Numerical solution of time fractional Kuramoto-Sivashinsky equation by Adomian decomposition method and applications

Sharvari Kulkarni¹*, Kalyanrao Takale², Shrikisan Gaikwad³

Abstract
In the paper, we develop the Adomian Decomposition Method for fractional order nonlinear Kuramoto-Sivashinsky (KS) equation. Caputo fractional derivatives are used to define fractional derivatives. We know that KS equation has many applications in physical phenomenon such as reaction diffusion system, long waves on the boundary of two viscous fluids and hydrodynamics. In this paper, we will solve time fractional KS equation which may help to researchers for their work. We solve some examples numerically, which will show the efficiency and convenience of Adomian Decomposition Method.

Keywords
Kuramoto-Sivashinsky equation, Fractional derivative, Adomian Decomposition Method, Convergence, Mathematica.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

1 Introduction
In the present scenario fractional calculus is useful in the various fields of science. In past few years, the increase of interest in the subject is witnessed by series of conferences, research papers and several monographs. The dynamic models of a large number of phenomena can be modeled by fractional order partial differential equations which are characterized by fractional space and/or time derivatives [2]. Fractional calculus is pragmatic to archetypal occurrences dependent damping behavior of many viscoelastic materials, continuum mechanics, statistical mechanics, economics etc [1]. But, many times it is difficult to obtain exact solutions, hence numerical methods must be used. Now a days, Adomian Decomposition Method (ADM) is used to obtain the solution of fractional differential equations . This method gives rapidly convergent series solutions by using a few iterations for both linear and nonlinear equations. This method is very useful to avoid linearization, perturbation, massive computation and transformations [3, 4]. Various instabilities and spatio- temporal chaotic behavior are exhibited in many thermodynamical systems. Pattern formation, travelling wave problems, reaction-diffusion systems, long waves on thin films, unstable drift waves in plasmas etc. are some of the physical phenomenon which arise from chaotic instabilities. In this context Kuramoto-Sivashinsky (KS) equation has a wide range of applicability in science. It is used to model fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, spatially uniform oscillating chemical reaction in a homogeneous medium, solitary pulses in a falling thin film [5] etc. It is also useful to physical problems such as viscous flow problems, hydrodynamics in thin films,
Belousov-Zhabotinsky reactions and instabilities of solidifica-
tion fronts of dilute binary alloys [6]. Kuramoto Sivashinsky
equation was developed by Kuramoto and Sivashinsky is writ-
ten as follow

\[ w_t + \lambda_1 w w_x + \lambda_2 w_{xx} + \lambda_3 w_{xxxx} = 0 \quad (1.1) \]

where \( \lambda_1, \lambda_2, \lambda_3 \) are unknown parameters. The second order
and fourth order spatial derivatives are making this equation’s
behaviour complicated and interesting. The nonlinear term
transforms energy from low to high wave numbers. Also,
Maziar Raissi and George have developed methodology ap-
plicated to the problem of learning, system identification or data-
driven discovery of partial differential equation and provides
new direction to design learning machines without requiring
large quantities of data. They gave following observations for
values of parameters \( \lambda_1, \lambda_2, \lambda_3 \) with clean and noisy data [7]:
(i) The correct KS partial differential equation is

\[ w_t + w_{xx} + w_{xx} + w_{xxxx} = 0 \quad (1.2) \]

(ii) The identified KS partial differential equation (clean data)
is

\[ w_t + 0.952 w_{xx} + 1.005 w_{xx} + 0.980 w_{xxxx} = 0 \quad (1.3) \]

(iii) The identified KS partial differential equation

\[ w_t + 0.908 w_{xx} + 0.951 w_{xx} + 0.927 w_{xxxx} = 0 \quad (1.4) \]

Recently, some researchers have used Homotopy Perturbation
Method [8], He’s Variational Iteration Method [9] and Lattice
Boltzmann method [10] to solve KS equation. Saad A. Mannan,
Fadhil H. Easif [11] have used ADM to solve KS equation.
Weishi Yin, Fei Xu et.al. found the asymptotic expansion of
solutions to time-space fractional KS equation by residual
power series method [12].

