamma\)-open set in \((X, \tau)\);

(iii) \(b\gamma\text{int}(X \setminus B) = X \setminus B\);

Proof. Evident. \(\square\)

4. \(b\gamma\)-boundary and \(b\gamma\)-exterior

Definition 4.1. Let \(C\) be a subset of a space \((X, \tau)\). Then the \(b\gamma\)-boundary of \(C\) (briefly, \(b\gamma\text{bd}(C)\)) is given by \(b\gamma\text{bd}(C) = b\gamma\text{cl}(C) \cap b\gamma\text{cl}(X \setminus C)\).

Theorem 4.2. Let \((X, \tau)\) be a space and \(B \subseteq X\). Then the below statements are hold:

\((1)\) \(b\gamma\text{bd}(B) = b\gamma\text{bd}(X \setminus B)\);

\((2)\) \(b\gamma\text{bd}(B) = b\gamma\text{cl}(B) \setminus \text{bint}(B)\);

\((3)\) \(b\gamma\text{bd}(B) \cap \text{bint}(B) = \emptyset\);

\((4)\) \(b\gamma\text{bd}(B) \cup \text{bint}(B) = \text{bcl}(A)\).
Then the below statements are hold:

(i) The set $B$ is a $b$-$\gamma$-open $\iff B \cap b_\gamma(B) = \emptyset$;

(ii) The set $B$ is a $b$-$\gamma$-closed $\iff b_\gamma(B) \subseteq B$;

(iii) The set $B$ is a $b$-$\gamma$-clopen $\iff b_\gamma(B) = \emptyset$.

Proof. (i) Suppose that $B$ be a $b$-$\gamma$-open set. Then $B = b_\gamma(B)$, Thus $B \cap b_\gamma(B) = b_\gamma(B) = \emptyset$. Conversely, let $B \cap b_\gamma(B) = \emptyset$. Then by Theorem 4.2, $B \cap [b_\gamma(B) \cap b_\gamma(B)] = B \cap b_\gamma(B) = \emptyset$. So, $B = b_\gamma(B)$ and hence $B$ is $b$-$\gamma$-open.

(ii) Suppose that $B$ be a $b$-$\gamma$-closed set. Then $B = b_\gamma(B)$. But $b_\gamma(B) \cap b_\gamma(B) = b_\gamma(B)$. Therefore $b_\gamma(B) \subseteq B$. Conversely, consider $b_\gamma(B) \subseteq B$. By Theorem 4.2, $b_\gamma(B) = b_\gamma(B) \cup b_\gamma(B) = b_\gamma(B)$. Therefore $b_\gamma(B) \subseteq B$ and $B \subseteq b_\gamma(B)$. Hence, $B = b_\gamma(B)$. Thus $B$ is $b$-$\gamma$-closed.

(iii) Suppose that $B$ be a $b$-$\gamma$-clopen set. Then $B = b_\gamma(B)$ and also $B = b_\gamma(B)$. Then by Theorem 4.2, $b_\gamma(B) = b_\gamma(B) \cap b_\gamma(B) = \emptyset$. Conversely, assume that $b_\gamma(B) = \emptyset$. Then $b_\gamma(B) = b_\gamma(B) \cap b_\gamma(B) = \emptyset$ and hence, $B$ is $b$-$\gamma$-clopen.

Definition 4.4. Let $(X, \tau)$ be a space and $B$ be a subset of a space $X$. Then the set $X \setminus b_\gamma(B)$ is said to be $b$-$\gamma$-exterior of $B$ and is denoted by $b_\gamma(B)$. Every point $x$ in $X$ is said to be a $b$-$\gamma$-exterior point of $B$, if it is a $b$-$\gamma$-interior point of $X \setminus B$.

Definition 4.5. Let $(X, \tau)$ be a space and $N$ be a subset of a space $X$. $N$ is said to be a $b$-$\gamma$-neighborhood of a point $x \in X$ if there exists an open set $P$ such that $x \in P \subseteq N$.

The class of all $b$-$\gamma$-nbds of $x \in X$ is called the $b$-$\gamma$-neighborhood system of $x$ and it is denoted by $b_\gamma-N_x$.

