\(\mu\)-Pre*-closed sets in generalized topological spaces

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Abstract
In this paper, we extend the notion of closed sets in ordinary topological spaces to \(\mu\)-pre*-closed sets in generalized topological spaces. Relationship between this closed set and other closed sets are established. Also we have investigated \(\mu\)-pre*-open sets, \(\mu\)-pre*-closure, \(\mu\)-pre*-interior and some of their basic properties.

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\(\mu\)-pre*-closed sets, \(\mu\)-pre*-open sets, \(\mu\)-pre*-closure, \(\mu\)-pre*-interior.

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1. Introduction
In topological spaces, generalized topology is an important generalization. According to the definition of generalized topological spaces, \(\emptyset\) may not be come in closed set. In the same way, \(X\) may not be come in the open set of generalized topological spaces. In general topology, frequent applications of interior and closure operators give rise to several different new classes of sets.

In 2002, A. Csaszar\([3,4]\) was initiated the concept and invented many \(\mu\)-closed sets like \(\mu\)-pre closed set, \(\mu\)-closed sets, \(\mu\)-\(\alpha\)-closed sets, \(\mu\)-\(\beta\)-closed sets etc., in generalized topological spaces. He also described some basic operators on a generalized topology. Thereafter in 2007, P. Sivagami\([10]\) developed generalized topological spaces and revealed many more results of it. Then a huge number of papers are devoted to the study of generalized topological spaces like topological spaces. G. Hari siva Annam\([6]\) progressed kuratowski operator in general topology which is motivated to us analysis on \(\mu\)-pre*-closed sets in generalized topological spaces. In this paper, we announce the notion of \(\mu\)-pre*-closed set in generalized topological spaces, and also come out with some of their properties.

2. Preliminaries:
Definition 2.1. [3] Let \(X\) be a non-empty set and \(\mu\) be a collection of subsets of \(X\). Then the pair \((X, \mu)\) is called as a generalized topological space (short on GTS) on \(X\) if \(\mu\) has the following properties

(i) \(\emptyset \in \mu\)

(ii) Any union of elements of \(\mu\) belongs to \(\mu\).

The elements of \(\mu\) are called \(\mu\)-open sets and the complement of \(\mu\)-open sets are called \(\mu\)-closed sets. The family of all \(\mu\)-closed sets in \(X\) is denoted by \(\mu(X)\).

In GTS \((X, \mu)\), \(M_\mu\) is defined as \(M_\mu = \bigcup_{i \in I} U_i\).

If \(M_\mu = X\) then \((X, \mu)\) is called as a strongly generalized topological space.

Definition 2.2. [3] For \(A \subseteq X\), the generalized closure \(c_\mu(A)\) is the intersection of all \(\mu\)-closed sets containing \(A\), which is the smallest \(\mu\)-closed set containing \(A\) \((A \subseteq c_\mu(A))\) and the generalized interior \(i_\mu(A)\), the union of all \(\mu\)-open sets contained in \(A\), that is the largest \(\mu\)-open set contained in \(A\) \((A \supseteq i_\mu(A))\).

Note that \(c_\mu(A) = X - i_\mu(X - A)\) and \(i_\mu(A) = X - c_\mu(X - A)\).
Definition 2.3 (9). Let \((X, \mu)\) be a GTS. Then the subset \(A\) of \(X\) is called a \(\mu\)-generalized closed set (in short, \(\mu-g\)-closed set) iff \(c_\mu(A) \subseteq U\) whenever \(A \subseteq U\) where \(U\) is \(\mu\)-open set in \(X\). The complement of a \(\mu-g\)-closed set is called a \(\mu-g\)-open set.

Result 2.4. [4] Every \(\mu\)-closed set is \(\mu-g\)-closed.

Definition 2.5. Let \(A\) be a subset of \(X\), then the \(\mu-g\) closure of \(A\) is denoted by \(c_\mu^*(A)\) and it is defined by the intersection of all \(\mu-g\)-closed set containing \(A\).

Result 2.6. Let \(A\) be a subset of \(X\), then \(c_\mu^*(A) \subseteq c\mu(A)\)

Proof. Suppose \(x \notin c_\mu(A)\). Then \(x \notin F\), for some \(\mu-g\)-closed set containing \(A\). By Result 2.4, \(x \notin F\), for some \(\mu-g\)-closed set containing \(A\). Therefore \(x \notin c_\mu^*(A)\).