Therefore, models which represent wave phenomenon needs
to study travelling wave solutions. As per Abdul Wazwaz,
in the study of solitary wave theory, we can obtain travelling
wave solutions. These solutions are used by scientists to study
various physical applications in plasma physics. In [13], the
researchers used Bogning-Dijeumend Tchal-Kofane method
(BDK Method) to solve very strong nonlinear KS equation.
By applying BDK method they make up modulated soliton
solution of KS equation. In paper [14] KS equation was solved
by truncated expansion method and compared with Ansatz
method. Also researchers analyzed new solitary wave solu-
tions of KS equation with comparison of solutions given by
Chen and Zhand, Wazwaz and Wazaan. In this connection in
our paper, we used ADM to solve time fractional KS equation
because ADM is a powerful method to obtain the solution of
linear and nonlinear fractional partial differential equations of
higher order.

We organize the paper as follows: We have given some for-

mulae and theorem in Section 2, which are useful for further
developments. Section 3, is devoted for ADM to solve time
fractional KS equation and prove convergence. In section 4,
numerical problems are solved and presented their solutions
graphically by using mathematica software.

2. Preliminaries

Some basic concepts, which we will be using are as follows:-

Definition 2.1. The Caputo fractional derivative \( J_0^\alpha \) of the
function \( f(x) \) is defined as

\[ D_0^\alpha f(x) = J_0^{\alpha-m} D^m f(x) \]

\[ = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{(m-\beta)} f(t) dt, \]

for \( m-1 < \beta \leq m, m \in N, x > 0, f \in C^m_0. \)

Properties:

For \( f(x) \in C_0, \mu \geq 1, \alpha, \beta \geq 0 \) and \( \gamma > -1 \) [15], we have

(i) \( J_0^\alpha J_0^\beta f(x) = J_0^{\alpha+\beta} f(x), \)

(ii) \( J_0^\alpha J_0^\beta f(x) = J_0^{\beta \alpha} f(x), \)

(iii) \( J_0^\alpha f^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma)} x^{\alpha+\gamma}. \)

Lemma 2.2. If \( m-1 < \alpha \leq m, m \in N \) and \( f \in C_0^m, \mu \geq 1, \)

then

\[ D_0^\alpha J_0^\alpha f(x) = f(x) \]

\[ J_0^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0+) x^k \]

\[ k! k > 0. \]

3. Fractional Adomian Decomposition

Method

We consider following time fractional KS equation to de-
velop the time Fractional ADM [15] for solving KS equation,

\[ w_t^\alpha + \lambda_1 w w_x + \lambda_2 w_{xx} + \lambda_3 w_{xxxx} = 0, \quad 0 < \alpha \leq 1, t > 0 \quad (3.1) \]

initial condition : \( w(x,0) = f(x) \quad (3.2) \)

We will operate \( J_0^\alpha \) on R.H.S. and L.H.S. of equation,

\[ J_0^\alpha \left[ w_t^\alpha + \lambda_1 w w_x + \lambda_2 w_{xx} + \lambda_3 w_{xxxx} = 0 \right] = 0, \quad 0 < \alpha \leq 1, t > 0 \]

Now, consider following decomposition series:-

\[ w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \quad (3.3) \]

The decomposed series of nonlinear terms \( Nw(x,t) \) are:

\[ Nw(x,t) = \sum_{n=0}^{\infty} A_n \quad (3.4) \]
where the formula for Adomian polynomial is as follows:

$$A_n = \frac{1}{n!} \left[ \frac{d^n N}{d \lambda^n} \left( \sum_{k=0}^{n} \lambda^k u_k \right) \right]_{\lambda=0} \tag{5.3}$$

From (3.3) and using lemma (2.1), we get

$$\sum_{n=0}^{\infty} w_n(x,t) = \sum_{n=0}^{\infty} \frac{\partial^k w(x,0)}{\partial t^k} \frac{t^k}{k!} - J^\alpha \left[ \lambda_2 \sum_{n=0}^{\infty} D^2_t w_n(x,t) + \lambda_3 \sum_{n=0}^{\infty} D^3_t w_n(x,t) \right] \tag{3.7}$$

