The multiplicative reformulated first Zagreb index of some graph operations

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Abstract
In this paper, the multiplicative reformulated first Zagreb index is presented and the sharp upper bound for the multiplicative reformulated first Zagreb index of various graph operations for example, join, composition, cartesian and corona product of graphs are derived.

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Topological indices and graph operations.

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1. Introduction

All graphs observed here are simple, connected and finite. Let $V(G), E(G)$ and $d_G(w)$ indicate the vertex set, the edge set and the degree of a vertex of a graph $G$ respectively. A graph with $p$ vertices and $q$ edges is known as a $(p, q)$ graph.

A topological index of a graph $G$ is a real number which is invariant under automorphism of $G$ and does not depend on the labeling or pictorial representation of a graph.

Gutman et.al.[1] introduced the first and second Zagreb indices of a graph $G$ as follows:

\[ M_1(G) = \sum_{w \in V(G)} (d_G(w) + d_G(z)) = \sum_{w \in E(G)} d_G^2(w) \]

and

\[ M_2(G) = \sum_{wz \in E(G)} d_G(w)d_G(z) \]

Shirdel et.al. in [6] found Hyper-Zagreb index $HM(G)$ which is established as

\[ HM(G) = \sum_{wz \in E(G)} [d_G(w) + d_G(z)]^2. \]

Also, they have computed the hyper - Zagreb index of the cartesian product, composition, join and disjunction of graphs.

Milicevic et al [3] reformulated the Zagreb indices in terms of edge degrees instead of vertex degrees, where the degree of an edge $e = wz$ is defined as $d(e) = d(w) + d(z) - 2$. Thus, the reformulated first and second Zagreb indices of a graph $G$ are defined as

\[ EM_1(G) = \sum_{e \in E(G)} d^2(e) \text{ and } EM_2(G) = \sum_{e \sim f} d(e)d(f) \]

where $e \sim f$ means that the edges $e$ and $f$ share a common vertex in $G$. That is, they are adjacent.

Nilanjan De. et.al., [4] computed precise formulae for the reformulated first Zagreb index of some graph operations.

Recently, Todeschine et al [7, 8] have presented the multiplicative variants of ordinary Zagreb indices, which are defined as follows:
\[ \prod_1 = \prod_1(G) = \prod_{w \in V(G)} d_G(w)^2 = \prod_{w \in E(G)} [d_G(w) + d_G(z)] \]

and \[ \prod_2 = \prod_2(G) = \prod_{w \in E(G)} d_G(w)d_G(z) \]

In this paper, we introduce a new graph invariant namely multiplicative reformulated Zagreb indices, denoted by

\[ \prod EM_1(G) = \prod_{w \in E(G)} (d_G(w) + d_G(z) - 2)^2 \]

The join \( G = G_1 + G_2 \) of two graphs \( G_1 \) and \( G_2 \) is a graph formed from disjoint copies of \( G_1 \) and \( G_2 \) by connecting each vertex of \( G_1 \) to each vertex of \( G_2 \).

The composition \( G = G_1[2] \) of graphs \( G_1 \) and \( G_2 \) with vertex set \( V(G_1) \times V(G_2) \) and \((w_1, z_1), (w_2, z_2) \in G_1[G_2] \) iff \( w_1w_2 \in E(G_1) \) or \( w_1 = w_2 \) and \( z_1z_2 \in E(G_2) \).

The Cartesian product of the graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \times G_2 \) with vertex set \( V(G_1) \times V(G_2) \) and for which \((w_1, w_2), (z_1, z_2) \in G_1 \times G_2 \) iff \( w_1 = z_1 \) and \( w_2z_2 \in E(G_2) \) or (ii) \( w_1 = z_1 \) and \( w_2z_2 \in E(G_1) \).

The corona product of the graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \circ G_2 \) obtained by taking one copy of \( G_1 \) and \( |V(G_1)| \) disjoint copies of \( G_2 \), and then joining the \( pth \) vertex of \( G_1 \) to every vertex in \( pth \) copy of \( G_2 \).

The aim of this paper is to continue this program for computing the sharp upper bound for the multiplicative reformulated first Zagreb index of these operations on graphs and to prove our bound is tight.