Result 2.7. If \(A \subseteq B\), then \(c_\mu^*(A) \subseteq c_\mu^*(B)\), where \(A\) and \(B\) are subsets of \(X\).

Proof. Let \(x \notin c_\mu^*(B)\). By definition, \(x \notin F\), for some \(\mu-g\)-closed set containing \(B\). Since \(A \subseteq B\), \(x \notin F\), for some \(\mu-g\)-closed set containing \(A\). Hence \(x \notin c_\mu^*(A)\).

Result 2.8. If \(A\) and \(B\) are subsets of \(X\), then \(c_\mu^*(A \cap B) \subseteq c_\mu^*(A) \cap c_\mu^*(B)\).

Proof. We know that, \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\). By Result 2.7, \(c_\mu^*(A \cap B) \subseteq c_\mu^*(A) \cap c_\mu^*(B)\). Hence \(c_\mu^*(A \cap B) \subseteq c_\mu^*(A) \cap c_\mu^*(B)\).

Result 2.9. Let \(A\) be a subset of \(X\) then \(A\) is \(\mu-g\)-closed if and only if \(c_\mu^*(A) = A\).

Proof. Suppose \(A\) is a \(\mu-g\)-closed set of \(X\). By definition of \(c_\mu^*(A)\), \(c_\mu^*(A) = \cap\{F/A \subseteq F\text{ and }F\text{ is }\mu-g\text{-closed}\}\). Since \(A\) is \(\mu-g\)-closed that implies \(c_\mu^*(A) = A\). Conversely, suppose \(c_\mu^*(A) = A\). By definition of \(c_\mu^*(A)\), \(c_\mu^*(A)\) is \(\mu-g\)-closed. Clearly, \(A\) is a \(\mu-g\)-closed set.

Definition 2.10. Let \((X, \mu)\) be a generalized topological space. Then \(A \subseteq X\) is said to be

(i) \(\mu\)-semi closed [5] if \(c_\mu^*(i_\mu(A)) \subseteq A\)

(ii) \(\mu\)-pre closed [3] if \(c_\mu (i_\mu(A)) \subseteq A\)

(iii) \(\mu-\alpha\) closed [4] if \(c_\mu (i_{(i_\mu(A)))} \subseteq A\)

(iv) \(\mu-\beta\) closed [4] if \(i_\mu (c_\mu (i_\mu(A))) \subseteq A\)

(v) \(\mu\)-regular closed [8] if \(A = c_\mu (i_\mu(A))\)

(vi) \(\mu-\pi\) closed if \(A = \mu\text{-union of finitely many }\mu\text{-regular open sets.}\)

The complement of \(\mu\)-semi closed (resp., \(\mu\)-pre closed, \(\mu\)-\(\alpha\) closed, \(\mu\)-\(\beta\) closed, \(\mu\)-regular closed, \(\mu\)-\(\pi\) closed) is said to be \(\mu\)-semi open (resp., \(\mu\)-pre open, \(\mu\)-\(\alpha\) open, \(\mu\)-\(\beta\) open, \(\mu\)-regular open, \(\mu\)-\(\pi\) open) in \(X\).

3. \(\mu\)-pre-closed set:

Definition 3.1. A subset \(A\) of a generalized topological space \((X, \mu)\) is called a \(\mu\)-pre-closed set if \(c_\mu^*(i_\mu(A)) \subseteq A\).

The collection of all \(\mu\)-pre-closed sets in \(X\) is denoted by \(P^*c_\mu(X)\) or \(pre^*c_\mu(X)\).

Theorem 3.2. Arbitrary intersection of \(\mu\)-pre-closed sets is \(\mu\)-pre-closed.

Proof. Let \(\{ F_\alpha \}\) be the collection of \(\mu\)-pre-closed sets. Then \(c_\mu^*(i_\mu(F_\alpha)) \subseteq F_\alpha\) for each \(\alpha\). Now, \(c_\mu^*(i_\mu(\cap F_\alpha)) \subseteq c_\mu^*(\cap i_\mu(F_\alpha)) \subseteq \cap c_\mu(i_\mu(F_\alpha)) \subseteq \cap F_\alpha\). Therefore \(\cap F_\alpha\) is \(\mu\)-pre-closed.