The value of $w_n(x,t), n \geq 0$ can be obtained as follows:

$$w_0(x,t) = w(x,0) = f(x) \tag{3.6}$$

$$w_{n+1}(x,t) = -J^\alpha \left[ \lambda_2 D^2_t w_n(x,t) + \lambda_3 D^3_t w_n(x,t) + \lambda_4 A_n \right] \tag{3.7}$$

For $x > 0$, now we can obtain solution by calculating value of each component.

$$\phi_0(x,t) = \sum_{n=0}^{N-1} w_n(x,t) \tag{3.8}$$

$$\lim_{N \to \infty} \phi_0 = w(x,t) \tag{3.9}$$

**Theorem 3.1. Uniqueness Theorem [16]**

Consider time fractional KS equation for $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 1$, as follows

$$w_t^\alpha + w_{xx} + w_{xxx} = 0, \quad 0 < \alpha \leq 1, t > 0 \tag{3.10}$$

**initial condition**: $w(x,0) = f(x)$ \tag{3.11}

The equation has a unique solution whenever $0 < \gamma < 1$ where

$$\gamma = \frac{C_1 + C_2 + C_3}{\alpha + 1}.$$  

**Proof**: Let $X = (C(I), \| \cdot \|)$ be the Banach space of all continuous functions on $I = [0, T]$ with norm

$$\|w(t)\| = \max_{i \in I} |w(t)|.$$

We define a mapping $M : X \to X$, such that

$$M(w(t)) = f(x) - J^\alpha N(w(t)) - J^\alpha S(w(t)) - J^\alpha F(w(t)).$$

Now, $N(w(t))$ denotes nonlinear term and $S(w(t))$ denotes second order spatial term and $F(w(t))$ denotes fourth order spatial term. Also nonlinear term $N(w(t))$ is Lipschitzian and is

$$|N(w) - N(p)| \leq C_1 |w - p|$$

where $C_1$ is Lipschitz constant. Let $w, w' \in X$, we have

$$\|M(w) - M(w')\| = \max_{i \in I} |J^\alpha N(w(t)) - J^\alpha S(w(t)) - J^\alpha F(w(t))|$$

$$= \max_{i \in I} | -J^\alpha(Nw - Nw') - J^\alpha(Sw - Sw') - J^\alpha(Fw - Fw')\|$$

$$= \max_{i \in I} | J^\alpha(Nw - Nw') + J^\alpha(Sw - Sw') + J^\alpha(Fw - Fw')\|$$

$$\leq \max_{i \in I} | J^\alpha(Nw - Nw') | + | J^\alpha(Sw - Sw') | + | J^\alpha(Fw - Fw')\|$$

Now suppose $S(w(t))$ and $F(w(t))$ are also Lipschitzian that is

$$\|S(w) - S(p)\| \leq C_2 |w - p|$$

and

$$\|F(w) - F(p)\| \leq C_3 |w - p|,$$

where $C_2$ and $C_3$ are Lipschitz constants. Therefore

$$\|M(w) - M(w')\| \leq \max_{i \in I} |C_1 J^\alpha |w - w'| + C_2 J^\alpha |w - w'|$$

$$+ C_3 J^\alpha |w - w'| \leq (C_1 + C_2 + C_3) \|w - w'\||\alpha + 1\|$$

$$\|M(w) - M(w')\| \leq \gamma \|w - w'\|$$

Therefore, whenever $0 < \gamma < 1$, the mapping is contraction. Hence with the reference of Banach fixed point theorem for contraction, we proved that equation has unique solution.

**Theorem 3.2. Convergence Theorem [16]**

Let $Q_n$ be the $n$th partial sum, that is

$$Q_n = \sum_{i=0}^{n} w_i(x,t) \tag{3.12}$$

Then we shall prove that $\{Q_n\}$ is a Cauchy sequence in Banach space $X$.

