A study on fuzzy semi-stratifiable and fuzzy stratifiable spaces

M.S. Jisha 1* and R. Sreekumar2

Abstract
The notion of stratifiable space and semi-stratifiable space were studied by Gary Gruenhage [4]. Alexander P. Sostak extended the concept of stratifiable space to the fuzzy case [1]. In this paper we investigate some properties of fuzzy closure-preserving collection, fuzzy stratifiable space and fuzzy semi-stratifiable space.

Keywords
Fuzzy closure-preserving collection, fuzzy stratifiable space, fuzzy semi-stratifiable space and fuzzy $M_1$-space.

AMS Subject Classification
54A40, 54E35, 03E72.

1 Department of Mathematics, M.S.M. College, Kayamkulam-690502, Kerala, India.
2 Department of Mathematics, S.D.College, Alappuzha-688003, Kerala, India.
*Corresponding author: jishamkrishnan@gmail.com; dr.r.sreekumar@gmail.com

Article History: Received 03 April 2020; Accepted 12 July 2020

Contents
1 Introduction ............................................. 1240
2 Preliminaries ........................................... 1240
3 Fuzzy closure-preserving collection .......... 1240
4 Fuzzy semi-stratifiable and fuzzy stratifiable spaces 1241
References ...................................................... 1242

1. Introduction

The concepts like stratifiable space and semi-stratifiable space were extensively studied by Gary Gruenhage. For a detailed discussion reference may be made of [4]. The class of stratifiable spaces does not differ too much from the class of metrizable spaces. Alexander P. Sostak extended the concept of stratifiable space to the fuzzy case [1]. In this paper we establish some properties of fuzzy closure-preserving collection, fuzzy stratifiable space and fuzzy semi-stratifiable space.

2. Preliminaries

We use the term fuzzy topological space in Chang’s sense [2]. A fuzzy point $x_\alpha$ is a fuzzy set which takes every element in $X$ to $0$ except one element $x \in X$ and its value at $x$ is $\alpha$ where $(0 < \alpha \leq 1)$. $\emptyset$ is the fuzzy set which maps all element in $X$ to $0$ and $1$ is the fuzzy set which maps every element in $X$ to $1$. The closure of a fuzzy set $U$ in a fuzzy topological space is denoted by $\overline{U}$. In this paper each fuzzy topological space is assumed to be $T_1$ and regular. $\mathbb{N}$ denotes set of all natural numbers. We list some of the definitions which we are using in this paper.

Definition 2.1. [11] A collection $\mathcal{A}$ of fuzzy subsets of a fuzzy topological space $(X, F)$ is called discrete if for each fuzzy point $x_\alpha$, there is a $G \in F$ with $x_\alpha \leq G$ and $G \land A \neq \emptyset$ holds for at most one element $A$ of $\mathcal{A}$.

Definition 2.2. [9] Let $\mathcal{A}$ be a cover of a fuzzy topological space $(X, F)$. For $\alpha \in (0, 1]$ and a fuzzy point $x_\alpha$, $st(x_\alpha, \mathcal{A}) = \bigvee \{B : B \in \mathcal{A}, B(x) \geq \alpha\}$ and for a fuzzy set $G$, $st(G, \mathcal{A}) = \bigvee \{B : B \in \mathcal{A} \land B \cap G \neq \emptyset\}$.

Definition 2.3. [9] Let $(X, F)$ be a fuzzy topological space. A fuzzy point $x_\alpha, \alpha \in (0, 1]$ is called a cluster point of the set $\{(x_n)_\alpha : n \in \mathbb{N}\}$, where $(x_n)_\alpha$ is a fuzzy set with support $x_n$ and value $\alpha$, if for each fuzzy set $G \in F$ such that $x_\alpha \leq G$, there exists $n_0 \in \mathbb{N}$ with $x_{n_0} \neq x$ and $(x_{n_0})_\alpha \leq G$.

Definition 2.4. [7] Let $(X, F)$ be a fuzzy topological space and let $F_0 \subset F$. Then $F_0$ is called a fuzzy base of $F$ if $F = \{\bigvee \mathcal{A} : \mathcal{A} \subset F_0\}$.

