Mapping properties of some integral operators associated with generalized Bessel functions

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**Abstract**
In the present paper, we study the mapping properties of some integral operators on certain classes of harmonic univalent functions associated with generalized Bessel functions of the first kind. To be more precise, we study the mapping properties of Goodman-Rønning-type harmonic univalent functions in the open unit disc \(U\).

**Keywords**
Harmonic, univalent functions, generalized Bessel functions.

**AMS Subject Classification**
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#### 1. Introduction

Let \(A\) denote the class of functions \(f\) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \(U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}\) and satisfy the normalization condition \(f(0) = f'(0) - 1 = 0\). Now, we recall that the generalized Bessel function of the first kind \(w = w_{p,b,c}\) is defined as the particular solution of the second-order linear homogeneous differential equation

\[
z^2 \omega''(z) + b z \omega'(z) + \left[ c z^2 - p^2 + (1 - b) p \right] \omega(z) = 0,
\]

where \(b, p, c \in \mathbb{C}\), which is a natural generalization of Bessel’s differential equation. This function has the familiar representation

\[
\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n^p}{n! \Gamma(p+n+\frac{b}{2})} \left( \frac{2}{z} \right)^{2n+p}, \quad z \in \mathbb{C}.
\]

The differential equation (1.2) permits the study of Bessel, modified Bessel, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.2) are referred to as the generalized Bessel function of order \(p\). The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order \(p\). Although the series defined above is convergent everywhere, the function \(\omega_{p,b,c}\) is generally not univalent in \(U\). It is worth mentioning that, in particular, when \(b = c = 1\), we reobtain the Bessel function \(\omega_{p,1,1} = J_p\) and for \(c = -1, b = 1\) the function \(\omega_{p,1,-1}\) becomes the modified Bessel function \(I_p\). Now, consider the function \(u_{p,b,c}\) defined by the transformation

\[
u_{p,b,c}(z) = 2^p \Gamma \left( p + \frac{b+1}{2} \right) z^{-p/2} \omega_{p,b,c}(z^{1/2}).
\]

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for \(a \neq 0, -1, -2, \ldots\) by

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & : \text{if } n = 0 \\ a(a+1) \cdots (a+n-1) & : \text{if } n = 1, 2, 3, \ldots \end{cases}
\]

We obtain for the function \(u_{p,b,c}\) by the following representa-
The classes consisting of functions of the form
\[ (1.5) \]
\[ h \alpha \in \Omega, \ z \in \mathbb{U}. \]

Further we define \( TG_H(\gamma) \equiv G_H(\gamma) \cap T \). The class \( G_H(\gamma) \) is called Goodman-Rønning-type harmonic univalent functions in \( U \).

For complex parameters \( c_1, k_1, c_2, k_2 \) \((k_1, k_2 \neq -1, -2, \ldots)\), we introduce the following convolution operator

\[ \Omega \equiv \Omega \left( k_1, \ c_1 \atop k_2, \ c_2 \right) : H \to H \]

defined by

\[ \Omega \left( k_1, \ c_1 \atop k_2, \ c_2 \right) f = h(z) \ast \int_0^z u_{p_1}(t) dt + g(z) \ast \int_0^z u_{p_2}(t) dt \]

for any function \( f = h + g \) in \( H \).

Letting

\[ \Omega \left( k_1, \ c_1 \atop k_2, \ c_2 \right) f(z) = H(z) + \overline{G(z)}, \]

where

\[ H(z) = z + \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{n-1} A_n z^n \]

and

\[ G(z) = \sum_{n=1}^{\infty} \frac{(-c_2/4)^{n-1}}{n-1} B_n z^n. \]

Similarly we define the Libera type integral operator

\[ L \left( k_1, \ c_1 \atop k_2, \ c_2 \right) f(z) = H(z) + \overline{G(z)}, \]

where

\[ H(z) = z + \sum_{n=2}^{\infty} \frac{2(-c_1/4)^{n-1}}{(k_1)_{n-1}(n+1)(n-1)!} A_n z^n, \]

and

\[ G(z) = \sum_{n=1}^{\infty} \frac{2(-c_2/4)^{n-1}}{(k_2)_{n-1}(n+1)(n-1)!} B_n z^n. \]