Remark 3.3. Union of two \(\mu\)-pre-closed sets need not be \(\mu\)-pre-closed set as shown in the following example. And hence finite union of \(\mu\)-pre-closed sets need not be \(\mu\)-pre-closed set.

Example 3.4. Consider the GTS \((X, \mu)\), where \(X = \{ a, b, c, d \}\) and \(\mu = \{ \emptyset, \{ b \}, \{ a, b \}, \{ a, d \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \}, X \}\). Then \(\{ a, c \}\) and \(\{ b, c \}\) are \(\mu\)-pre-closed but \(\{ a, c \} \cup \{ b, c \} = \{ a, b, c \}\) is not \(\mu\)-pre-closed.

Theorem 3.5. Every \(\mu\)-pre-closed set is \(\mu\)-pre-closed.

Proof. Suppose \(A\) is a \(\mu\)-pre-closed set. That implies \(c_\mu^*(i_\mu(A)) \subseteq A\). Now, \(c_\mu^*(i_\mu(A)) \subseteq c_\mu^*(i_\mu(A)) \subseteq A\). Therefore \(\cap A\) is \(\mu\)-pre-closed.

Theorem 3.6. Every \(\mu\)-g-closed set is \(\mu\)-pre-closed.

Proof. By Result 2.9, \(A\) is \(\mu-g\)-closed then \(c_\mu^*(A) = A\). Now, \(c_\mu^*(i_\mu(A)) \subseteq c_\mu^*(A) = A\). Therefore \(A\) is \(\mu\)-pre-closed. But the converse is not true.

Example 3.7. Let \(X = \{ a, b, c, d \}\) and \(\mu = \{ \emptyset, \{ b \}, \{ a, b \}, \{ a, d \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \}, X \}\). Then \(\{ a, b \}\) and \(\{ c, d \}\) are \(\mu\)-pre-closed but not \(\mu\)-g-closed.

Theorem 3.8. Every \(\mu\)-g-closed set is \(\mu\)-pre-closed.

Proof. Let \(A\) be a \(\mu\)-closed set. By Result 2.4, \(A\) is a \(\mu\)-g-closed set. By theorem 3.6, the result is clearly true. But the converse is not true.

Example 3.9. Consider the GTS \((X, \mu)\), where \(X = \{ a, b, c, d \}\) and \(\mu = \{ \emptyset, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, X \}\). The sets \(\{ a, b \}, \{ a, c, d \}\) and \(\{ b, c, d \}\) are \(\mu\)-pre-closed but not \(\mu\)-g-closed.

Theorem 3.10. Every \(\mu\)-\(\alpha\)-closed set is \(\mu\)-pre-closed.

Proof. Suppose \(A\) is a \(\mu\)-\(\alpha\)-closed set which implies \(c_\mu^*(i_\mu(A)) \subseteq A\). Now, \(c_\mu^*(i_\mu(A)) \subseteq c_\mu^*(i_\mu(A)) \subseteq A\). Therefore we have \(A\) is \(\mu\)-pre-closed. But the converse is not true.
Example 3.11. Let $X=\{a,b,c,d\}$ and $\mu = \{\phi, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. Then $\{a,b,d\}, \{a,c,d\}$ are $\mu - \text{pre}^*\mu$-closed but not $\mu - \alpha$-closed.

Theorem 3.12. Every $\mu$-regular closed set is $\mu - \text{pre}^*\mu$-closed.

Proof. Let A be a $\mu$-regular closed set then $\mu \cap (i \mu (A)) = A$. Since $\mu \cap (i \mu (A)) \subseteq \mu \cap (i \mu (A)) = A$. Clearly, the result is true.

Remark 3.13. The converse of the above theorem is not true may be seen by the following example.

Example 3.14. Let $X=\{a,b,c,d\}$ and $\mu = \{\phi, \{a,c\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$. Here $\{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}$ and $\{a,c,d\}$ are $\mu - \text{pre}^*\mu$-closed but not $\mu$-regular closed.

Theorem 3.15. Every $\mu - \pi$-closed set is $\mu - \text{pre}^*\mu$-closed.

Proof. Suppose A is $\mu - \pi$-closed, then $\mu \cap (i \mu (A)) = A$. By theorem 3.12, $\mu \cap (i \mu (A)) = A$. Since $\mu \cap (i \mu (A)) \subseteq \mu \cap (i \mu (A)) = A$. Hence A is $\mu - \text{pre}^*\mu$-closed. But the converse is not true as shown in the ensuing example.