### 2. Main Results

**Lemma 2.1.** \([2, 5]\)

1. \( d_{G_1+G_2}(w) = \begin{cases} \begin{align*} d_{G_1}(w) + V(G_2), & w \in V(G_2) \\ d_{G_2}(w) + V(G_1), & w \in V(G_2) \end{align*} \end{cases} \)

2. \( d_{G_1[G_2]}(w, z) = p_2d_{G_1}(w) + d_{G_2}(z) \)

3. \( d_{G_1 \odot G_2}((w_i, z_j)) = d_{G_1}(w_i) + d_{G_2}(z_j), \text{ where } (w_i, z_j) \in V(G_1 \odot G_2) \).

4. \( d_{G_1 \circ G_2}(w) = \begin{cases} \begin{align*} d_{G_1}(w) + p_2 & \text{if } w \in V(G_1) \\ d_{G_1}(w) + p_2 & \text{if } w \in V(G_2), \text{ for some } 0 \leq i \leq p_1 - 1, \end{align*} \end{cases} \)

where \( w \in V(G_1 \odot G_2) \) \( G_{2,i} \) is the \( ith \) copy of the graph \( G_2 \) in \( G_1 \odot G_2 \).

**Lemma 2.2** (Arithmetic geometric Inequality). Let \( y_1, y_2, \ldots, y_n \) be non-negative numbers. Then \( \frac{y_1 + y_2 + \cdots + y_n}{n} \geq \sqrt[n]{y_1y_2 \cdots y_n} \)

### 3. The multiplicative reformulated first Zagreb index of join of graphs

**Theorem 3.1.** Let \( G_1, G_2 \) be a \((p_1, q_1)\) graph. Then

\[ \prod EM_1(G_1 + G_2) \leq \prod EM_1(G_1) + 4p_2q_1 + 4p_2(M_1(G_1) - 2q_1) \]

\[ \times \prod EM_1(G_2) + 4p_1q_2 + 4p_1(M_1(G_2) - 2q_2) \]

\[ \times \frac{p_2M_1(G_1) + p_1M_1(G_2)}{p_1p_2} + \frac{p_2M_1(G_1) + p_1M_1(G_2)}{p_1p_2} + \frac{4(p_1 + p_2 - 2)(p_1q_2 + p_2q_1)}{p_1p_2} \]

**Proof.** From the definition of the multiplicative first Zagreb index,

\[ \prod EM_1(G_1 + G_2) = \prod_{w \in E(G_1 + G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \]

\[ = \prod_{w \in E(G_1)} [d_{G_1}(w) + d_{G_1}(z) - 2]^2 \]

\[ \times \prod_{w \in E(G_2)} [d_{G_2}(w) + d_{G_2}(z) - 2]^2 \]

\[ \times \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \]

\[ = A \times B \times C \]

where \( A, B \) and \( C \) indicate the products of the above terms in order.

Now we calculate \( A \).

\[ A = \prod_{w \in E(G_1)} [d_{G_1}(w) + d_{G_1}(z) + 2p_2 - 2]^2 \]

\[ \leq \left[ \sum_{w \in E(G_1)} [d_{G_1}(w) + d_{G_1}(z) + 2p_2 - 2]^2 \right]^{\frac{q_1}{2}} \]

\[ = \left[ \sum_{w \in E(G_1)} [d_{G_1}(w) + d_{G_1}(z) - 2]^2 + 4p_2^2 \right]^{\frac{q_1}{2}} \]

\[ = \left[ EM_1(G_1) + 4p_2^2q_1 + 4p_2(M_1(G_1) - 2q_1) \right]^{\frac{q_1}{2}} \]

1190
Next we calculate $B$.

\[
B = \prod_{w \in E(G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \\
= \prod_{w \in E(G_2)} [d_{G_2}(w) + d_{G_2}(z) + 2p_1 - 2]^2 \\
\leq \left[ \sum_{w \in E(G_2)} [d_{G_2}(w) + d_{G_2}(z) + 2p_1 - 2]^2 \right]^{q_2} \\
= \left[ \sum_{w \in E(G_2)} [d_{G_2}(w) + d_{G_2}(z) - 2]^2 + 4p_1^2 \\
+ 4p_1 [d_{G_2}(w) + d_{G_2}(z) - 2] \right]^{q_2} \\
= \left[ EM_1(G_2) + 4p_1^2 q_2 + 4p_1 (M_1(G_2) - 2q_2) \right]^{q_2}
\]