**Proof**: For proving this theorem, we consider

$$\|Q_{n+p} - Q_n\| = \max_{i \in I} |Q_{n+p} - Q_n|$$

$$= \max_{i \in I} | \sum_{i=n+1}^{n+p} w_i(x,t) |$$

$$= \max_{i \in I} | -J^\alpha \sum_{i=n+1}^{n+p} S_{w_{i-1}}(x,t) - J^\alpha \sum_{i=n+1}^{n+p} F_{w_{i-1}}(x,t)$$

$$- J^\alpha \sum_{i=n+1}^{n+p} N_{w_{i-1}}(x,t) |$$

$$= \max_{i \in I} | J^\alpha S_{Q_{n+p-1}} - S_{Q_{n-1}} + J^\alpha F_{Q_{n+p-1}} - F_{Q_{n-1}} + J^\alpha N_{Q_{n+p-1}} - N_{Q_{n-1}} |$$

$$\leq \max_{i \in I} | J^\alpha |(S_{Q_{n+p-1}} - S_{Q_{n-1}}) + J^\alpha |(F_{Q_{n+p-1}} - F_{Q_{n-1}}) + J^\alpha |(N_{Q_{n+p-1}} - N_{Q_{n-1}}) |$$

$$+ \max_{i \in I} | J^\alpha N_{Q_{n+p-1}} - N_{Q_{n-1}} |$$

$$\leq C_2 \max_{i \in I} | (Q_{n+p-1} - Q_{n-1}) + C_3 \max_{i \in I} | (Q_{n+p-1} - Q_{n-1}) |$$
\[ + C_1 \max_{t \in J} J^{\alpha} \| Q_{n+p-1} - Q_{n-1} \| \]
\[ \leq (C_1 + C_2 + C_3) J^{\alpha} \| Q_{n+p-1} - Q_{n-1} \| \]
\[ \| Q_{n+p} - Q_n \| \leq \gamma \| Q_{n+p-1} - Q_{n-1} \|, \]

where \( \gamma = (C_1 + C_2 + C_3) J^{\alpha} \)
\[ \| Q_{n+p} - Q_n \| \leq \gamma \| Q_{n+p-1} - Q_{n-1} \| \]

Similarly, we have
\[ \| Q_{n+p} - Q_n \| \leq \gamma^2 \| Q_{n+p-2} - Q_{n-2} \| \]
\[ \vdots \]
\[ \leq \gamma^p \| Q_p - Q_0 \| \text{ for } p = 1 \]
\[ \leq \gamma^p \| w_1 \| \]

Now, for \( n > m \), where \( n, m \in N \),
\[ \| Q_n - Q_m \| \leq \| Q_{m+1} - Q_m \| + \| Q_{m+2} - Q_{m+1} \| + \cdots + \| Q_n - Q_{n-1} \| \]
\[ \leq (\gamma^m + \gamma^{m+1} + \cdots + \gamma^{n-1}) \| w_1 \| \]
\[ \leq \gamma^m \left[ \frac{1 - \gamma^{n-m}}{1 - \gamma} \right] \| w_1 \| \]

Since, \( 0 < \gamma < 1 \), then \( 1 - \gamma^{n-m} < 1 \), so we have,
\[ \| Q_n - Q_m \| \leq \frac{\gamma^m}{1 - \gamma} \| w_1 \| \]

Since, \( w(t) \) is bounded, therefore \( \| w_1 \| < \infty \)
\[ \lim_{n \to \infty} \| Q_n - Q_m \| = 0 \]

Hence, we proved that solution is convergent because \( \{ Q_n \} \) is a Cauchy sequence in \( X \).

### 4. Numerical Examples

**Example 4.1:** We will consider the following time fractional KS equation

\[ w_t^\alpha + w_{xx} + w_{xxx} + w_{xxxx} = 0, \quad 0 < \alpha \leq 1, t > 0 \quad (4.1) \]

Initial condition: \( w(x, 0) = \sec h^2 \left( \frac{x}{2} \right) \) \quad (4.2)

