Malaya Journal of Matematik, Vol. 8, No. 3, 1301-1305, 2020
https://doi.org/10.26637/MJM0803/0100

Some new oscillation criteria of third-order half-linear neutral difference equations

T. Gopal[1], G. Ayyappan[2]* and R. Arul[3]

Abstract
In this article, we introduce the oscillation of all solutions of third-order half-linear neutral difference equation(OSTOHLDE)

$$\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}) + f(n)y^{\alpha}(n+1) = 0,$$

where $z(n) = y(n) + e(n)y(n-k)$ and α is a ratio of odd positive integers(PI). Our results are new and complement to the existing ones.

Keywords
Third-order, half-linear difference equation(DE), neutral, oscillation.

AMS Subject Classification
39A10, 39A21.

[1] *Department of Mathematics, Periyar University, Salem-636011, Tamil Nadu, India.*
[2] *Department of Mathematics, Periyar University College of Arts and Science, Pappireddipatti -636905, Tamil Nadu, India.*
[3] *Department of Mathematics, Kandaswami Kandar's College, Velur-638182, Tamil Nadu, India.*
***Corresponding author**: [1]gopalmaths@gmail.com; [2]ayyapmath@gmail.com; [3] rarulkkc@gmail.com;
Article History: Received **30** Jun **2020**; Accepted **18** August **2020**

Contents

1. Introduction

This paper investigated the OSTOHLDE

$$\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}) + f(n)y^{\alpha}(n+1) = 0 \qquad (E),$$

where $n \in \mathbb{N}(n_0) = \{n_0,\ n_0+1,\ \dots\}$, n_0 is a PI, and $z(n) = y(n) + e(n)y(n-k)$. Throughout, we use the following assumptions:

(A_1) $\{g(n)\}$, $\{h(n)\}$, $\{e(n)\}$ and $\{f(n)\}$ are positive real sequences with $0 \le e(n) \le p < 1$, and α is a ratio of odd PI;

(A_2) k is a PI;

(A_3) $\sum_{n=n_0}^{\infty}(g(n))^{-\frac{1}{\alpha}} = \sum_{n=n_0}^{\infty}\frac{1}{h(n)} = \infty.$

The real sequence $\{y(n)\}$ is a solution of (E) if it is defined and satisfies (E) for all $n \in \mathbb{N}(n_0)$. A nontrivial solution of (E) is called oscillatory if the terms of the sequence $\{y(n)\}$ are neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation (E) is called oscillatory.

In recent years, many researchers studying the OSTOHLDE. The monographs and the references [1–3, 6, 8–11, 13] cited therein as examples of recent results on this OSTOHLDE. However, the sufficient conditions established in these papers except [9] ensure that every solution of equations concerned either oscillatory or tends to zero.

For the case $0 \le e(n) \le p < 1$, we used the following relation

$$y(n) \ge (1-e(n))z(n) \tag{1.1}$$

when $y(n)$ is positive and $z(n)$ is positive and increasing. But if $z(n)$ is positive and decreasing there is no such relation of the form (1.1) is found. However, if $z(n)$ is positive and decreasing then [12, 14] used $y(n)$ is also decreasing and found a relation of the form (1.1) between $y(n)$ and $z(n)$. The following example shows that if $z(n)$ is decreasing then $y(n)$ is not decreasing. Let

$$y(n) = \frac{1}{3^n}\left(\frac{3}{2} + (-1)^n\right) > 0$$

then
$$z(n) = y(n) + \frac{1}{3}y(n-1) = \frac{1}{3^{n-1}} > 0.$$
Also
$$\Delta z(n) = -\frac{2}{3^n} < 0 \text{ and } \Delta y(n) = \frac{1}{3^{n+1}}(4(-1)^{n+1} - 3)$$
which is oscillatory and hence $y(n)$ is not decreasing.

The above observation in this article by using a different method, first we obtain a relation between $y(n)$ and $z(n)$ when $z(n)$ is positive and decreasing, then using this relation we establish new oscillation criteria ensuring all solutions of (E) are oscillatory. Thus, the results presented in this article are new and complement to the existing results reported in the literature. The article we are provided to other examples and showed in the importance of the main results.