Finally, we compute $C$

\[
C = \prod_{w \in V(G_1) \in V(G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \\
= \prod_{w \in V(G_1) \in V(G_2)} [d_{G_1}(w) + d_{G_2}(z) - 2]^2 \\
\leq \left[ \sum_{w \in V(G_1) \in V(G_2)} [d_{G_1}(w) + d_{G_2}(z) - 2]^2 + 4p_1^2 \\
+ 4p_1 (d_{G_1}(w) + d_{G_2}(z) - 2) \right]^{p_1 p_2} \\
= \left[ \sum_{w \in V(G_1) \in V(G_2)} [d_{G_1}(w) + d_{G_2}(z) + 2d_{G_1}(w)d_{G_2}(z) + (p_1 + p_2 - 2)^2 \\
+ 2(d_{G_1}(w) + d_{G_2}(z)) (p_1 + p_2 - 2)] \right]^{p_1 p_2} \\
= \left[ p_2 M_1(G_1) + p_1 M_2(G_2) + 8q_1 q_2 \\
+ p_1 p_2 (p_1 + p_2 - 2)^2 \right]^{p_1 p_2} \\
= \left[ 4(p_1 + p_2 - 2)(p_1 q_2 + p_2 q_1) \right]^{p_1 p_2}
\]

Now using $A, B$ and $C$ we get the desired results.

\[
\prod_{w \in E(G_1)} (d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2)^2 \\
= EM_1(G_1 + G_2) \\
\times \prod_{w \in E(G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \\
= \prod_{w \in V(G_1) \in V(G_2)} [d_{G_1+G_2}(w) + d_{G_1+G_2}(z) - 2]^2 \\
= \prod_{w \in V(G_1)} (r_1 + r_1 + 2p_2 - 2)^2 \\
\times \prod_{w \in V(G_2)} (r_2 + r_2 + 2p_1 - 2)^2 \\
= (2r_1 + 2p_2 - 2)^{2q_1} \\
\times (2r_2 + 2p_1 - 2)^{2q_2} \\
\times (r_1 + r_2 + p_1 + p_2 - 2)^{2p_1 p_2}
\]

(3.1)

\[
EM_1(G_1) = q_1 (2r_1 - 2)^2 \\
= 2p_1 r_1 (r_1 - 1) \\
EM_1(G_2) = q_2 (2r_2 - 2)^2 \\
= 2p_2 r_2 (r_2 - 1)
\]

Corollary 3.4. Using Remark 3.3 in Theorem 3.1, then we get

\[
\prod_{w \in E(G_1 + G_2)} (2r_1 + 2p_2 - 2)^{2q_1} \\
\times (2r_2 + 2p_1 - 2)^{2q_2} \\
\times (r_1 + r_2 + p_1 + p_2 - 2)^{2p_1 p_2}
\]

From (3.1) and (3.2) the bound is tight.
4. The multiplicative reformulated first Zagreb index of composition of graphs

Theorem 4.1. Let $G_i, i = 1, 2$ be a $(p_i,q_i)$-graph. Then

$$\prod_{\text{EM}_1(G_i [G_2])} \leq \frac{4p_i^2q_2M_1(G_1) + p_1EM_1(G_2) + 8p_2q_1M_1(G_2) - 16p_2q_1q_2}{p_1q_2} \cdot \frac{p_2EM_1(G_1) + 2p_2(2M_1(G_1) - 2q_1)}{(4p_2q_2 + 2p_2^2(p_2 - 1))} \cdot \frac{2p_2q_1M_1(G_2) + 8q_1q_2^2 + 16pq_1q_2(p_2 - 1)}{q_1p_2^2} \cdot \frac{4p_2q_1(p_2 - 1)^2}{q_1p_2^2}$$

Proof.

$$\prod_{\text{EM}_1(G_i [G_2])} = \prod_{(w,k)\in(V(G_1)\times V(G_2))} [d_{G_1[G_2]}(w,k) + d_{G_1[G_2]}(z,l) - 2]^2 \cdot \prod_{w\in V(G_1)} \prod_{q\in V(G_2)} [d_{G_1[G_2]}(w,k) + d_{G_1[G_2]}(z,l) - 2]^2 \cdot \prod_{k\in V(G_1)} \prod_{l\in V(G_2)} [d_{G_1[G_2]}(w,k) + d_{G_1[G_2]}(z,l) - 2]^2$$

where $A$ and $B$ indicate the products of the above terms in order.