3. Fuzzy closure-preserving collection

In this section we define fuzzy closure-preserving collection and fuzzy $\sigma$-closure-preserving collection.
**Definition 3.1.** A collection $\mathcal{H}$ of fuzzy subsets of a fuzzy topological space $(X,F)$ is called fuzzy closure-preserving if $\bigvee_{H \in \mathcal{H}} = \bigvee\{H : H \in \mathcal{H}\}$ for each $\mathcal{H} \subseteq \mathcal{H}$. If $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ where each $\mathcal{H}_n$ is fuzzy closure-preserving, then $\mathcal{H}$ is called a fuzzy $\sigma$-closure-preserving.

**Theorem 3.2. Fuzzy locally finite collections are fuzzy closure-preserving.**

Proof. Let $\mathcal{A}$ be a fuzzy locally finite collection of fuzzy subsets of $X$. Let $\mathcal{A} \subseteq \mathcal{A}$. Let $A \in \mathcal{A}$. Now $A \subseteq \bigvee\{A : A \in \mathcal{A}\}$ which implies $A \subseteq \bigvee\{A : A \in \mathcal{A}\}$. Thus we need to prove the other side. Clearly, $\bigvee\{A : A \in \mathcal{A}\} \subseteq \bigvee\{A : A \in \mathcal{A}\}$. If $\bigvee\{A : A \in \mathcal{A}\}$ is closed, then $\bigvee\{A : A \in \mathcal{A}\}$ is a fuzzy topological space $(X,F)$ is fuzzy developable if it has a fuzzy development.

**Theorem 4.2.** Every fuzzy developable space is fuzzy semi-stratifiable.

Proof. Suppose $(X,F)$ is a fuzzy developable space. Let $\mathcal{H}_n$ be a fuzzy development for $X$. Let $n \in N$ and $H$ be a fuzzy closed subset of $X$. Let $G(n,H) = st(H, \mathcal{H}_n)$. Clearly $H \subseteq st(H, \mathcal{H}_n) = G(n,H)$ and $H = \bigcup_{n=1}^{\infty} H \subseteq \bigcup_{n=1}^{\infty} G(n,H)$.

**Definition 4.4.** A fuzzy topological space $(X,F)$ is called fuzzy semi-stratifiable if there is a function $G$ which assigns to each $n \in N$ and fuzzy closed subset $H$ of $X$, a fuzzy open set $G(n,H)$ with $H \leq G(n,H)$ such that

1. $H = \bigcup_{n=1}^{\infty} G(n,H)$
2. $H \leq K \implies G(n,H) \leq G(n,K)$

**Definition 4.5.** A sequence $(\alpha_n)$ of fuzzy open covers of $(X,F)$ is called a fuzzy development for $X$; if for $\alpha \in [0,1]$, a fuzzy point $x_\alpha$, the set $\{st(x_\alpha, \mathcal{H}_n) : n \in N\}$ is a base at $x_\alpha$. A fuzzy topological space $(X,F)$ is fuzzy developable if it has a fuzzy development.

**Theorem 4.6.** Every fuzzy developable space is fuzzy semi-stratifiable.

Proof. Suppose $(X,F)$ is a fuzzy developable space. Let $\mathcal{H}_n$ be a fuzzy development for $X$. Let $n \in N$ and $H$ be a fuzzy closed subset of $X$. Let $G(n,H) = st(H, \mathcal{H}_n)$. Clearly $H \subseteq st(H, \mathcal{H}_n) = G(n,H)$ and $H = \bigcup_{n=1}^{\infty} H \subseteq \bigcup_{n=1}^{\infty} G(n,H)$.

**Definition 4.7.** A fuzzy topological space $(X,F)$ is called a fuzzy $\alpha$-M$_1$-space if $X$ has a fuzzy $\sigma$-closed preserving base.

**Theorem 4.8.** Every fuzzy $\alpha$-M$_1$-space is fuzzy stratifiable.

Proof. Suppose $(X,F)$ is a fuzzy $\alpha$-M$_1$-space. Let $\mathcal{B}$ be a fuzzy $\sigma$-closed preserving base for $X$. Let $H$ be a fuzzy closed subset of $X$ and $n \in N$. Let $G(n,H) = \{V \in B \subseteq \mathcal{B} : B \subseteq \mathcal{B} \land H \cap B = 0\}$. Then $H \leq G(n,H)$. Clearly $H \subseteq G(n,H)$.

