Digraph homomorphisms on $Z_n$-digraph

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Abstract
A graph homomorphism is a mapping between two graphs that respect their structure. In this paper we develop some results related to digraph homomorphisms for the class of $Z_n$-digraphs. We will begin by giving some standard definitions, then expanding our focus to specifically study different types of digraph homomorphisms. In particular we will be discussing the idea of automorphism, isomorphism, weak homomorphism and its relation with the corresponding group homomorphisms defined on $Z_n$.

Keywords
Group homomorphisms, Automorphism, Digraph homomorphisms, Weak homomorphism.

AMS Subject Classification
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1. Introduction

We are considering finite directed graph throughout this paper unless otherwise specified. We are assuming the basics of Graph Theory and Group Theory and use the notations of Harary [1] for digraphs and John B Fraleigh [2] for groups. For positive integers $n, Z_n$ [2] be the cyclic group of order $n$ under addition modulo $n$. The Euler phi – function is defined for positive integers $n$ by $\phi(n) = s$ where $s$ is the number of positive integers less than or equal to $n$ that are relatively prime to $n$. The number of generators of $Z_n$ is $\phi(n)$. If $G$ is a group, then the order of $G$ is the number of elements in $G$. An element $a$ of a group $G$ is a generator for $G$ if the cyclic subgroup of $G$ generated by $a$ is $G$ itself.

A homomorphism [2] $f$ from a group $G$ to a group $G'$ is a mapping from $G$ to $G'$ that preserves the group operation; that is, $f(ab) = f(a)f(b)$ for all $a,b$ in $G$. An isomorphism $f$ from a group $G$ to a group $G'$ is a one-one mapping from $G$ onto $G'$ that preserves the group operation; that is, $f(ab) = f(a)f(b)$ for all $a,b$ in $G$. An isomorphism from a group $G$ onto itself is called an Automorphism of $G$. It is also proved that for every positive integer $n$, $Aut(G)$ is isomorphic to $U(n)$, where $Aut(G)$ is the set of all automorphisms of $Z_n$, a group under the operation function composition and $U(n)$ is the set of all positive integers less than $n$ and relatively prime to $n$, a group under multiplication modulo $n$. So $|Aut(Z_n)| = \phi(n)$.

A digraph [1] $D$ consists of a finite set $V$ of points and a collection of ordered pair of distinct points. Any such pair $(u,v)$ is called an arc or directed line and usually be denoted by $uv$. The outdegree $od(v)$ of a point $v$ is the number of points adjacent from it, and the indegree $id(v)$ is the number of points adjacent to it.

Given digraphs $G$ and $G'$, a homomorphism [8] $f: A \rightarrow B$ is a map $f: V(G) \rightarrow V(G')$ for which $ad' \in A(G') \implies f(a)f(a') \in A(G')$. An isomorphism between two digraphs $G$ and $G'$ is a bijective mapping $f: V(G) \rightarrow V(G')$ with the property $ad' \in A(G)$ if and only if $f(a)f(a') \in A(G')$. A weak homomorphism $f: A \rightarrow B$ is map $f: V(G) \rightarrow V(G')$ for which $ad' \in A(G) \implies f(a)f(a') \in A(G')$ or $f(a) = f(a')$.

2. The Main Results

**Definition 2.1.** Let $Z_n$ be the cyclic group of order $n$. The corresponding digraph $\overrightarrow{Z_n}$-digraph [3] is a simple digraph with vertex set equal to $Z_n$ and two vertices $u,v \in \overrightarrow{Z_n}, u \neq v$ are joined by a directed edge or arc from $u$ to $v$ if and only if there exists $r, 0 \leq r \leq n-1$ such that $v \equiv ru (\text{mod } n)$. That
For example the digraph induced by the vertices
where