Now we compute $A$.

$$A = \prod_{w\in V(G_1)\times V(G_2)} [d_{G_1[G_2]}(w,k) + d_{G_1[G_2]}(w,l) - 2]^2 \cdot \prod_{w\in V(G_1)\times V(G_2)} [2p_2d_1(w) + d_2(k) + d_2(l) - 2]^2$$

Using $A$ and $B$, we get the required results.

Lemma 4.2. Let $G_i, i = 1, 2$ be two regular graphs of degree $r_i$ and let $G_i, i = 1, 2$ be a $(p_i,q_i)$-graph. Then

$$\prod_{\text{EM}_1(G_1 [G_2])} = (2p_2q_1 + 2r_2 - 2)^2(2p_2q_2 + p_2^2q_1)$$

Proof.

$$\prod_{\text{EM}_1(G_1 [G_2])} = \prod_{w\in V(G_1)\times V(G_2)} [d_{G_1[G_2]}(w,k) + d_{G_1[G_2]}(w,l) - 2]^2 \cdot \prod_{w\in V(G_1)\times V(G_2)} [2p_2d_1(w) + d_2(k) + d_2(l) - 2]^2 \cdot \prod_{w\in V(G_1)\times V(G_2)} [(2p_2r_1 + 2r_2 - 2)^2] \cdot \prod_{w\in V(G_1)\times V(G_2)} [(2p_2r_2 + 2p_2^2q_1)]$$

Corollary 4.3. Using the Remark 3.3 in Theorem 4.1, we have

$$\prod_{\text{EM}_1(G_1 [G_2])} \leq \left(2(p_2r_1 + r_2 - 2)\right)^2(2p_2q_2 + p_2^2q_1) \quad \text{(4.2)}$$

From (4.1) and (4.2) our bound is tight.
5. The multiplicative reformulated first Zagreb index of cartesian product of graphs

Theorem 5.1. Let $G_i, i = 1, 2$ be a $(p_i, q_i) - \text{graph}$. Then

\[
\prod EM_1(G_1 \square G_2) \\
\leq \left[ \frac{p_1EM_1(G_2) + 8q_1M_1(G_2) + 4q_2M_1(G_1) - 16q_1q_2}{p_1q_2} \right]^{p_1q_2} \\
\times \left[ \frac{p_2EM_1(G_1) + 8q_2M_1(G_1) + 4q_1M_1(G_2) - 16q_1q_2}{p_2q_1} \right]^{p_2q_1}
\]

Proof.

\[
\prod EM_1(G_1 \square G_2) = \prod_{(w,k)\in E(G_1 \square G_2)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(z, l) - 2]^2
\]
\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(w, l) - 2]^2
\]
\[
\times \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} [d_{G_1\square G_2}(w) + d_{G_1, G_2}(z, k) - 2]^2
\]
\[
= A \times B
\]

where $A$ and $B$ indicate the products of the above terms in order.

Now we calculate $A$.

\[
A = \prod_{w \in V(G_1)} \prod_{l \in E(G_2)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(w, l) - 2]^2
\]
\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l) - 2]^2
\]
\[
\leq \left[ \frac{\sum_{w \in V(G_1)} \sum_{k \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k)]}{p_1q_2} + \frac{d_{G_2}(l) - 2]^2}{p_1q_2} \right]^{p_1q_2}
\]
\[
= \left[ \frac{\sum_{w \in V(G_1)} \sum_{k \in E(G_2)} [4d_{G_1}^2(w) + (d_{G_2}(k))]}{p_1q_2} + \frac{d_{G_2}(l) - 2]^2}{p_1q_2} \right]^{p_1q_2}
\]
\[
= \left[ \frac{p_1EM_1(G_2) + 8q_1M_1(G_2) + 4q_2M_1(G_1) - 16q_1q_2}{p_1q_2} \right]^{p_1q_2}
\]

Now we compute $B$.