Example 3.16. Consider the GTS $(X, \mu)$, where $X=\{a,b,c,d\}$ and $\mu = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$. The sets $\{a,c\}, \{c,d\}$ and $\{b,c,d\}$ are $\mu - \text{pre}^*\mu$-closed but not a $\mu - \pi$-closed set.

4. $\mu - \text{pre}^*\mu$-closure:

Definition 4.1. Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then the $\mu - \text{pre}^*\mu$-closure of A, denoted by $\text{pre}^*\mu(A)$ or $\mu^*\mu(A)$, is the intersection of all $\mu - \text{pre}^*\mu$-closed sets containing A ($A \subseteq \mu^*\mu(A)$).

i) $\mu^*\mu(A) = \cap \{F / F \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\}

$\mu^*\mu(A)$ is the smallest $\mu - \text{pre}^*\mu$-closed set containing A.

Theorem 4.2. Let A be a subset of X then A is $\mu - \text{pre}^*\mu$-closed if $\mu^*\mu(A) = A$.

Proof. Suppose A is $\mu - \text{pre}^*\mu$-closed. By definition, $\mu^*\mu(A) = \cap \{F / F \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\} then $\mu^*\mu(A) = A$. Conversely, suppose $\mu^*\mu(A) = A$. By definition $\mu^*\mu(A) = \cap \{F / F \subseteq F$, $F$ is $\mu - \text{pre}^*\mu$-closed\} and By theorem 3.2, $\mu^*\mu(A)$ is $\mu - \text{pre}^*\mu$-closed. Clearly, A is $\mu - \text{pre}^*\mu$-closed.

Theorem 4.3. If A and B are subsets of $(X, \mu)$ then the following results hold

i) $\mu^*\mu(X) = X$

ii) $A \subseteq \mu^*\mu(A)$

iii) If $A \subseteq B$ then $\mu^*\mu(A) \subseteq \mu^*\mu(B)$

iv) $A \subseteq \mu^*\mu(A) \subseteq \mu^*\mu(A) \subseteq \mu^*\mu(A)$

v) $\mu^*\mu(\mu^*\mu(A)) = \mu^*\mu(A)$

Proof. (i) follows from definition. (ii) Suppose $x \notin \mu^*\mu(A)$ then $x \notin \cap \{F : A \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\}. That implies $x \notin F$, for some $\mu - \text{pre}^*\mu$-closed set containing A. Hence $x \notin A$. Therefore $A \subseteq \mu^*\mu(A)$. (iii) Let $A \subseteq B$, suppose $x \notin \mu^*\mu(B)$ then $x \notin \cap F$ such that $F$ is the $\mu - \text{pre}^*\mu$-closed set containing B. $x \notin F$, for some $\mu - \text{pre}^*\mu$-closed set containing B. Since $A \subseteq B$ Therefore $x \notin F$, for some $\mu - \text{pre}^*\mu$-closed set containing A. Hence $x \notin \mu^*\mu(A)$. Thus, the result is true.

(iv) Assume $x \notin \mu^*\mu(A)$, implies $x \notin \cap \{F : A \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\}. By definition, $x \notin \cap \{F : A \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\}. Then by theorem 3.5, $x \notin \cap \{F : A \subseteq F$ and $F$ is $\mu - \text{pre}^*\mu$-closed\} Therefore $x \notin \cap F$. Where F is in A. Hence, it is proved. (v) In theorem 4.2, replace A by $\mu^*\mu(A)$, then clearly get the result.

Remark 4.4. In GTS$(X, \mu)$, $\mu^*\mu(\phi) \neq \phi$ if $\mu$ does not having X.

Consider the GTS $(X, \mu)$, where $X=\{1, 2, 3, 4, 5\}$ and $\mu = \{\phi, \{2\}, \{1, 3\}, \{3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}\}$. In this example, $\mu^*\mu(\phi) = \{1\}$. In general, $\mu^*\mu(\phi) \neq \phi$ if X is not a member of $\mu$.

Theorem 4.5. If A and B are subsets of $(X, \mu)$ then $\mu^*\mu(A \cup B) = \mu^*\mu(A) \cup \mu^*\mu(B)$.