**Definition 4.9.** A fuzzy topological space $(X,F)$ is fuzzy stratifiable if and only if for each fixed $\alpha \in [0,1]$ there is a function $g : N \times \{x_\alpha : x \in X\} \rightarrow F$ such that

(i) $x_\alpha = \land_n g(n,x_\alpha)$
(ii) $y_\alpha \leq g(n,x_\alpha) \implies (x_\alpha)_\alpha \rightarrow y_\alpha$

$(X,F)$ is fuzzy stratifiable is we can also obtain

(iii) If $y_\alpha \not\leq H$ where $H$ is fuzzy-closed subset, then $y_\alpha \not\leq g(n,x_\alpha) : x_\alpha \leq H$ for some $n \in N$.

Proof. Suppose $(X,F)$ is fuzzy semi-stratifiable. Fix $\alpha \in [0,1]$. Let $g(n,x_\alpha) = G(n,x_\alpha)$, where $G$ satisfies definition 4.4. with $G(n+1,H) \leq G(n,H)$ for each $n$. Since $H = \land_n G(n,H)$, $x_\alpha = \land_n g(n,x_\alpha)$.

**Remark 4.3.** we can assume the following condition satisfies:

(4) $G(n+1,H) \leq G(n,H)$ for each $n \in N$.

**Definition 4.4.** A fuzzy topological space $(X,F)$ is called fuzzy semi-stratifiable if there is a function $G$ which assigns to each $n \in N$ and fuzzy closed subset $H$ of $X$, a fuzzy open set $G(n,H)$ with $H \leq G(n,H)$ such that

1. $H = \land_n G(n,H)$
2. $H \leq K \implies G(n,H) \leq G(n,K)$
3. $H = \land_n G(n,H)$

**Remark 4.3.** we can assume the following condition satisfies:

(4) $G(n+1,H) \leq G(n,H)$ for each $n \in N$.

**Definition 4.4.** A fuzzy topological space $(X,F)$ is called fuzzy semi-stratifiable if there is a function $G$ which assigns to each $n \in N$ and fuzzy closed subset $H$ of $X$, a fuzzy open set $G(n,H)$ with $H \leq G(n,H)$ such that

1. $H = \land_n G(n,H)$
2. $H \leq K \implies G(n,H) \leq G(n,K)$

**Definition 4.5.** A sequence $(\alpha_n)$ of fuzzy open covers of $(X,F)$ is called a fuzzy development for $X$; if for $\alpha \in [0,1]$, a fuzzy point $x_\alpha$, the set $\{st(x_\alpha, \mathcal{H}_n) : n \in N\}$ is a base at $x_\alpha$.

A fuzzy topological space $(X,F)$ is fuzzy developable if it has a fuzzy development.

**Theorem 4.6.** Every fuzzy developable space is fuzzy semi-stratifiable.

Proof. Suppose $(X,F)$ is a fuzzy developable space. Let $\mathcal{H}_n$ be a fuzzy development for $X$. Let $n \in N$ and $H$ be a fuzzy closed subset of $X$. Let $G(n,H) = st(H, \mathcal{H}_n)$. Clearly $H \subseteq st(H, \mathcal{H}_n) = G(n,H)$ and $H = \land_n st(H, \mathcal{H}_n) = \land_n G(n,H)$. If $H \leq K$ then $st(H, \mathcal{H}_n) \leq st(K, \mathcal{H}_n)$. Therefore $(X,F)$ is fuzzy semi-stratifiable.

**Definition 4.7.** A fuzzy topological space $(X,F)$ is called a fuzzy $\alpha$-M$_1$-space if $X$ has a fuzzy $\sigma$-closed preserving base.