Throughout this paper, we will frequently use the notation

\[ \Omega (f) = \Omega \left( k_1, \ c_1 \atop k_2, \ c_2 \right) f \]

and

\[ L(f) = L \left( k_1, \ c_1 \atop k_2, \ c_2 \right) f. \]
The generalized Bessel function is a recent topic of study in
the Geometric Function Theory (e.g. see the work of [3], [7]-[11]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [1], [4], [12], [13], [15] and [16]), we establish a number of connections between the classes $G_H(\gamma)$, $K_H^0$, $S_H^{0,0}$, $C_H^0$ and $N_H(\beta)$ by applying the convolution operators $\Omega$ and $L$.

2. Main Results

In order to establish connections between harmonic convex functions and Goodman-Ronneing-type harmonic univalent functions, we shall require the following lemmas.

Lemma 2.1. ([5], [6]). If $f = h + \overline{g} \in K_H^0$ where $h$ and $g$ are given by (1.5) with $B_1 = 0$, then

$$|A_n| \leq \frac{n+1}{2}, \quad |B_n| \leq \frac{n-1}{2}.$$ 

Lemma 2.2. ([14]). Let $f = h + \overline{g}$ be given by (1.5). If $0 \leq \gamma < 1$ and

$$\sum_{n=2}^{\infty} (2n-1-\gamma) |A_n| + \sum_{n=1}^{\infty} (2n+1+\gamma) |B_n| \leq 1 - \gamma,$$

then $f$ is sense-preserving, Goodman-Ronneing-type harmonic univalent functions in $U$ and $f \in G_H(\gamma)$.

Remark 2.3. In [14], it is also shown that $f = h + \overline{g}$ given by (1.6) is in the family $T G_H(\gamma)$ if and only if the coefficient condition (2.1) holds. Moreover, if $f \in T G_H(\gamma)$, then

$$|A_n| \leq \frac{1-\gamma}{2n-1-\gamma}, \quad n \geq 2,$$

and

$$|B_n| \leq \frac{1-\gamma}{2n+1+\gamma}, \quad n \geq 1.$$

Lemma 2.4. ([3]). If $b, p, c \in C$ and $k \neq 0, -1, -2, \cdots$ then the function $u_p$ satisfies the recursive relation $4 k u_p(z) = -c u_{p+1}(z)$ for all $z \in C$.

Lemma 2.5. If $c < 0$ and $k > 1$, then

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n(n+1)!} = \frac{4(k-1)}{c} [u_{p-1}(1) - 1].$$

Proof. We can write

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n(n+1)!} = \frac{(k-1)}{(-c/4)} \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_n(n+1)(n+2)!}$$

$$= \frac{4(k-1)}{c} [u_{p-1}(1) - 1].$$

Theorem 2.6. Let $c_1, c_2 < 0$, $k_1, k_2 > 1$. If for some $\gamma(0 \leq \gamma < 1)$ and the inequality

$$2u_{p_1}(1) + (3-\gamma)u_{p_1}(1) - (1+\gamma)$$

$$\left[ -\frac{4(k_1-1)}{c_1} [u_{p_1-1}(1) - 1] \right] + 2u_{p_2}(1) + (1+\gamma)u_{p_2}(1)$$

$$- (1+\gamma) \left[ -\frac{4(k_2-1)}{c_2} [u_{p_2-1}(1) - 1] \right] \leq 4(1-\gamma)$$

is satisfied then $\Omega (K_H^0) \subset G_H(\gamma)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where $h$ and $g$ are of the form (1.5) with $B_1 = 0$. We need to show that $\Omega (f) = H + \overline{C} \in G_H(\gamma)$, where $H$ and $G$ defined by (1.7) with $B_1 = 0$ are analytic functions in $U$.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1 - \gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left[ -\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} \right] A_n$$

$$+ \sum_{n=2}^{\infty} (2n+1+\gamma) \left[ -\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} \right] B_n.$$

In view of Lemma 2.1, we have

$$P_1 \leq \frac{1}{2} \left[ \sum_{n=2}^{\infty} (n+1)(2n-1-\gamma) \left[ -\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} \right] \right]$$

$$+ \sum_{n=2}^{\infty} (n+1)(2n+1+\gamma) \left[ -\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} \right]$$

$$\times \left[ -\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} \right]$$

$$+ \sum_{n=2}^{\infty} (n+1)(\gamma+1)n - (\gamma+1) \left[ -\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} \right]$$

$$\times \left[ -\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} \right].$$
Theorem 2.8. \( \Omega(\mathcal{H}) \subseteq G_H(\gamma) \) and \( \Omega(\mathcal{C}(\mathcal{H})) \subseteq G_H(\gamma) \).