Proof. Let $x \in \mu^*\mu(A \cup B)$ then $x \in \cap F$ such that $F$ is $\mu - \text{pre}^*\mu$-closed containing A $\cup$ B. Thus $x \in \cap F$ such that $F$ is a $\mu - \text{pre}^*\mu$-closed set containing A or B. Therefore $x \in \mu^*\mu(A) \cup \mu^*\mu(B)$.

Conversely, we know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By theorem 4.3, $\mu^*\mu(A) \subseteq \mu^*\mu(A \cup B)$ and $\mu^*\mu(B) \subseteq \mu^*\mu(A \cup B)$. Therefore, $\mu^*\mu(A) \cup \mu^*\mu(B) \subseteq \mu^*\mu(A \cup B)$.

Remark 4.6. In general, $\mu^*\mu(U A_i) = \cap \mu^*\mu(A_i)$.

Theorem 4.7. $\mu^*\mu(A \cap B) \subseteq \mu^*\mu(A) \cap \mu^*\mu(B)$ where A and B are in $(X, \mu)$.

Proof. As We know that, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By theorem 4.3, $\mu^*\mu(A \cap B) \subseteq \mu^*\mu(A)$ and $\mu^*\mu(A \cap B) \subseteq \mu^*\mu(B)$. Therefore, $\mu^*\mu(A \cap B) \subseteq \mu^*\mu(A) \cap \mu^*\mu(B)$.

Remark 4.8. In (ii), (iii) and (iv) of theorem 4.3, the inclusions may be strict and equality may also hold. This can be seen from the ensuing illustration.
Example 4.9. Let \( X = \{ a, b, c, d \} \) be endowed with \( \mu = \{ \phi, \{ b \}, \{ a, b \}, \{ a, b, d \} \} \).

For (ii), Let \( A = \{ a, c \} \) then \( \mu^*_{cA} = \{ a, c \} \) Therefore, \( A = \mu^*_{cA} \).

Example 4.10. Let \( X = \{ a, b, c, d \} \) be endowed with \( \mu = \{ \phi, \{ b \}, \{ a, b \}, \{ a, b, c, d \} \} \).

For (iv), Let \( A = \{ a, b \} \) then \( \mu^*_{cA} = \{ a, b, d \} \) and \( \mu^*_{cA} = \{ a, b, c, d \} \) Therefore \( A = \mu^*_{cA} \).

Let \( B = \{ b, c \} \) then \( \mu^*_{cB} = \{ b, c, d \} \) and \( \mu^*_{cB} = \{ b, c \} \) Therefore \( B = \mu^*_{cB} \).

Let \( C = \{ c \} \) then \( \mu^*_{cC} = \{ c \} \). Therefore, \( C = \mu^*_{cC} \).

The above results can be represented in the succeeding diagram.

\[ \begin{array}{ccc}
\mu - \text{closed} & \mu - \text{pre closed} & \mu - \text{g closed} \\
\mu - \text{pre}^* \text{ closed} & \mu - \text{regular closed} & \mu - \text{closed}
\end{array} \]

Where \( A \rightarrow B \) represents \( A \) implies \( B \) and \( A \rightarrow B \) represents \( A \) does not imply \( B \).

Remark 4.11. \( \mu - \beta \)-closed set, \( \mu - \delta \)-closed set, \( \mu - \theta \)-closed set and \( \mu - \alpha^* \)-closed set are independent with \( \mu - \text{pre}^* \)-closed.

5. \( \mu - \text{pre}^* \)-open set

In this sector, we bring up \( \mu - \text{pre}^* \)-open sets and analyze their basic properties.

Definition 5.1. Let \( (X, \mu) \) be a GTS. A subset \( A \) of \( X \) is a \( \mu - \text{pre}^* \)-set if \( X \setminus A \) is a \( \mu - \text{pre}^* \)-clos.

\( \mu^* \text{O}_\mu(X) \) or \( \text{P}_\mu(X) \) is the collection of all \( \mu - \text{pre}^* \)-clos sets in \( X \).

Theorem 5.2. Arbitrary union of \( \mu - \text{pre}^* \)-open set is \( \mu - \text{pre}^* \)-open.