Theorem 4.6. Let $B$ and $C$ be two subsets of a space $(X, \tau)$. Then the below statements are hold:

(i) $b_\gamma(B) = \emptyset$ and $b_\gamma(B) = \emptyset$;

(ii) $b_\gamma(B) = b_\gamma(X \setminus B)$;

(iii) $b_\gamma(B) \cap b_\gamma(B) = \emptyset$;

(iv) $b_\gamma(B) \cup b_\gamma(B) = b_\gamma(X \setminus B)$;

(v) $\{b_\gamma(B), b_\gamma(B) \text{ and } b_\gamma(B)\}$ form a partition of $X$;

(vi) If $B \subseteq C$, then $b_\gamma(C) \subseteq b_\gamma(B)$;

(vii) $b_\gamma(B \cup C) = b_\gamma(B) \cup b_\gamma(C)$;

(viii) $b_\gamma(B \cap C) \supseteq b_\gamma(B) \cap b_\gamma(C)$.

Proof. (i) Evident.

(ii) Evident from Definition 4.4.

(iii) From statement (ii) and Theorem 4.2, we have $b_\gamma(B \cup b_\gamma(B)) = b_\gamma(X \setminus B \cap b_\gamma(X \setminus B) = \emptyset$.

(iv) Also, From statement (ii) and Theorem 4.2, we have $b_\gamma(B \cup b_\gamma(B)) = b_\gamma(X \setminus B \cup b_\gamma(X \setminus B) = b_\gamma(X \setminus B)$.

Proof. (ii) Evident.

(vi) Evident from Definition 4.4.

(vii) By definition, $b_\gamma(B \cup C) = X \setminus b_\gamma(B \cup C) \subseteq X \setminus b_\gamma(B \cup C) \subseteq X \setminus b_\gamma(B \cup C) = [\gamma(B \cup C) \cap X \setminus b_\gamma(B \cup C)] = b_\gamma(B \cup C) \cap b_\gamma(C) = b_\gamma(B) \cup b_\gamma(C) = b_\gamma(B \cap b_\gamma(C) = \emptyset$.

(viii) Also by definition, $b_\gamma(B \cap C) = X \setminus b_\gamma(B \cap C) = X \setminus b_\gamma(B \cap C) = X \setminus b_\gamma(B \cap C) = b_\gamma(B) \cup b_\gamma(B) \cap b_\gamma(C) = b_\gamma(B) \cup b_\gamma(B) \cap b_\gamma(C)$.

Remark 4.7. In the above Theorem 4.6, the inclusion relation of the statement (vi), (vii) cannot be replaced by equality as shown in the below example.

Example 4.8. Let $X = \{a, b, c\}$ with topology $\tau_X = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. Define an operation $\gamma$ on $\tau_X$ by

$$\gamma(B) = \begin{cases} \text{int}(cl(B)), & \text{if } a \in B; \\ cl(B), & \text{if } a \notin B. \end{cases}$$

Let $C = \{a, b\}$ and $D = \{b, c\}$. Then $b_\gamma(C) = \emptyset$ and $b_\gamma(D) = \emptyset$. But $b_\gamma(C \cup D) = \emptyset$. Also, $b_\gamma(C \cap D) = \emptyset$. Therefore, $b_\gamma(C \cap D) = \emptyset$.

Definition 4.9. Let $X$ be a space and $B \subseteq X$. Then a point $x \in X$ is said to be a $b$-$\gamma$-limit point of a set $B \subseteq X$ if every $b$-$\gamma$-open set $P \subseteq X$ containing $x$ contains a point of $B$ other than $x$.

The collection of all $b$-$\gamma$-limit points of $B$ is said to be a $b$-$\gamma$-derived set of $B$ and it is mentioned by $b_\gamma-DS(B)$.

Proposition 4.10. Let $B$ be a subset of a space $(X, \tau)$. Then, the below statements are hold:

(i) The set $B$ is $b$-$\gamma$-closed $\iff b_\gamma-DS(B) \subseteq B$;

(ii) The set $B$ is $b$-$\gamma$-open $\iff B$ is $b$-$\gamma$-neighborhood, $\forall$ point $x \in B$;

(iii) $b_\gamma(B) = B \cup b_\gamma-DS(B)$.

Proof. (i) Let $B$ be a $b$-$\gamma$-closed set and $x \in B$. Then $x \in X \setminus B$, which is open. Thus $\exists$ a $b$-$\gamma$-open set $(X \setminus B)$ such that $(X \setminus B) \cap B = \emptyset$. Therefore $x \notin b_\gamma-DS(B)$. Thus, $b_\gamma-DS(B) \subseteq B$.

Conversely, assume that $b_\gamma-DS(B) \subseteq B$ and $x \notin B$. Then $x \notin b_\gamma-DS(B)$. Thus $\exists$ a $b$-$\gamma$-open set $V$ containing $x$ such that...
Then the below statements are hold:

(i) Let $B$ be a $b$-$\gamma$-open set. Then $B$ is a $b$-$\gamma$-neighborhood, $\forall x \in B$.