Theorem 2.2. [10] The function \( f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \) given by \( f(x) = ax \) for some \( a \in \mathbb{Z}_n \) fixed is a homomorphism if and only if \( na \equiv 0 \pmod{m} \). It is also proved that the number of group
homomorphism \( f : \mathbb{Z}_n \rightarrow \mathbb{Z}_m \), \( f(x) = ax \) is \( d = \gcd(m,n) \), where

\[
a = \frac{m}{d} k, \quad k = 0, 1, \ldots, d - 1.
\]

Definition 2.3. \( \mathbb{Z}_n \)-subdigraph \( \mathbb{H} \) is a subdigraph of \( \mathbb{Z}_n \)-digraph induced by the subgroup \( H \) of \( \mathbb{Z}_m \).
For example the digraph induced by the vertices \( \{0, 2, 4\} \) is a \( \mathbb{Z}_6 \)-subdigraph, but the digraph induced by the vertices \( \{0, 2, 3\} \) is not a \( \mathbb{Z}_6 \)-subdigraph.

Theorem 2.4. There exists a one-one correspondence between \( \text{Aut}(\mathbb{Z}_n) \) and \( \text{Aut}(\mathbb{Z}_n^+) \), for all \( n \).

Proof. Suppose
\[
\text{Aut}(\mathbb{Z}_n) = \{f_1, f_2, \ldots, f_{\phi(n)}\}
\]

Then
\[
\begin{align*}
f_1(1) &= a_1 \\
f_2(1) &= a_2 \\
f_3(1) &= a_3 \\
&\vdots \\
f_{\phi(n)}(1) &= a_{\phi(n)}
\end{align*}
\]

where \( a_1, a_2, \ldots, a_{\phi(n)} \) are the generators of \( \mathbb{Z}_n \). Also we have
\[
f_i(x) = xa_i \equiv 0 \pmod{n}, \quad \forall x \in \mathbb{Z}_n
\]

Define a mapping \( F_i : V(\mathbb{Z}_n) \rightarrow V(\mathbb{Z}_n^+) \) as,
\[
F_i(x) = xa_i \equiv 0 \pmod{n}, \quad \forall x \in V(\mathbb{Z}_n)
\]

Since \( f_i \) is an automorphism of \( \mathbb{Z}_n \), \( F_i \) is well defined, one-one and onto. Now our claim is to check the digraph homomorphism condition. Let \( uv \in A(\mathbb{Z}_n) \). Then

\[
v \equiv ru \pmod{n}, \quad 0 \leq r \leq n - 1
\]

and
\[
a_i \equiv a_i \pmod{n}
\]

Hence,
\[
v a_i \equiv ru a_i \pmod{n}, \quad 0 \leq r \leq n - 1
\]

\[
\implies F_i(v) \equiv rF_i(u) \pmod{n}, \quad 0 \leq r \leq n - 1
\]

By the definition of \( \mathbb{Z}_n \)-digraph, there exists an arc from \( F_i(u) \) to \( F_i(v) \) in \( \mathbb{Z}_n \)-digraph.
That is,
\[
F_i(u)F_i(v) \in A(V(\mathbb{Z}_n))
\]

Thus \( F_i \) is a digraph automorphism on \( \mathbb{Z}_n \)-digraph. \( \Box \)

Theorem 2.5. Let \( \mathbb{H} \) be a subdigraph of \( \mathbb{Z}_n \)-digraph having order \( m \). Then there exists a digraph isomorphism between \( \mathbb{H} \) and \( \mathbb{Z}_m \).

Proof. Let \( a \) be a generator of \( H \), then \( H = \{ka/k = 0, 1, \ldots, m-1\} \).

The mapping \( f : H \rightarrow Z_m \) defined by \( f(ka) = k, \quad k = 0, 1, 2, \ldots, m-1 \) is an isomorphism from \( H \) to \( Z_m \).

Now define \( F : V(\mathbb{H}) \rightarrow V(\mathbb{Z}_m) \) as \( F(ka) = k, \forall k \). So \( F \) is well defined and both one to one and onto.