Proof. Assume \( \{ U_{\alpha} \} \) is a collection of \( \mu - \text{pre}^* \)-open sets then \( \bigcup \{ U_{\alpha} \} \) is a collection of \( \mu - \text{pre}^* \)-closed sets. By theorem 3.2, \( \cap \{ X \setminus \{ U_{\alpha} \} \} \) is \( \mu - \text{pre}^* \)-closed. It follows that \( \cap \{ U_{\alpha} \} \) is \( \mu - \text{pre}^* \)-open.

Theorem 5.3. Intersection of two \( \mu - \text{pre}^* \)-open sets is not \( \mu - \text{pre}^* \)-open which is shown from the following example.

Example 5.4. Let \( X = \{ a, b, c, d \} \) and \( \mu = \{ \phi, \{ b \}, \{ a, c \}, \{ a, b, c, d \} \} \). Here \( \mu - \text{pre}^* \)-open but not \( \mu - \text{pre}^* \)-open.

Theorem 5.5. Suppose \( (X, \mu) \) is a GTS, then the following theorems are hold

(i) All \( \mu \)-open is \( \mu - \text{pre}^* \)-open.

(ii) All \( \mu - \text{g} \)-open is \( \mu - \text{pre}^* \)-open.

(iii) All \( \mu - \text{pre} \)-open is \( \mu - \text{pre}^* \)-open.

(iv) All \( \mu - \alpha \)-open is \( \mu - \text{pre}^* \)-open.

(v) All \( \mu - \text{regular} \)-open is \( \mu - \text{pre}^* \)-open.

(vi) All \( \mu - \pi \)-open is \( \mu - \text{pre}^* \)-open.

Proof. Clearly, the result is true by the complement concept of theorem 3.8, 3.6, 3.5, 3.10, 3.12 and 3.15. But the converse of all the above statements are not true which are shown form the following examples.

Example 5.6. Consider the GTS \( (X, \mu) \), \( X = \{ a, b, c, d \} \) and \( \mu = \{ \phi, \{ b \}, \{ a, c \}, \{ a, d \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \} \} \). Here \( \mu - \text{pre}^* \)-open but not \( \mu - \text{pre} \)-open and \( \mu - \text{g} \)-open.

Example 5.7. Consider the GTS \( (X, \mu) \), \( X = \{ a, b, c, d \} \) and \( \mu = \{ \phi, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \} \} \). Here \( \mu - \text{pre}^* \)-open but not \( \mu - \text{pre} \)-open, \( \mu - \alpha \)-open, \( \mu - \text{regular} \)-open and \( \mu - \pi \)-open.

6. \( \mu - \text{pre}^* \)-Interior

Definition 6.1. Let \( A \) be a subset of \( (X, \mu) \), then the interior of \( \mu - \text{pre}^* \)-open set is defined by \( \bigcup \{ U \mid U \subseteq A \text{ and } U \in \mu^* \text{O}_\mu(X) \} \) and denoted as \( \mu^*_{\text{int}}(A) \) or \( \mu^*_{\text{int}}(A) \). \( \mu^*_{\text{int}}(A) \) is the largest \( \mu - \text{pre}^* \)-open set contained in \( A \).
Theorem 6.2. If $(X, \mu)$ is a GTS, then the following results hold:

1. $\text{pre}^* i_\mu(\emptyset) = \emptyset$
2. $\text{pre}^* i_\mu(A) \subseteq A$
3. If $A \subseteq B$ then $\text{pre}^* i_\mu(A) \subseteq \text{pre}^* i_\mu(B)$
4. $\text{pre}^* i_\mu(\text{pre}^* i_\mu(A)) = \text{pre}^* i_\mu(A)$

Proof. Obviously, the result is true by compliment concept of theorem 4.3.

Remark 6.3. In GTS$(X, \mu)$, $\text{pre}^* i_\mu(X) \neq X$ if $\mu$ does not having $X$.

Consider the example, Let $(X, \mu)$ be GTS, where $X = \{1, 2, 3, 4, 5\}$ and $\mu = \{\emptyset, \{2\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}\}$. In this example, $\text{pre}^* i_\mu(X) = \{2, 3, 4, 5\}$ In general, $\text{pre}^* i_\mu(X) \neq X$, $X$ is not a member of $\mu$.

Theorem 6.4. Let $A$ be a subset of $X$ then $A$ is $\mu-\text{pre}^*$-open if and only if $\text{pre}^* i_\mu(A) = A$.