**Proof.** Let \( f = h + g \in \mathcal{H} \) where \( h \) and \( g \) are given by (1.5) with \( B_1 = 0 \). We need to show that \( \Omega(f) = H + G \in G_H(\gamma) \), where \( H \) and \( G \) defined by (1.7) with \( B_1 = 0 \) are analytic functions in \( U \). In view of Lemma 2.2, it is enough to show that \( P_1 \leq 1 - \gamma \), where

\[
P_1 = \sum_{n=2}^{\infty} (2n + 1) - \gamma \left( \frac{-c_1/4}{(k_n)_{n+1}} \right) |A_n| \quad + \sum_{n=2}^{\infty} (2n + 1 - \gamma) \left( \frac{c_1/4}{(k_n)_{n+1}} \right) |B_n| .
\]

In view of Lemma 2.7, we have

\[
P_1 \leq \frac{1}{b} \left\{ 4 \sum_{n=0}^{\infty} \left( \frac{-c_1/4}{(k_n)_{n+1}} \right)^n + (16 - 2\gamma) \sum_{n=0}^{\infty} \left( \frac{-c_1/4}{(k_n)_{n+1(n+1)}} \right)^n + (1 + \gamma) \sum_{n=0}^{\infty} \left( \frac{-c_1/4}{(k_n)_{n+1(n+1)}} \right)^n \right\} .
\]

Now \( P_1 \leq 1 - \gamma \) follows from the given condition.

This completes the proof of Theorem 2.6. \( \square \)

Analogous to Theorem 2.6, we next find conditions of the classes \( \mathcal{S}_H, \mathcal{C}_H \) with \( G_H(\gamma) \). However we first need the following result which may be found in [5], [6].

**Lemma 2.7.** If \( f = h + g \in \mathcal{S}^0_H \) or \( \mathcal{C}^0_H \) where \( h \) and \( g \) are given by (1.5) with \( B_1 = 0 \), then

\[
|A_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |B_n| \leq \frac{(2n-1)(n-1)}{6} ,
\]

Theorem 2.8. If \( c_1, c_2 < 0, k_1, k_2 > 1 \). If some \( \gamma(0 \leq \gamma < 1) \) and the inequality

\[
4u''_{p_1}(1) + (16 - 2\gamma)u'_{p_1}(1) + (7 - 5\gamma)u_{p_1}(1)
\]

\[
-(1 + \gamma) \left[ \frac{-4k_1-1}{c_1} [u_{p_1-1}(1) - 1] \right]
\]

\[
+4u''_{p_2}(1) + (8 + 2\gamma)u'_{p_2}(1) - (1 + \gamma)u_{p_2}(1)
\]

\[
+(1 + \gamma) \left[ \frac{-4k_1-1}{c_2} [u_{p_2-1}(1) - 1] \right] \leq 12(1 - \gamma) \quad (2.3)
\]

is satisfied, then

\[
\Omega(\mathcal{S}^0_H) \subseteq G_H(\gamma) \quad \text{and} \quad \Omega(\mathcal{C}^0_H) \subseteq G_H(\gamma).
\]

Now \( P_1 \leq 1 - \gamma \) follows from the given condition. \( \square \)

In order to determine connection between \( TN_H(\beta) \) and \( G_H(\gamma) \), we need the following results in Lemma 2.9.
Lemma 2.9. (12). Let $f = h + g$ where $h$ and $g$ are given by (1.6) with $B_1 = 0$, and suppose that $0 \leq \beta < 1$. Then

$$f \in TN_H(\beta) \iff \sum_{n=2}^{\infty} n|A_n| + \sum_{n=2}^{\infty} n|B_n| \leq 1 - \beta.$$ 

Remark 2.10. If $f \in TN_H(\beta)$, then $|A_n| \leq \frac{1 - \beta}{n}$ and $|B_n| \leq \frac{1 - \beta}{n^2}$, $n \geq 2$.