Let \( u = k_1a \) and \( v = k_2a \) are any two elements of \( H \). Now,
\[
v \equiv ru \pmod{m}, \quad 1 \leq r \leq m - 1
\]
\[
\iff k_2a \equiv k_1a \pmod{m}, \quad 1 \leq r \leq m - 1
\]
\[
\iff k_2 \equiv k_1 \pmod{m}, \quad 1 \leq r \leq m - 1; \quad \text{since} \gcd(a,m) = 1
\]
\[
\iff F(v) \equiv rF(u) \pmod{m}, \quad 1 \leq r \leq m - 1
\]

That is \( uv \in A(\mathbb{H}) \) if and only if \( F(u)F(v) \in A(\mathbb{Z}_m) \). Hence \( F \) is a digraph isomorphism. \( \Box \)

Example 2.6. Consider \( \mathbb{Z}_{12} \) and \( H = \langle 3 \rangle = \{0, 3, 6, 9\} \). Hence the order of \( H \) is 4 and \( \mathbb{H} \) is isomorphic to \( \mathbb{Z}_4 \).
For, define \( F : V(\mathbb{H}) \rightarrow V(\mathbb{Z}_4) \) as follows,
\[
F(0) = F(0 \times 3) = 0
\]
\[
F(3) = F(1 \times 3) = 1
\]
\[
F(6) = F(2 \times 3) = 2
\]
\[
F(9) = F(3 \times 3) = 3
\]

Figure 1. Digraph isomorphism
Proof. We have \( d = \text{g.c.d.}(m, n) \) group homomorphisms from \( Z_n \) to \( Z_m \). Let \( f : Z_n \rightarrow Z_m \) be the homomorphism defined by \( f(x) = ax \), where,
\[
a = \frac{m}{d}k, k = 0, 1, \ldots, d - 1.
\]
and
\[
na \equiv 0(\text{mod } m)
\]
Define \( F : \overrightarrow{Z_n} \rightarrow \overrightarrow{Z_m} \) as \( F(x) = ax, \forall x \in V(\overrightarrow{Z_n}) \). Let \( x \in \overrightarrow{Z_n} \). Then,
\[
y \equiv rx(\text{mod } n), 0 \leq r \leq n - 1
\]
\[
\Rightarrow y - rx \equiv 0(\text{mod } n), 0 \leq r \leq n - 1
\]
\[
\Rightarrow a(y - rx) \equiv 0(\text{mod } an), 0 \leq r \leq n - 1
\]
\[
\Rightarrow a(y - rx) \equiv 0(\text{mod } m), 0 \leq r \leq n - 1
\]
\[
ay \equiv rax(\text{mod } m), 0 \leq r \leq n - 1
\]
Let \( r \equiv r_1(\text{mod } m) \)
Then
\[
ay \equiv rax(\text{mod } m), 0 \leq r \leq n - 1
\]
\[
\Rightarrow ay \equiv r_1 ax(\text{mod } m), 0 \leq r_1 \leq m - 1
\]
Since \( F(x) \equiv ax(\text{mod } m) \) and \( F(y) \equiv ay(\text{mod } m) \), we get
\[
F(y) \equiv r_1 F(x)(\text{mod } m), 0 \leq r_1 \leq m - 1
\]
Then either there exists an arc from \( F(x) \) to \( F(y) \) or \( F(x) = F(y) \) in \( Z_m \).
For,
Let \( x, y \in Z_n \) such that \( x \equiv y(\text{mod } d) \)
Then,
\[
ax \equiv ay(\text{mod } ad)
\]
\[
\Rightarrow F(x) \equiv F(y)(\text{mod } ad)
\]
\[
\Rightarrow F(x) \equiv F(y)(\text{mod } m)
\]
Then \( F(x) = F(y) \) in \( Z_m \). Thus \( F : \overrightarrow{Z_n} \rightarrow \overrightarrow{Z_m} \) is a weak homomorphism.