**Theorem 4.8.** Every fuzzy $\alpha$-M$_1$-space is fuzzy stratifiable.

Proof. Suppose $(X,F)$ is a fuzzy $\alpha$-M$_1$-space. Let $\mathcal{B}$ be a fuzzy $\sigma$-closed preserving base for $X$. Let $H$ be a fuzzy closed subset of $X$ and $n \in N$. Let $G(n,H) = \{V \in B \subseteq \mathcal{B} : B \subseteq \mathcal{B} \land H \cap B = 0\}$.

Then $H \leq G(n,H)$. Clearly $H \subseteq G(n,H)$.

**Theorem 4.9.** A fuzzy topological space $(X,F)$ is fuzzy stratifiable if and only if for each fixed $\alpha \in [0,1]$ there is a function $g : N \times \{x_\alpha : x \in X\} \rightarrow F$ such that

(i) $x_\alpha = \land_n g(n,x_\alpha)$
(ii) $y_\alpha \leq g(n,x_\alpha) \implies (x_\alpha)_\alpha \rightarrow y_\alpha$

$(X,F)$ is fuzzy stratifiable is we can also obtain

(iii) If $y_\alpha \not\leq H$ where $H$ is fuzzy-closed subset, then $y_\alpha \not\leq g(n,x_\alpha) : x_\alpha \leq H$ for some $n \in N$.

Proof. Suppose $(X,F)$ is fuzzy semi-stratifiable. Fix $\alpha \in [0,1]$. Let $g(n,x_\alpha) = G(n,x_\alpha)$, where $G$ satisfies definition 4.4. with $G(n+1,H) \leq G(n,H)$ for each $n$. Since $H = \land_n G(n,H)$, $x_\alpha = \land_n g(n,x_\alpha)$.

Then there is an infinite subset $A \subseteq N$ with $y_\alpha \not\leq g((n,x_\alpha) : x_\alpha \in A)$. Then $y_\alpha \not\leq G(m,\{x_\alpha \in A\} : x_\alpha \in A)$ for some $m \in N$. Choose $n \in A$ with $n \geq m$. Then $y_\alpha \not\leq g(n,x_\alpha) = G(n,x_\alpha) \subseteq G(m,\{x_\alpha \in A\} : x_\alpha \in A)$, since $n \geq m$. This is a contradiction. Therefore $(x_\alpha)_\alpha \rightarrow y_\alpha$.

Suppose $(X,F)$ is fuzzy stratifiable. To check (iii), assume that $y_\alpha \not\leq H$ where $H$ is a closed fuzzy subset. Since $H = \land_n G(n,H)$, there exists $n \in N$ with $y_\alpha \not\leq G(n,H)$.

But $\land_n G(n,x_\alpha) \leq G(n,x_\alpha)$ and hence $G(n,x_\alpha) \leq G(n,H)$. So $y_\alpha \not\leq \land_n G(n,x_\alpha)$ and hence
Conversely, assume that there is a function \( g \) satisfying the conditions of theorem. Let \( n \in \mathbb{N} \) and \( H \) be a closed fuzzy subset. Let \( G(n,H) = \bigvee_{\alpha = H(x)} g(n,x_{\alpha}) \). Clearly \( H \leq G(n,H) \).

Then \( \bigwedge G(n,H) = \bigwedge n \bigvee_{\alpha = H(x)} g(n,x_{\alpha}) = \bigvee_{\alpha = H(x)} \big( \bigwedge n g(n,x_{\alpha}) \big) = \bigwedge n H(x_{\alpha}) = H \).

Suppose \( H \leq K \) where \( K \) is a closed fuzzy subset. Then \( G(n,H) = \bigvee_{\alpha = H(x)} g(n,x_{\alpha}) \leq \bigvee_{\alpha = K(x)} g(n,x_{\alpha}) = G(n,K) \). Therefore \((X,F)\) is fuzzy semi-stratifiable. Clearly \( \bigwedge G(n,H) = \bigwedge n \bigvee_{\alpha = H(x)} g(n,x_{\alpha}) = H \) by (iii). Hence \((X,F)\) is fuzzy stratifiable.

\[ y_{\alpha} \notin \bigvee \{ g(n,x_{\alpha}) : x_{\alpha} \leq H \} . \]

**Acknowledgment**

The author is thankful to the reviewers for their valuable suggestions and comments to improve the quality of the paper.

**References**