Now, using equations (3.6) and (3.7), we have

\[ w_0(x,t) = w(x,0) = f(x) \]
\[ w_{k+1}(x,t) = -J^{\alpha} \left[ A_k + D^2_{x}w_k(x,t) + D^4_{x}w_k(x,t) \right], \quad x > 0 \]
\[ w_0(x,t) = w(x,0) = \sec h^2 \left( \frac{x}{2} \right) \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ A_0 = w_0(w_0) = \left[ -\frac{1}{2} \tan h \left( \frac{x}{4} \right) + \tan h^3 \left( \frac{x}{4} \right) \right] \]
\[ D^2_{x}w_0(x,t) = \left[ -\frac{1}{8} \sec h^2 \left( \frac{x}{4} \right) + \frac{3}{8} \tan h^2 \left( \frac{x}{4} \right) \right] \]
\[ D^4_{x}w_0(x,t) = \left[ -\frac{1}{8} \left( 1 - \tan h^2 \left( \frac{x}{4} \right) \right) + \frac{3}{8} \tan h^2 \left( \frac{x}{4} \right) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]
\[ w_1(x,t) = -J^{\alpha} \left[ A_0 + D^2_{x}w_0(x,t) + D^4_{x}w_0(x,t) \right] \]

After calculating and substituting values of various components, we have

\[ w(x,t) = w_0(x,t) + w_1(x,t) + \cdots \]
\[ w(x,t) = \sec h^2 \left( \frac{x}{2} \right) + \left[ -\frac{1}{16} + \frac{1}{2} \tan h \left( \frac{x}{4} \right) + \frac{1}{32} \tan h^3 \left( \frac{x}{4} \right) \right] \]
\[ -\tan h^3 \left( \frac{x}{4} \right) - \frac{9}{16} \tan h^3 \left( \frac{x}{4} \right) + \frac{1}{2} \tan h^3 \left( \frac{x}{4} \right) \]
\[ + \frac{15}{32} \tan h^3 \left( \frac{x}{4} \right) \]
Example 4.2: We will solve the following time fractional KS equation

\[ w^\alpha_t + w w_x + w_{xx} + w_{xxxx} = 0, \quad 0 < \alpha \leq 1, t > 0 \quad (4.3) \]

initial condition : \[ w(x,0) = \cos \left( \frac{x}{2} \right) \quad (4.4) \]

Now, using equation (3.3) and (3.6), we have

\[ w_0(x,t) = w(x,0) = f(x) \]
\[ w_{k+1}(x,t) = -\frac{1}{\Gamma(\alpha)} \int_0^t w_k(x,\tau) \ d\tau \quad (4.5) \]

\[ w_1(x,t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \left( A_k + D^\gamma_{xx} w_k(x,\tau) + D^\lambda_{xx} w_k(x,\tau) \right) \ d\tau \quad (4.6) \]
\[ w_2(x,t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \left( A_k + 64 x \sin x + \frac{1}{4} \cos \left( \frac{x}{2} \right) \right) \ d\tau \quad (4.7) \]
\[ w_3(x,t) = -\frac{1}{\Gamma(\alpha)} \int_0^t \left( A_k + 64 x \sin x + \frac{1}{4} \cos \left( \frac{x}{2} \right) \right) \ d\tau \quad (4.8) \]

After calculating and substituting values of various components, we have

\[ w(x,t) = w_0(x,t) + w_1(x,t) + \cdots \quad (4.9) \]

\[ w(x,t) = \cos \left( \frac{x}{2} \right) + \frac{1}{\Gamma(\alpha+1)} \left( \frac{1}{4} \sin x + \frac{3}{16} \cos \left( \frac{x}{2} \right) \right) t^\alpha + \cdots \]

\[ + \frac{9}{\Gamma(2\alpha+1)} \left( \frac{9}{64} \sin x + \frac{9}{256} \cos \left( \frac{x}{2} \right) \right) t^{2\alpha} + \cdots \]
5. Conclusion

The time fractional KS equation is solved by using ADM and we can say that the formula of ADM polynomials is powerful to obtain the solution of nonlinear fractional partial differential equation. The graphical presentation of solutions of time fractional KS equation reveals the reliability of the mathematical procedure. We also prove the uniqueness and convergence theorem for time fractional KS equation.

Acknowledgment

The authors are thankful to all referee’s for their useful comments and suggestions.

References

[12] W. Yin, F. Xu, Asymptotic Expansion of the Solutions to Time-Space Fractional Kuramoto-Sivashinsky equation,
Numerical solution of time fractional Kuramoto-Sivashinsky equation by Adomian decomposition method and applications — 1084