Example 2.8. Consider the group homomorphisms from \( Z_{12} \) to \( Z_{10} \).
There are \( d = \text{g.c.d.}(12, 10) = 2 \) group homomorphisms.
They are
\[
f(x) = ax, a = \frac{10}{2}k, k = 0, 1
\]
That is either \( a = 0 \) or \( a = 5 \). Define \( F_1 : \overrightarrow{Z_{12}} \rightarrow \overrightarrow{Z_{10}} \) by
\[
F_1(x) = \begin{cases} 
0, & \text{if } i = 1 \\
5x, & \text{if } i = 2
\end{cases}
\]
Clearly \( F_1 \) is the trivial group homomorphism and a weak digraph homomorphism. Consider \( F_2(x) = 5x, \forall x \in \overrightarrow{Z_{12}} \). Here \( \text{g.c.d.}(5, 10) \neq 1 \), and
\[
F_2(x) = \begin{cases} 
0, & \text{if } x \text{ is even} \\
5, & \text{if } x \text{ is odd}
\end{cases}
\]
\( F_2 \) is a weak homomorphism.

Theorem 2.9. Let \( p \) and \( q \) are two prime numbers such that \( q \geq p \). Then there exists a digraph homomorphism from \( \overrightarrow{Z_p} \) to \( \overrightarrow{Z_q} \) which is independent of the group homomorphism from \( Z_p \) to \( Z_q \).

Proof. Since \( \text{g.c.d.}(p, q) = 1 \), there exists only the trivial homomorphism from \( Z_p \) to \( Z_q \), which induces a weak homomorphism from \( \overrightarrow{Z_p} \) to \( \overrightarrow{Z_q} \). Let
\[
f : V(\overrightarrow{Z_p}) \rightarrow V(\overrightarrow{Z_q})
\]
be defined by
\[
f(0) = 0
\]
and
\[
f(i) = i, \forall i = 1, 2, \ldots, p - 1
\]
The arcs in \( \overrightarrow{Z_p} \) are of the form \( i0, \forall i = 1, 2, \ldots, p - 1 \) and \( ij, \forall i \neq j, i = 1, 2, \ldots, p - 1 \). Then corresponding to the arcs \( i0 \in \overrightarrow{Z_p} \) there exists arcs \( f(i)f(0) = i0 \in \overrightarrow{Z_q} \). Now corresponding to the arcs of the form \( ij \in \overrightarrow{Z_p} \), since \( q \geq p \), there exists arcs
\[
f(i)f(j) = ij \in \overrightarrow{Z_q}
\]
Therefore the mapping \( f : V(\overrightarrow{Z_p}) \rightarrow V(\overrightarrow{Z_q}) \) is a digraph homomorphism.

Remark 2.10. In the above theorem if we define a one-one function \( f : V(\overrightarrow{Z_p}) \rightarrow V(\overrightarrow{Z_q}) \) by \( f(i) = \begin{cases} 
0, & i = 0 \\
j, & i, j \neq 0
\end{cases} \) then also \( f \) is a digraph homomorphism. The number of different such homomorphisms \( f : V(\overrightarrow{Z_p}) \rightarrow V(\overrightarrow{Z_q}) \) that fix zero is \((q - 1)(q - 2)\ldots(q - p + 1)\).

Example 2.11. Consider the digraph homomorphism \( f : V(\overrightarrow{Z_3}) \rightarrow V(\overrightarrow{Z_5}) \), then the 12 different homomorphisms that fix zero are,
\[
f_1(0) = 0, f_1(1) = 1, f_1(2) = 2
\]
\[
f_2(0) = 0, f_2(1) = 1, f_2(2) = 3
\]
\[
f_3(0) = 0, f_3(1) = 1, f_3(2) = 4
\]
\[
f_4(0) = 0, f_4(1) = 2, f_4(2) = 1
\]
\[
f_5(0) = 0, f_5(1) = 2, f_5(2) = 3
\]
$f_6(0) = 0, f_1(1) = 2, f_1(2) = 4$
$f_7(0) = 0, f_1(1) = 3, f_1(2) = 1$
$f_8(0) = 0, f_1(1) = 3, f_1(2) = 2$
$f_9(0) = 0, f_1(1) = 3, f_1(2) = 4$
$f_{10}(0) = 0, f_1(1) = 4, f_1(2) = 1$
$f_{11}(0) = 0, f_1(1) = 4, f_1(2) = 2$
and
$f_{12}(0) = 0, f_1(1) = 4, f_1(2) = 3$