2. Main Results

In this section, we present sufficient conditions for the oscillation of all solutions of (E). We may deal only with the positive solutions(PS) of (E) since the proof for the negative case is similar. We begin with the following lemma, these lemmas gives the basic properties of nonoscillatory, let us say PS of (E).

Lemma 2.1. *Let $\{y(n)\}$ be a PS of (E). Then there are only two cases for the corresponding sequence $\{z(n)\}$*

(I) $z(n) > 0,\ \Delta z(n) < 0,\ \Delta(h(n)\Delta z(n)) > 0,$ $\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}) < 0$;

(II) $z(n) > 0,\ \Delta z(n) > 0,\ \Delta(h(n)\Delta z(n)) > 0,$ $\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}) < 0$,

eventually.

Proof. The proof is standard, and also similar to that of [10, Lemma 2.1] and thus is excluded. ☐

Lemma 2.2. *Let $\{y(n)\}$ be a PS of (E) and $z(n)$ satisfies Case (II) of Lemma 2.1. Then*
$$y(n+1) \geq (1 - e(n+1))z(n-k) \tag{2.1}$$
for all $n \geq N \in \mathbb{N}(n_0)$.

Proof. The proof is similar to that of in [12, Lemma 2.2], and thus excluded. ☐

Our convenience, let us define
$$G(n) = \sum_{s=N}^{n-1} \frac{1}{g^{\frac{1}{\alpha}}(s)},\quad H(n) = \sum_{s=N}^{n-1} \frac{G(s)}{h(s)},$$
$$Q(n) = \frac{1}{h(n)} \sum_{s=n}^{\infty} \left(\frac{1}{g(s)} \sum_{t=s}^{\infty} f(t) \right)^{\frac{1}{\alpha}},$$
and
$$\phi(n) = \prod_{s=n-k}^{n} \left(\frac{1}{1+Q(s)} \right),\quad \psi(n) = (\phi(n) - e(n+1))$$
for all $N \in \mathbb{N}(n_0)$.

Lemma 2.3. *Let $\{y(n)\}$ be a positive solution of (E) with $\{z(n)\}$ satisfies Case (I) of Lemma 2.1. Then*
$$z(n+1) \geq \phi(n) z(n-k) \tag{2.2}$$
for all $n \geq N \in \mathbb{N}(n_0)$.

Proof. Assume that $\{y(n)\}$ is a PS of (E) with the corresponding sequence $\{z(n)\}$ belongs to Case (I) of Lemma 2.1. Then it is easy to verify that $\lim_{n\to\infty} h(n)\Delta z(n) = 0$, and $\lim_{n\to\infty} g(n)(\Delta(h(n)\Delta z(n)))^{\alpha} = 0$. Thus, a summation of (E) yields
$$\begin{aligned} g(n)(\Delta(h(n)\Delta z(n)))^{\alpha} &= \sum_{s=n}^{\infty} f(s) y^{\alpha}(s+1) \\ &\leq \sum_{s=n}^{\infty} f(s) z^{\alpha}(s+1) \\ &\leq z^{\alpha}(n+1) \sum_{s=n}^{\infty} f(s). \end{aligned} \tag{2.3}$$
Summing up again, one obtains
$$h(n)\Delta z(n) \geq -z(n+1) \sum_{s=n}^{\infty} \left(\frac{1}{g(s)} \sum_{t=s}^{\infty} f(t) \right)^{\frac{1}{\alpha}},$$
or
$$\Delta z(n) \geq -Q(n) z(n+1).$$
Hence
$$\frac{z(n+1)}{z(n)} \geq \frac{1}{1+Q(n)}.$$
Summing the above inequality from $n-k$ to n, we have
$$z(n+1) \geq \phi(n) z(n-k)$$
for all $n \geq N \in \mathbb{N}(n_0)$. This completes the proof. ☐