Theorem 2.11. If $c_1, c_2 < 0, k_1, k_2 > 1$ ($k_1, k_2 \neq -1, -2, \ldots$). If for some $\beta(0 \leq \beta < 1)$, $\gamma(0 \leq \gamma < 1)$ and the inequality

$$(1 - \beta) [2 \{ u_{p_1}(1) - 1 \} + (1 + \gamma) \frac{4(k_1-1)}{c_1} u_{p_2}(1) - (1 + \gamma) \frac{4(k_1-1)}{c_2} [u_{p_2}(1) - 1]] \leq 1 - \gamma$$

is satisfied then

$$\Omega(TN_H(\beta)) \subset G_H(\gamma).$$

Proof. Let $f = h + g \in TN_H(\beta)$ where $h$ and $g$ are given by (1.6). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1 - \gamma$, where

$$P_2 = \sum_{n=2}^{\infty} (2n - 1 + \gamma) \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} |B_n|.$$ 

Using Remark 2.10, we have

$$P_2 \leq (1 - \beta) \left[ \sum_{n=2}^{\infty} \left( 2 - \frac{(1 + \gamma)}{n} \right) \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} \right]$$

$$+ \sum_{n=1}^{\infty} \left( 2 + \frac{(1 + \gamma)}{n} \right) \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}$$

$$= (1 - \beta) \left[ 2 \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!} + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)_{n+1}n!} \right]$$

$$+ 2 \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)n!} + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)_{n+1}n!}$$

$$\leq (1 - \beta) \left[ 2 \{ u_{p_1}(1) - 1 \} - (1 + \gamma) \frac{(k_1-1)}{(-c_1/4)} \right]$$

$$+ 2u_{p_2}(1) + (1 + \gamma) \left[ \frac{(k_2-1)}{(-c_2/4)} \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!(n+2)!} \right]$$

$$= (1 - \beta) \left[ 2 \{ u_{p_1}(1) - 1 \} - (1 + \gamma) \frac{(k_1-1)}{(-c_1/4)} \right]$$

$$+ 2u_{p_2}(1) + (1 + \gamma) \left[ \frac{(k_2-1)}{(-c_2/4)} \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!(n+2)!} \right]$$

by the given hypothesis.

In next theorem, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$.

Theorem 2.12. Let $c_1, c_2 < 0, k_1, k_2 > 1$. If for some $\gamma(0 \leq \gamma < 1)$ the inequality

$$\frac{4(k_1-1)}{c_1} [u_{p_1-1}(1) - 1] + \frac{4(k_2-1)}{c_2} [u_{p_2-1}(1) - 1] \geq -2$$

is satisfied, then $\Omega(TG_H(\gamma)) \subset G_H(\gamma)$.

Proof. Let $f = h + g \in TG_H(\gamma)$ where $h$ and $g$ are given by (1.6). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1 - \gamma$, where

$$P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} |B_n|.$$ 

Using Remark 2.3, it follows that

$$P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} |B_n|$$

$$\leq (1 - \gamma) \left[ \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} + \sum_{n=1}^{\infty} \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} \right]$$

$$= (1 - \gamma) \left[ 2 \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!} + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)_{n+1}n!} \right]$$

$$+ \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)n!} + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n}}{(k_2)_{n+1}n!}$$

$$\leq (1 - \gamma) \left[ 2 \{ u_{p_1}(1) - 1 \} - (1 + \gamma) \frac{(k_1-1)}{(-c_1/4)} \right]$$

$$+ 2u_{p_2}(1) + (1 + \gamma) \left[ \frac{(k_2-1)}{(-c_2/4)} \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!(n+2)!} \right]$$

$$= (1 - \beta) \left[ 2 \{ u_{p_1}(1) - 1 \} - (1 + \gamma) \frac{(k_1-1)}{(-c_1/4)} \right]$$

$$+ 2u_{p_2}(1) + (1 + \gamma) \left[ \frac{(k_2-1)}{(-c_2/4)} \sum_{n=0}^{\infty} \frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!(n+2)!} \right]$$

by the given condition and this completes the proof.
In next theorem, we present conditions on the parameters \(k_1,k_2,c_1,c_2\) and obtain a characterization for operator \(\Omega\) which maps \(T G_H(\gamma)\) on to itself.