\[
B = \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(z, k) - 2]^2
\]
\[
= \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} [2d_{G_2}(k) + d_{G_2}(w) + d_{G_2}(z) - 2]^2
\]
\[
\leq \left[ \frac{\sum_{k \in V(G_2)} \sum_{w \in E(G_1)} [2d_{G_2}(k) + d_{G_2}(w)]}{p_2q_1} + \frac{d_{G_2}(z) - 2]^2}{p_2q_1} \right]^{p_2q_1}
\]
\[
= \left[ \frac{\sum_{k \in V(G_2)} \sum_{w \in E(G_1)} [4d_{G_2}^2(k) + (d_{G_2}(w))] + \frac{d(G_2)(z) - 2]^2}{p_2q_1} + 4d_{G_2}(k)(d_{G_2}(w) + d_{G_2}(z) - 2)]}{p_2q_1} \right]^{p_2q_1}
\]
\[
= \left[ \frac{p_2EM_1(G_1) + 8q_2M_1(G_1) + 4q_1M_1(G_2) - 16q_1q_2}{p_2q_1} \right]^{p_2q_1}
\]

Using $A$ and $B$ we get the required result. \(\blacksquare\)

Lemma 5.2. Let $G_i, i = 1, 2$ be two regular graphs of degree $r_i$ and let $G_i; i = 1, 2$ be a $(p_i, q_i) - \text{graph}$. Then $\prod EM_1(G_1 \square G_2) = 2(r_1 + r_2 - 2)^2(p_1q_2 + p_2q_1)$

Proof.

\[
\prod EM_1(G_1 \square G_2) = \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(w, l) - 2]^2
\]
\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l) - 2]^2
\]
\[
\times \prod_{k \in E(G_2)} \prod_{w \in V(G_1)} [d_{G_1\square G_2}(w) + d_{G_1\square G_2}(z, k) - 2]^2
\]
\[
= \prod_{w \in V(G_1)} \prod_{l \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l) - 2]^2
\]
\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l) - 2]^2
\]
\[
\leq \left[ \frac{\sum_{w \in V(G_1)} \sum_{k \in E(G_2)} [2d_{G_1}(w) + d_{G_2}(k)]}{p_1q_2} + \frac{d_{G_2}(l) - 2]^2}{p_1q_2} \right]^{p_1q_2}
\]
\[
= \left[ \frac{\sum_{w \in V(G_1)} \sum_{k \in E(G_2)} [4d_{G_1}^2(w) + (d_{G_2}(k))]}{p_1q_2} + \frac{d_{G_2}(l) - 2]^2}{p_1q_2} \right]^{p_1q_2}
\]
\[
= \left[ \frac{p_1EM_1(G_2) + 8q_1M_1(G_2) + 4q_2M_1(G_1) - 16q_1q_2}{p_1q_2} \right]^{p_1q_2}
\]

Corollary 5.3. Using Remark 3.3 in Theorem 5.1, we get

\[
\prod EM_1(G_1 \square G_2) \leq 2(r_1 + r_2 - 2)^2(p_1q_2 + p_2q_1)
\]

From (5.1) and (5.2) the bound is tight.
6. The multiplicative reformulated first Zagreb index of corona product of graphs

Theorem 6.1. Let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod EM_1 (G_1 \odot G_2) \leq \left[ EM_1 (G_1) + 4p_2^2q_1 + 4p_2M_1 (G_1) - 8p_2q_1 \right]^{q_1} \times \left[ HM_2 (G_2) \right]^{p_1q_2} \times \left[ \frac{p_2M_1 (G_1) + p_1M_1 (G_2) + 4p_1p_2 (p_2 - 1)^2 + 4p_2q_1 (p_2 - 1) + 8p_1q_2}{p_1p_2} \right]^{p_1p_2}$$

Proof.

$$\prod EM_1 (G_1 \odot G_2) = \prod_{w \in E (G_1)} \left[ d_{G_1} (w) + d_{G_1} (z) + 2n_2 - 2 \right]^2 \times \prod_{w \in V (G_1)} \prod_{k \in E (G_2)} \left[ d_{G_2} (k) + d_{G_2} (l) \right]^2 \times \prod_{w \in V (G_1)} \prod_{k \in V (G_2)} \left[ d_{G_1} (w) + d_{G_1} (k) + p_2 - 1 \right]^2 = A \times B \times C$$

where $A, B$ and $C$ are the products of the about terms in order.