Conversely, let $B$ be a $b$-$\gamma$-neighborhood, $\forall x \in G$. Then $\exists$ a $b$-$\gamma$-open set $V_x$ containing $x$ such that $x \in V_x \subseteq B$. Therefore $B = \bigcup \{V_x\}$. Thus, $B$ is a $b$-$\gamma$-open.

(iii) Since, $b$-$\gamma$Ds$(B) \subseteq bcl_{\gamma}(B)$ and $B \subseteq bcl_{\gamma}(B)$, $B \cup b$-$\gamma$Ds$(B) \subseteq bcl_{\gamma}(B)$.

Conversely, assume that $x \notin b$-$\gamma$Ds$(B) \cup B$. Then $x \notin b$-$\gamma$Ds$(B)$, $x \notin B$. Then $\exists$ a $b$-$\gamma$-open set $V$ containing $x$ such that $V \cap B = \emptyset$. Therefore $x \notin bcl_{\gamma}(B)$ which implies that $bcl_{\gamma}(B) \subset B \cup b$-$\gamma$Ds$(B)$. Thus, $bcl_{\gamma}(B) = B \cup b$-$\gamma$Ds$(B)$.

**Theorem 4.11.** Let $B$ and $C$ be two subsets of a space $(X, \tau)$. Then the below statements are hold:

(i) If $B \subset C$, then $b$-$\gamma$Ds$(B) \subset b$-$\gamma$Ds$(C)$.

(ii) $B$ is a $b$-$\gamma$-closed set $\iff$ $B$ contains each of its $b$-$\gamma$-limit points.

(iii) $bcl_{\gamma}(B) = B \cup b$-$\gamma$Ds$(B)$.

**Proof.** (i) Evident.

(ii) If $B$ be a $b$-$\gamma$-closed set, then $X \setminus B$ is $b$-$\gamma$-open. If $x \notin B$, then $x \in X \setminus B$. Then $\exists$ a $b$-$\gamma$-open $(X \setminus B)$ such that $(X \setminus B) \cap B = \emptyset$. Therefore $x \notin b$-$\gamma$Ds$(B)$. Hence, $b$-$\gamma$Ds$(B) \subset B$.

Conversely, assume that $b$-$\gamma$Ds$(B) \subset B$ and $x \notin B$. Then $x \notin b$-$\gamma$Ds$(B)$. Then $\exists$ a $b$-$\gamma$-open set $M$ containing $x$ such that $M \cap B = \emptyset$ and therefore

$$X \setminus B = \bigcup_{x \in B} \{M, M \text{ is } b$-$\gamma$-open \}.$$ 

Hence $B$ is $b$-$\gamma$-closed.

(iii) Since, $b$-$\gamma$Ds$(B) \subset bcl_{\gamma}(B)$ and $B \subseteq bcl_{\gamma}(B)$, $b$-$\gamma$Ds$(B) \cup B \subset bcl_{\gamma}(B)$. Conversely, assume that $x \notin b$-$\gamma$Ds$(B) \cup B$. Then $x \notin b$-$\gamma$Ds$(B)$, $x \notin B$. Then $\exists$ a $b$-$\gamma$-open set $M$ containing $x$ such that $M \cap B = \emptyset$. Thus $x \notin bcl_{\gamma}(B)$. This gives that $bcl_{\gamma}(B) \subset b$-$\gamma$Ds$(B) \cup B$. Hence, $bcl_{\gamma}(B) = b$-$\gamma$Ds$(B) \cup B$.

**Theorem 4.12.** Let $X$ be a space and $B \subseteq X$. $B$ is $b$-$\gamma$-open $\iff$ $B$ is $b$-$\gamma$-neighborhood, $\forall$ point $x \in H$.

**Proof.** Let $B$ be a $b$-$\gamma$-open set. Then clearly $B$ is a $b$-$\gamma$-neighborhood, $\forall x \in B$. Conversely, let $B$ be a $b$-$\gamma$-neighborhood, $\forall x \in B$. Then $\exists$ a $b$-$\gamma$-open set $U_x$ containing $x$ such that $x \in U_x \subseteq B$. Therefore, $B = \bigcup_{x \in B} U_x$. Hence, $B$ is a $b$-$\gamma$-open.

**Theorem 4.13.** Let $(X, \tau)$ be space. If $b$-$\gamma$-$N_x$ be the $b$-$\gamma$-neighborhood systems of a point $x \in X$, then the below statements are hold:

(1) Every member of $b$-$\gamma$-$N_x$ contains a point $x$ and $b$-$\gamma$-$N_x$ is not empty;

(2) Every superset of members of $N_x$ belongs $b$-$\gamma$-$N_x$;

(3) Every member $N \in b$-$\gamma$-$N_x$ is a superset of a member $V \in b$-$\gamma$-$N_x$, where $V$ is $b$-$\gamma$-neighborhood of every point $x \in V$.