Proof. If $A$ is $\mu-\text{pre}^*$-open, by definition $\text{pre}^* i_\mu(A) = A(95,53)

Remark 6.6. The inclusion of theorem 6.5 may be strict and equality may also hold. This can be seen from the ensuing illustration.

Example 6.7. Consider the GTS$(X, \mu)$, where $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{b\}, \{a, b\}, \{a, b, d\}\}$.

Let $A = \{a\}$ and $B = \{b, d\}$ then $A \cup B = \{a, b, d\}$. Also $\text{pre}^* i_\mu(A) = \{a\}$, $\text{pre}^* i_\mu(B) = \{b, d\}$, and $\text{pre}^* i_\mu(A \cup B) = \{a, b, d\}$. Here $\text{pre}^* i_\mu(A) \cup \text{pre}^* i_\mu(B) = \{a, b, d\}$, hence $\text{pre}^* i_\mu(A) \cup \text{pre}^* i_\mu(B) = \text{pre}^* i_\mu(A \cup B)$.

Let $C = \{b\}$ and $D = \{c, d\}$ then $C \cup D = \{b, c, d\}$. Also $\text{pre}^* i_\mu(C) = \emptyset$, $\text{pre}^* i_\mu(D) = \{b\}$ and $\text{pre}^* i_\mu(C \cup D) = \{b, d\}$. Here $\text{pre}^* i_\mu(C) \cup \text{pre}^* i_\mu(D) = \{b\}$, hence $\text{pre}^* i_\mu(C) \cup \text{pre}^* i_\mu(D) \subseteq \text{pre}^* i_\mu(C \cup D)$.

Theorem 6.8. Let $x \in X$, then $x \in \text{pre}^* c_\mu(A)$ if and only if $V \cap A \neq \emptyset$ for every $\mu-\text{pre}^*$-open set containing $x$.

Proof. Let $x \in \text{pre}^* c_\mu(A)$. Suppose $V \cap A = \emptyset$ for every $\mu-\text{pre}^*$-open set $V$ containing $x$. Clearly $A \subseteq X \setminus V$ and $X \setminus V$ is $\mu-\text{pre}^*$-closed. Since $\text{pre}^* c_\mu(A)$ is the smallest $\mu-\text{pre}^*$ closed set containing $A$. Therefore $\text{pre}^* c_\mu(A) \subseteq X \setminus V$. Hence $x \in X \setminus V$, which is contradiction to $x \in V$.

Suppose $x \notin \text{pre}^* c_\mu(A)$, then there exists $\mu-\text{pre}^*$ closed set $F$ such that $A \subseteq F$ and $x \notin F$. Obviously, $x \notin A$. If $x \in X \setminus F$ and $X \setminus F$ is $\mu-\text{pre}^*$ open. It follows that $A \cap X \setminus F = \emptyset$, which is contradiction to our assumption. Therefore, $x \in \text{pre}^* c_\mu(A)$.

Theorem 6.9. If $A$ is a subset of $(X, \mu)$. Prove that:

1. $\text{pre}^* i_\mu(X \setminus A) = X \setminus \text{pre}^* c_\mu(A)$
2. $\text{pre}^* c_\mu(X \setminus A) = X \setminus \text{pre}^* i_\mu(A)$

Proof. (i) Let $x \in \text{pre}^* i_\mu(X \setminus A)$, then there exists $\mu-\text{pre}^*$ open set $V$ such that $x \in V \subseteq X \setminus A$. Therefore, $x \in V$ and $V \cap A = \emptyset$. By theorem 6.6, $x \notin \text{pre}^* c_\mu(A)$. Hence $x \in X \setminus \text{pre}^* c_\mu(A)$. Conversely, Suppose $x \in X \setminus \text{pre}^* c_\mu(A)$, that implies $x \notin \text{pre}^* c_\mu(A)$. By theorem 6.6, there exists $\mu-\text{pre}^*$ open set $V$ such that $V \cap A = \emptyset$. This shows that $V \subseteq X \setminus A$. Hence $x \in \text{pre}^* i_\mu(X \setminus A)$.

(ii) It follows from (i).

7. Conclusion

The goal of the paper is to find basic properties of $\mu$–$\text{pre}^*$-closed sets in generalized topological spaces. Further, I raise my research towards some special areas in generalized topological spaces.

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