Lemma 2.4. *Assume that $z(n)$ satisfies Case (II) of Lemma 2.1 for all $n \geq N \in \mathbb{N}(n_0)$. Then*
$$\Delta z(n) \geq \frac{G(n)}{h(n)} g^{\frac{1}{\alpha}}(n) \Delta(h(n)\Delta z(n)), \tag{2.4}$$
$$z(n) \geq H(n) g^{\frac{1}{\alpha}}(n) \Delta(h(n)\Delta z(n)), \tag{2.5}$$
and
$$z(n-k) \geq H(n-k) \frac{h(n)\Delta z(n)}{G(n)}, \tag{2.6}$$
for all $n \geq N$.

Proof. The proof is similar to that of Lemma 2.5 of [11] and Lemma 2.4 of [11] and hence the details are omitted. ☐

Now, we prove our main results.

Theorem 2.5. *If both the first order delay DE*

$$\Delta w(n)+\frac{f(n)\psi^{\alpha}(n)}{F(n)}w(n-k)=0, \tag{2.7}$$

and

$$\Delta x(n)+f(n)(1-e(n+1))^{\alpha}H^{\alpha}(n-k)x(n-k)=0 \tag{2.8}$$

where $F(n)=\sum_{s=n-k}^{\infty}f(s)$*, is oscillatory, then equation (E) is oscillatory.*

Proof. Let $\{y(n)\}$ be a PS of (E). Then there is an integer $N\in\mathbb{N}(n_0)$ such that $y(n)>0$ and $y(n-k)>0$ for all $n\geq N$. From the definition of $z(n)$, we have $z(n)>0$ and it satisfies two cases of Lemma 2.1 for all $n\geq N$.
Case (I) The definition of $z(n)$ and Lemma 2.3, we have

$$y(n+1)\geq z(n+1)-e(n+1)z(n+1-k)\geq\psi(n)z(n-k) \tag{2.9}$$

for all $n\geq N$. Using (2.9) in (E), we obtain

$$\Delta\left(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}\right)+f(n)\psi^{\alpha}(n)z^{\alpha}(n-k)\leq 0,\ \ n\geq N. \tag{2.10}$$

From (2.3), we have

$$z^{\alpha}(n)\sum_{s=n}^{\infty}f(s)\geq z^{\alpha}(n+1)\sum_{s=n}^{\infty}f(s)\geq g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}$$

and using this in (2.10), we obtain

$$\begin{aligned}&\Delta\left(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}\right)\\&\quad+\frac{f(n)\psi^{\alpha}(n)}{\sum_{s=n-k}^{\infty}f(s)}g(n-k)(\Delta(h(n-k)\Delta z(n-k)))^{\alpha}\leq 0.\end{aligned}$$

Let $w(n)=g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}>0$ be a PS of the inequality

$$\Delta w(n)+\frac{f(n)\psi^{\alpha}(n)}{F(n)}w(n-k)\leq 0.$$

But by Lemma 5 of Section 2 in [7] $\Longrightarrow$ that the corresponding difference equation (2.7) also has a PS, which is $\Longrightarrow\Longleftarrow$.
Case(II) Using (2.1) in (E), we obtain

$$\Delta\left(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}\right)+f(n)(1-e(n+1))^{\alpha}z^{\alpha}(n-k)\leq 0 \tag{2.11}$$

for $n\geq N$. In view of (2.5), one obtains

$$z^{\alpha}(n-k)\geq H^{\alpha}(n-k)g(n-k)(\Delta(h(n-k)\Delta z(n-k)))^{\alpha},\ \ n\geq N. \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$\begin{aligned}&\Delta\left(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}\right)+f(n)(1-e(n+1))^{\alpha}H^{\alpha}(n-k)\\&\quad g(n-k)(\Delta(h(n-k)\Delta z(n-k)))^{\alpha}\leq 0,\end{aligned}$$

for $n\geq N$. Let $x(n)=g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}>0$. Then, we see that $\{x(n)\}$ is a PS of the inequality

$$\Delta x(n)+f(n)\psi^{\alpha}(n)H^{\alpha}(n-k)x(n-k)\leq 0.$$

But by Lemma 5 of Section 2 in [7] $\Longrightarrow$ that the corresponding difference equation (2.8) also has a PS, which is $\Longrightarrow\Longleftarrow$. This completes the proof. □