**Theorem 2.13.** If \(c_1,c_2 < 0, k_1,k_2 > 1\) and \(\gamma(0 \leq \gamma < 1)\). Then

\[
\Omega(T G_H(\gamma)) \subset T G_H(\gamma),
\]

if and only if the inequality (2.4) is satisfied.

**Proof.** The proof of above theorem is similar to that of Theorem 2.4. Therefore we omit the details involved.

**Theorem 2.14.** Let \(c_1,c_2 < 0, k_1 > 0,k_2 > 2\). If for some \(\gamma(0 \leq \gamma < 1)\) and the inequality

\[
2u'_{p_1}(1) + (1 - \gamma) \left( u_{p_1}(1) - 1 \right) + 2u'_{p_2}(1) + (1 - \gamma) \\
(u_{p_2}(1) - 1) + 2(1 - \gamma) \left[ \frac{(-c_2/4)^{n-1}}{k_2(n-1)!} \right] u_{p_2-1}(1) - 1 \\
-2(1 - \gamma) \left[ \frac{(-c_2/4)^{n-1}}{k_2(n-1)!} \right] u_{p_2-2}(1) - 1 \\
+ \frac{c_2/4}{k_2-2} \left[ \frac{(-c_2/4)^{n-1}}{k_2(n-1)!} \right]
\]

is satisfied then \(L(K_H)^T \subset G_H(\gamma)\).

**Proof.** Let \(f = h + \overline{g} \in K^0_H\), where \(h\) and \(g\) are of the form (1.5) with \(B_1 = 0\). We need to show that \(L(f) = H + G \in G_H(\gamma)\), where \(H\) and \(G\) defined by (1.9) with \(B_1 = 0\) are analytic functions in \(U\).

In view of Lemma 2.2, we need to prove that

\[
P_3 \leq 1 - \gamma,
\]

where

\[
P_3 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \left( \frac{(-c_1/4)^{n-1}}{(k_1(n-1)!)} \right) \frac{2k_1}{n+n} + \\
\sum_{n=2}^{\infty} (2n + 1 + \gamma) \left( \frac{(-c_2/4)^{n-1}}{(k_2(n-1)!)} \right) \frac{2k_2}{n+n}
\]

In view of Lemma 2.1, we have

\[
P_3 \leq \sum_{n=2}^{\infty} (2n - 1 - \gamma) \left( \frac{(-c_1/4)^{n-1}}{(k_1(n-1)!)} \right) + \\
\sum_{n=2}^{\infty} (2n + 1 + \gamma) (n-1) \left( \frac{(-c_2/4)^{n-1}}{(k_2(n-1)!)} \right)
\]

\[
= \sum_{n=2}^{\infty} \left( 2(n-1) + (1 - \gamma) \right) \left( \frac{(-c_1/4)^{n-1}}{(k_1(n-1)!)} \right) + \\
\sum_{n=2}^{\infty} \{ 2(n+1) (n-1) + (\gamma - 1) (n+1)n \} + \\
2(1 - \gamma) (n+1) (2(1 - \gamma) (n+1) + 2(\gamma - 1) \left( \frac{(-c_2/4)^{n-1}}{(k_2(n-1)!)} \right)
\]

performing the similar calculation as in Theorem 2.6 we obtain the required condition.

This completes the proof.

**Theorem 2.15.** Let \(c_1,c_2 < 0, k_1,k_2 > 2, 0 \leq \beta < 1\). If for some \(\gamma(0 \leq \gamma < 1)\) and the inequality

\[
2(1 - \beta) \left( \frac{-4(k_1 - 1)}{c_1} \right) (u_{p_1-1}(1) - 1) \\
-3 + \gamma) \left( \frac{16(k_2 - 2)(k_2 - 1)}{c_2} \right) (u_{p_2-1}(1) - 1) \\
+ 2 \left( \frac{-4(k_2 - 1)}{c_2} \right) (u_{p_2-1}(1) - 1)
\]

is satisfied then \(L(T N_H(\beta)) \subset G_H(\gamma)\).

**Proof.** The proof of above theorem is similar to that of Theorem 2.14, therefore we omit the details involved.

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**References**