**Theorem 2.12.** Let $p$ be a prime number such that $n \leq p$, then there exists a digraph homomorphism $f : V(\overrightarrow{Z}_n) \to V(\overrightarrow{Z}_p)$.

*Proof.* Let $f : V(\overrightarrow{Z}_n) \to V(\overrightarrow{Z}_p)$ be defined by

$$f(i) = \begin{cases} 0, & i = 0 \\ j, & i, j \neq 0 \end{cases}$$

Since $\{1, 2, \ldots, p-1\}$ is the set of generators of $Z_p$, then $f$ is a digraph homomorphism. $\square$

**Theorem 2.13.** Let $r$ be a positive integer, then there exists a digraph homomorphism $f : V(\overrightarrow{Z}_n) \to V(\overrightarrow{Z}_m)$.

*Proof.* Since $rn$ is a multiple of $n$, the subgroup diagram $\overrightarrow{Z}_n$ of $\overrightarrow{Z}_n$ is embeded in the subgroup diagram $\overrightarrow{Z}_m$ of $\overrightarrow{Z}_m$. Draw the subgroup diagram of $\overrightarrow{Z}_n$ using the strong components of $\overrightarrow{Z}_m$. Define

$$f : V(\overrightarrow{Z}_n) \to V(\overrightarrow{Z}_m)$$

as follows,

$$f(0) = 0$$

and

$$f(a_i) = b_i$$

where $a_i$ are the generators of $Z_n$ and $b_i$ are the generators of $Z_m$. Now let $c_i \in S_j$, the strong component containing $j$ in $\overrightarrow{Z}_n$ then there exists a strong component $S_j'$ in $\overrightarrow{Z}_m$ containing the same $j$. Let

$$f(c_i) = d_i, \forall c_i \in S_j; d_i \in S_j'$$

Since the number of elements of $S_j'$ is greater than that of $S_j$, this $f$ can be one-one also. Repeat this in all the strong components of $\overrightarrow{Z}_n$. Then this mapping is a homomorphism. $\square$

Homomorphism from $\overrightarrow{Z}_6$ to $\overrightarrow{Z}_{12}$ is given in figure 2

**Theorem 2.14.** Let $Z_n$ and $Z_m$ be two groups such that the structure of the subgroup diagram of $Z_n$ is embedable in the subgroup diagram of $Z_m$. If $\phi(n) \leq \phi(m)$, then there exists a digraph homomorphism $f : V(\overrightarrow{Z}_n) \to V(\overrightarrow{Z}_m)$.

Homomorphism from $\overrightarrow{Z}_6$ to $\overrightarrow{Z}_{15}$ is given in figure 3

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**Subgroup diagram of $Z_6$**

- $S_1 = \{1, 5\}$
- $S_2 = \{2, 4\}$
- $S_3 = \{3\}$
- $S_0 = \{0\}$

**Subgroup diagram of $Z_{12}$**

- $S_1 = \{1, 5, 7, 11\}$
- $S_2 = \{2, 10\}$
- $S_3 = \{3, 9\}$
- $S_4 = \{4, 8\}$
- $S_0 = \{6\}$

*Figure 2*
3. Conclusion

In this paper we have studied the relations between group homomorphisms and the corresponding homomorphisms on its related digraphs. We have established some results related to digraph homomorphisms and weak homomorphisms. Here we have concentrated only on finite group $Z_n$ and on its corresponding $Z_n^\rightarrow$-digraph. Study of generalization of these results is further area of research.

References