Corollary 2.6. *If*

$$\liminf_{n\to\infty}\sum_{s=n-k}^{n-1}\frac{f(s)}{F(s)}\psi^{\alpha}(s)>\left(\frac{k}{k+1}\right)^{k+1}, \tag{2.13}$$

and

$$\liminf_{n\to\infty}\sum_{s=n-k}^{n-1}f(s)(1-e(s+1))^{\alpha}H^{\alpha}(s-k)>\left(\frac{k}{k+1}\right)^{k+1} \tag{2.14}$$

are hold, then (E) oscillates.

Proof. The proof follows from Theorem 7.6.1 of [5] and Theorem 2.5 and hence the details are excluded. □

Theorem 2.7. *Assume that there exists a positive nondecreasing real sequence* $\{\rho(n)\}$ *such that*

$$\begin{aligned}\limsup_{n\to\infty}\sum_{s=N}^{n}&\left[\rho(s)f(s)(1-e(s+1))^{\alpha}\frac{H^{\alpha}(s-k)}{G^{\alpha}(s)}\right.\\&\left.-\frac{g(s)(\Delta\rho(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right]=\infty,\end{aligned} \tag{2.15}$$

and

$$\limsup_{n\to\infty}\sum_{t=n}^{n+k}\left[\frac{1}{h(t)}\left(\frac{1}{g(s)}\sum_{i=s}^{t}f(i)\psi^{\alpha}(i)\right)^{\frac{1}{\alpha}}\right]>1 \tag{2.16}$$

for all $n\geq N\in\mathbb{N}(n_0)$*, then (E) oscillates.*

Proof. Let $\{y(n)\}$ be a PS of (E). Then the integer $N\in\mathbb{N}(n_0)$ such that $y(n)>0$ and $y(n-k)>0$ for all $n\geq N$. The definition of $z(n)$, we have $z(n)>0$ and it satisfies two cases of Lemma 2.1 for all $n\geq N$.
Case(I) Summing (2.10) from n to j, we obtain

$$\begin{aligned}&g(j+1)(\Delta(h(j+1)\Delta z(j+1)))^{\alpha}\\&\quad-g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}+\sum_{t=n}^{j}f(t)\psi^{\alpha}(t)z^{\alpha}(t-k)\leq 0.\end{aligned}$$

Since $\{g(j)(\Delta(h(j)\Delta z(j)))^{\alpha}\}$ is positive and decreasing, the above inequality implies that, as $j\to\infty$,

$$-\Delta(h(n)\Delta z(n))+\left(\frac{1}{g(n)}\sum_{t=n}^{j}f(t)\psi^{\alpha}(t)z^{\alpha}(t-k)\right)^{\frac{1}{\alpha}}\leq 0.$$

Summing up again from n to j and rearranging, we obtain

$$-h(j+1)\Delta z(j+1)+h(n)\Delta z(n) + \sum_{t=n}^{j}\left(\frac{1}{g(t)}\sum_{s=n}^{t} f(s)\psi^{\alpha}(s)\right)^{\frac{1}{\alpha}} z(t-k) \leq 0.$$

Since $\{\Delta z(j)\}$ is negative and $h(j)\Delta z(j)$ is increasing, as $j \to \infty$, we have

$$\Delta z(n)+\frac{1}{h(n)}\sum_{t=n}^{j}\left(\frac{1}{g(t)}\sum_{s=n}^{t} f(s)\psi^{\alpha}(s)\right)^{\frac{1}{\alpha}} z(t-k) \leq 0.$$