Now calculate $A$,

$$A = \prod_{w \in E (G_1)} \left[ d_{G_1} (w) + d_{G_1} (z) + 2n_2 - 2 \right]^2 \leq \left[ \sum_{w \in E (G_1)} \left[ d_{G_1} (w) + d_{G_1} (z) + 2n_2 - 2 \right]^2 \right]^{q_1} \times \left[ \sum_{w \in E (G_1)} \left[ (d_{G_1} (w) + d_{G_1} (z) - 2)^2 + 4p_2^2 + 4p_2 (d_{G_1} (w) + d_{G_1} (z) - 2) \right] \right]^{q_1} = \left[ EM_1 (G_1) + 4p_2^2q_1 + 4p_2M_1 (G_1) - 8p_2q_1 \right]^{q_1}$$

Next compute $B$

$$B = \prod_{w \in V (G_1)} \prod_{k \in E (G_2)} \left[ d_{G_2} (k) + d_{G_2} (l) \right]^2 \leq \left[ \sum_{w \in V (G_1)} \sum_{k \in E (G_2)} \left[ d_{G_2} (k) + d_{G_2} (l) \right]^2 \right]^{p_1q_2} = \left[ p_1HM_2 (G_2) \right]^{p_1q_2} = \left[ \frac{HM_2 (G_2)}{q_2} \right]^{p_1q_2}$$

Finally, compute $C$

$$C = \prod_{w \in V (G_1)} \prod_{k \in V (G_2)} \left[ d_{G_1} (w) + d_{G_2} (k) + p_2 - 1 \right]^2 \leq \left[ \sum_{w \in V (G_1)} \sum_{k \in V (G_2)} \left[ d_{G_1} (w) + d_{G_2} (k) + p_2 - 1 \right]^2 \right]^{p_1p_2} = \left[ p_2M_2 (G_1) + p_1M_2 (G_2) + p_1p_2 (p_2 - 1) + 2d_{G_1} (m) + 2d_{G_2} (m) \right]^{p_1p_2}$$

The required result is obtained by multiplying $A, B$ and $C$. \(\square\)

Lemma 6.2. Let $G_i, i = 1, 2$ be two regular graph of degree $r_i$, and let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod EM_1 (G_1 \odot G_2) = \left[ 2(r_1 + p_2 - 1) \right]^{2q_1} \times \left[ 2r_2 \right]^{2p_1q_2} \times (r_1 + r_2 + p_2 - 1)^{2p_1p_2}$$
Proof.

\[
\prod_{E_{M_1}(G_1 \odot G_2)} = \prod_{w \in E(G_1)} (d_{G_1}(w) + d_{G_1}(z) + 2p_2 - 2)^2 \\
\times \prod_{w \in V(G_1) \times k \in E(G_2)} [d_{G_2}(k) + d_{G_2}(l)]^2 \\
\times \prod_{w \in V(G_1) \times k \in E(G_2)} [d_{G_1}(w) + d_{G_2}(k) + p_2 - 1]^2 \\
= \prod_{w \in E(G_1)} (2r_1 + 2p_2 - 2)^2 \times \prod_{w \in V(G_1) \times k \in E(G_2)} (2r_2)^2 \\
\times \prod_{w \in V(G_1) \times k \in E(G_2)} (r_1 + r_2 + p_2 - 1)^2 \\
= [2(r_1 + p_2 - 1)]^{2q_1} \times (2r_2)^{2p_1 q_2} (r_1 + r_2 + p_2 - 1)^{2p_1 p_2} \\
(6.1)
\]

\[
\prod_{E_{M_1}(G_1 \odot G_2)} \leq [2(r_1 + r_2 - 1)]^{2q_1} \times (2p_2)^{2p_1 q_2} \\
\times (r_1 + r_2 + p_2 - 1)^{2p_1 p_2} \\
(6.2)
\]

From (6.1) and (6.2) the bound is tight.

\[\square\]

Corollary 6.3. Using Remark 3.3 in Theorem 6.1, we get

\[\prod_{E_{M_1}(G_1 \odot G_2)} \leq \left[2(r_1 + r_2 - 1)\right]^{2q_1} \times (2p_2)^{2p_1 q_2} \times (r_1 + r_2 + p_2 - 1)^{2p_1 p_2} \]

References