**Proof.** Evident.

**Definition 4.14.** Let $X$ be a space. $B \subseteq X$ is called locally $b$-$\gamma$-closed if $B = V \cap K$, $\forall$ open set $V$ and $K$ is $b$-$\gamma$-closed set in $X$.

**Theorem 4.15.** Let $X$ be a space and $B \subseteq X$. The set $B$ is locally $b$-$\gamma$-closed $\iff$ $B = V \cap bcl_{\gamma}(B)$.

**Proof.** Suppose that $B$ is a locally $b$-$\gamma$-closed set. Then $B = V \cap K$, $\forall$ open set $V$ and $K$ is $b$-$\gamma$-closed set in $X$. Thus, $B \subseteq bcl_{\gamma}(B) \subseteq bcl_{\gamma}(K) = K$. Therefore $B \subseteq V \cap bcl_{\gamma}(B) \subseteq V \cap bcl_{\gamma}(K) = B \subseteq V \cap bcl_{\gamma}(B)$. Conversely, since the set $bcl_{\gamma}(B)$ is $b$-$\gamma$-closed and $B = U \cap bcl_{\gamma}(B)$. Then, clearly $B$ is locally $b$-$\gamma$-closed.

**Theorem 4.16.** Let $X$ be a space and $B$ be a locally $b$-$\gamma$-closed subset of $X$. Then the below statements are hold:

(i) The set $bcl_{\gamma}(B) \setminus B$ is a $b$-$\gamma$-closed set;

(ii) The set $B \cup (X \setminus bcl_{\gamma}(B))$ is a $b$-$\gamma$-open set;

(iii) $B \subseteq bint_{\gamma}(B \cup (X \setminus bcl_{\gamma}(B)))$.

**Proof.** (i) If $B$ is a locally $b$-$\gamma$-closed set, then $\exists$ an open set $V$ such that $B = V \cap bcl_{\gamma}(B)$. Therefore, $bcl_{\gamma}(B) \setminus B = bcl_{\gamma}(B) \setminus [V \cap bcl_{\gamma}(B)] = bcl_{\gamma}(B) \cap [X \setminus (V \cap bcl_{\gamma}(B))] = bcl_{\gamma}(B) \cap (X \setminus V)$, which is $b$-$\gamma$-closed.

(ii) By statement (i), we have $X \setminus [(bcl_{\gamma}(B) \setminus B)] = X \setminus bcl_{\gamma}(B) \cup (X \setminus B) = B \cup [X \setminus bcl_{\gamma}(B)]$. Thus $B \cup (X \setminus bcl_{\gamma}(B))$ is $b$-$\gamma$-open.

(iii) It is obvious that, $B \subseteq (B \cup (X \setminus bcl_{\gamma}(B))) = bint_{\gamma}(B \cup (X \setminus bcl_{\gamma}(B)))$.

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**5. $b$-$\gamma$-open and $b$-$\gamma$-closed sets**

**Definition 5.1.** Let $(X, \tau)$ be a space and $B \subseteq X$ is said to be $b$-$\gamma$-generalized closed set (for shortly, $b$-$\gamma$-closed) in $(X, \tau)$, if $bcl_{\gamma}(B) \subseteq V$ whenever $B \subseteq V$ and $V$ is a $b$-$\gamma$-open set of $(X, \tau)$.

The complement of $b$-$\gamma$-generalized closed set is called $b$-$\gamma$-generalized open (for shortly, $b$-$\gamma$-open) set.

**Remark 5.2.** Let $(X, \tau)$ be a space and $B \subseteq X$. Then:

(i) The set $B$ is $b$-$\gamma$-generalized open $\iff$ $B^c$ is $b$-$\gamma$-generalized closed;

(ii) The set $B$ is $b$-$\gamma$-generalized closed $\iff$ $B^c$ is $b$-$\gamma$-generalized open.

**Theorem 5.3.** Let $(X, \tau)$ be a space. $B \subseteq X$. is said to be $b$-$\gamma$-open $\iff C \subseteq bint_{\gamma}(B)$, whenever $C$ is $b$-$\gamma$-closed set and $C \subseteq B$. 