Summing the last inequality from n to j and rearranging, we obtain

$$z(j+1)-z(n) + \sum_{t=n}^{j}\left[\frac{1}{h(t)}\sum_{s=n}^{t}\left(\frac{1}{g(s)}\sum_{i=s}^{t} f(i)\psi^{\alpha}(i)\right)^{\frac{1}{\alpha}} z(t-k)\right] \leq 0.$$

Since $\{z(n)\}$ is positive and decreasing, the last inequality

$$\sum_{t=n}^{j}\left[\frac{1}{h(t)}\sum_{s=n}^{t}\left(\frac{1}{g(s)}\sum_{i=s}^{t} f(i)\psi^{\alpha}(i)\right)^{\frac{1}{\alpha}}\right] z(t-k) \leq z(n),$$

or

$$\sum_{t=n}^{n+k}\left[\frac{1}{h(t)}\sum_{s=n}^{t}\left(\frac{1}{g(s)}\sum_{i=s}^{t} f(i)\psi^{\alpha}(i)\right)^{\frac{1}{\alpha}}\right] \leq 1$$

which $\Longrightarrow\Longleftarrow$ (2.16) as $n \to \infty$.

Case(II) Define

$$w(n)=\frac{\rho(n)g(n)(\Delta(h(n)\Delta z(n)))^{\alpha}}{(h(n)\Delta z(n))^{\alpha}}, \quad n \geq N. \tag{2.17}$$

Then $w(n)>0$ for all $n \geq N$, and

$$\begin{aligned}\Delta w(n) \quad = \quad & \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ & +\rho(n)\frac{\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha})}{(h(n)\Delta z(n))^{\alpha}} \\ & -\rho(n)\frac{g(n+1)(\Delta(h(n+1)\Delta z(n+1)))^{\alpha}}{(h(n+1)\Delta z(n+1))^{\alpha}(h(n)\Delta z(n))^{\alpha}} \\ & \Delta(h(n)\Delta z(n))^{\alpha}. \end{aligned} \tag{2.18}$$

Using (2.6) in (2.11), we obtain

$$\Delta(g(n)(\Delta(h(n)\Delta z(n)))^{\alpha})+f(n)(1-e(n+1))^{\alpha} \frac{H^{\alpha}(n-k)}{G^{\alpha}(n)}(h(n)\Delta z(n))^{\alpha} \leq 0. \tag{2.19}$$

In view of (2.19), (2.18) becomes

$$\begin{aligned}\Delta w(n) \quad \leq \quad & \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ & -\rho(n)f(n)(1-e(n+1))^{\alpha}\frac{H^{\alpha}(n-k)}{G^{\alpha}(n)} \\ & -\frac{\rho(n)}{\rho(n+1)}w(n+1)\frac{\Delta(h(n)\Delta z(n))^{\alpha}}{(h(n)\Delta z(n))^{\alpha}}, \ n \geq N. \end{aligned} \tag{2.20}$$

By discrete Mean-Value theorem

$$\begin{aligned}\Delta(h(n)\Delta z(n))^{\alpha} \quad = \quad & \alpha\frac{t^{\alpha}}{t}\Delta(h(n)\Delta z(n)), \\ & h(n)\Delta z(n) < t < h(n+1)\Delta z(n+1)\end{aligned}$$

and hence

$$\Delta(h(n)\Delta z(n)) \geq \alpha\frac{(h(n)\Delta z(n))^{\alpha}}{h(n+1)\Delta z(n+1)}\Delta(h(n)\Delta z(n)), \quad n \geq N.$$

Using the last inequality in (2.20), and this in view of (2.17) we obtain

$$\begin{aligned}\Delta w(n) \quad \leq \quad & \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ & -\rho(n)f(n)(1-e(n+1))^{\alpha}\frac{H^{\alpha}(n-k)}{G^{\alpha}(n)} \\ & -\alpha\frac{\rho(n)}{\rho^{1+\frac{1}{\alpha}}(n+1)g^{\frac{1}{\alpha}}(n)}w^{1+\frac{1}{\alpha}}(n+1), \ n \geq N, \end{aligned} \tag{2.21}$$

where we have used $g^{\frac{1}{\alpha}}(n)\Delta(h(n)\Delta z(n))$ is positive and decreasing. Now we obtain

$$Cu-Du^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^{\alpha}}, \ D>0$$

in (2.21), with $C=\frac{\Delta\rho(n)}{\rho(n+1)}$ and $D=\alpha\frac{\rho(n)}{\rho^{1+\frac{1}{\alpha}}(n+1)g^{\frac{1}{\alpha}}(n)}$, we obtain

$$\begin{aligned}\Delta w(n) \quad \leq \quad & -\rho(n)f(n)(1-e(n+1))^{\alpha}\frac{H^{\alpha}(n-k)}{G^{\alpha}(n)} \\ & +\frac{g(n)(\Delta\rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(n)}, \ n \geq N.\end{aligned}$$

Summing the last inequality from N to n, one gets

$$\sum_{s=N}^{n}\Big[\rho(s)f(s)(1-e(s+1))^{\alpha}\frac{H^{\alpha}(s-k)}{G^{\alpha}(s)} - \frac{g(s)(\Delta\rho(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\Big] < w(N) < \infty$$

which $\Longrightarrow\Longleftarrow$ (2.15). This completes the proof. $\square$

3. Conclusion

This article, we obtain the new oscillation criteria for (E) using delay argument in the neutral term, when $0 \leq e(n) \leq p < 1$ and $\alpha \in (0, \infty)$. The obtain results ensure that all solutions are oscillatory. Compare to other results is improve in the sense that the existing results for the case $0 \leq e(n) \leq p < 1$ provided the solutions are either oscillatory or tends to zero.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.

[2] R.P. Agarwal, M. Bohner, S.R. Grace and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.

[3] S.R. Grace, R.P. Agarwal and J.R. Graef, Oscillation criteria for certain third-order nonlinear difference equation, *Appl. Anal. Discrete Math.,* 3(2009), 27–38.

[4] S.R. Grace, Oscillatory behavior of third-order nonlinear difference equations with a nonlinear nonpositive neutral term, *Mediterr. J. Math.,* (2019) 16:128.

[5] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, *Clarendan Press, Oxford,* 1991.

[6] S. Jaikumar, K.Alagesan and G.Ayyappan, Oscillation of nonlinear third-order delay difference equations with unbounded neutral coefficients, *J. Inf. Comput. Sci.,* 9(2019), 902-910.

[7] G.Ladas and C.Qian, Comparison results and linearized oscillations for higher order difference equations, *Int. J. Math. Math. Sci.,* 15(1992), 129-142.

[8] S.H. Sakar, Oscillation and asymptotic behavior of third-order nonlinear neutral delay difference equations, *Dynam. Systems. Appl.,* 15(2006), 549-568.

[9] S. Selvarangam, M. Madhan, E. Thandapani and S. Pinelas, Improved oscillation theorems for third-order neutral type difference equations, *Elec. J. Differential Equ.,* 2017(2017), No.90, pp. 1–13.

[10] E. Thandapani, S. Pandian and R.K. Balasubramaniam, Oscillatory behavior of solutions of third-order quasilinear delay difference equations, *Stud. Univ. Zilina, Math. Ser.,* 19(2005), 65–78.

[11] E.Thandapani, M. Vijaya and T. Li, On the oscillation of third order half-linear neutral type difference equations, *Electron. J. Qual. Theory Differ. Equ.,* 76(2011), 1–13.

[12] E.Thandapani and S.Selvarangam, Oscillation of second order Emden-Fowler type neutral difference equations, *Dynam. Cont. Disc. Impul. Sys. Ser. A: Math. Anal.,* 19(2012), 453–469.

[13] E. Thandapani and S. Selvarangam, Oscillation of third-order half-linear neutral difference equations, *Math. Bohemica,* 138(2013), 87–104.

[14] D.M. Wang and Z.T. Xu, Oscillation of second-order quasilinear neutral delay difference equations, *Acta Math. Appl. Sinica,* 27(2011), 93–104.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