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980/981
Proposition 5.6. Let $B$ be a $b\gamma$-generalized open set in $X$. Then $B'$ is $b\gamma$-generalized closed in $X$. Let $C$ be a $b\gamma$-closed set in $X$ such that $C \subseteq B$. Then $B' \subseteq C'$, $C' \in b\gamma \text{OC}(X)$. Since $B'$ is $b\gamma$-generalized closed, $bcl_{\gamma}(B') \subseteq C'$, which gives $\text{bint}_{\gamma}(B) \subseteq C$. Hence $C \subseteq \text{bint}_{\gamma}(B)$.

Conversely, suppose that $C \subseteq \text{bint}_{\gamma}(B)$, whenever $C \subseteq B$ and $C$ is a $b\gamma$-closed set in $X$. Then $(\text{bint}_{\gamma}(B))' \subseteq C' = D$, where $D$ is a $b\gamma$-open set in $X$. That is $bcl_{\gamma}(B') \subseteq D$, which gives $B'$ is $b\gamma$-generalized closed. Thus $B$ is $b\gamma$-generalized open.

Theorem 5.4. Let $X$ be a space with an operation $\gamma$ on the topology $\tau$. Then each $b\gamma$-closed set is $b\gamma$-g-closed.

Proof. Let $B$ be a $b\gamma$-closed set in a space $X$ and $B \subseteq C$, where $C$ is a $b\gamma$-open in $X$. Since $B$ is a $b\gamma$-closed, $bcl_{\gamma}(B) = B \subseteq C$. Thus $bcl_{\gamma}(B) \subseteq C$. Hence, $B$ is $b\gamma$-g-closed.

The converse of the above Theorem 5.4 may not be true as shown in the below example.

Example 5.5. Let $X = \{a, b, c\}$ and $\tau_X$ be the discrete topology. Define an operation $\gamma$ on $\tau_X$ by $\gamma(B) = X$. Here the set $\{a, b\}$ is $b\gamma$-generalized closed but not $b\gamma$-closed.

Proposition 5.6. Let $X$ be a space. $B \subseteq X$ is a $b\gamma$-generalized closed if and only if $B \cap bcl_{\gamma}(\{y\}) = \emptyset$, $\forall y \in bcl_{\gamma}(B)$.

Proof. Suppose that $V$ is a $b\gamma$-open set such that $B \subseteq V$. Take a point $y \in bcl_{\gamma}(B)$. By supposition $\exists x \in bcl_{\gamma}(\{y\})$ and $x \in B \subseteq V$. Then $V \cap \{y\} \neq \emptyset$. This implies $y \in V$. Therefore $bcl_{\gamma}(B) \subseteq V$. Hence, $B$ is $b\gamma$-generalized closed.

Conversely, suppose that $B$ is $b\gamma$-generalized closed set of $X$ and $y \in bcl_{\gamma}(B)$. Since $bcl_{\gamma}(B)$ is $b\gamma$-closed, $bcl_{\gamma}(B) \subseteq X \setminus bcl_{\gamma}(\{y\})$. Therefore $y \in bcl_{\gamma}(B)$, which is a contradiction. Thus $B \cap bcl_{\gamma}(\{y\}) \neq \emptyset$.

Theorem 5.7. If $B \cap bcl_{\gamma}(\{y\}) \neq \emptyset$, $\forall y \in bcl_{\gamma}(B)$, then $bcl_{\gamma}(B) \setminus B$ does not contain a non-empty $b\gamma$-closed set.

Proof. Assume that $\exists a$ non-empty $b\gamma$-closed set $G$ such that $G \subseteq bcl_{\gamma}(B) \setminus B$. Take $y \in G$, $y \in bcl_{\gamma}(B)$ holds. It follows that $B \cap G = B \cap bcl_{\gamma}(B) \supseteq B \cap bcl_{\gamma}(\{y\}) \neq \emptyset$. Therefore, $B \cap G \neq \emptyset$, which is a contradiction.

Corollary 5.8. A subset $B$ of $(X, \tau)$ is $b\gamma$-generalized closed if and only if $B \cap G \setminus H$, where $G$ is $b\gamma$-closed and $H$ contains no non-empty $b\gamma$-closed subsets.

Proof. Necessity follows from Theorem 5.7 and Proposition 5.6, with $G = bcl_{\gamma}(B)$ and $H = bcl_{\gamma}(B) \setminus B$.

6. Conclusion

In this paper, the ideas of $b\gamma$-boundary, $b\gamma$-exterior and locally $b\gamma$-closed sets are presented. Also some concepts and lemmas of $b\gamma$-open and $b\gamma$-g-closed sets are also investigated. The results are illustrated with a well-analyzed examples. For future study, some other fields such as Fuzzy topology, Intuitionistic topology, Nano topology and etc., can be considered for studying these sets.

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