Completely P-regular ternary semiring
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Abstract
Here, we presented the completely P-regular ternary semiring. A new form of P-regularity is defined which is complete with arbitrary ideal P. Also we defined some new concept on “completely P-regular” and discussed some theorem with suitable examples as well.

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Ternary semiring, Regular ternary semiring, Completely regular ternary semiring, P-regular ternary semiring, Completely P-regular ternary semiring.

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Contents
1 Introduction ........................................... 1509
2 Preliminaries ........................................... 1509
3 Completely p-regular ternary semiring .......... 1509
4 Conclusion ............................................. 1512
References .............................................. 1512

1. Introduction
The concept of semiring was first introduced by Vandiver in 1934. Actually, semiring is a postulation of ring. In [5], Lister introduced ternary ring and regular ternary rings be prepared by Vasile We presently introduced the opinion of ternary semiring which is a generalization of the ternary ring presented by Lister. In fact, ternary semi ring is an algebraic system dwelling of a set T composed by a binary operation, called addition and ternary multiplication, marked by juxtaposition, which forms a commutative semi group apropos to addition, ternary semi group relative to multiplication and left, lateral, right distributive laws hold. Let take Z is a ring of integer. Now Z⁺ is subset of all positive integer of Z is an additive semi group which is closed under the ring product such algebraic system is said to be semi ring and Z⁺ form a ternary semiring. T.K.Dutta and S.Kar [3] introduced and studied some properties of ternary semirings which is a generalization of ternary semiring. M.K. Sen, S.K. Maity and K.P. Shum [4] have deliberated in completely regular semiring, which developed in the other paper V.R. Daddi and Y.S. Power [1] was discussed. In this paper we are introduce completely P-regular ternary semiring, where P is an arbitrary ideal.

2. Preliminaries
In this section we introduce completely p-regular ternary semiring.

Definition 2.1. An element t₁ of T is completely P-regular, if there exists t₂ ∈ T and p₁ ∈ P is an arbitrary ideal satisfying the following conditions,
(i) t₁ + s₂ + t₂ + p₁ = t₁ + p₁
(ii) [t₁t₂(t₁ + s₂) + p₁ = t₁ + s₂ + p₁.

Example 2.2. If Z₅ = 0, 1, 2, 3, 4 ∈ T. It is completely regular as well as it is a completely P-regular.

3. Completely p-regular ternary semiring
In this section we discussed some of the theorems in completely p-regular ternary semiring.

Theorem 3.1. If T is a completely P-regular ternary semiring if and only if for any t₁ ∈ T there exists s₁ ∈ T such that the following conditions are,
(i) t₁ + s₁ + t₁ + p₁ = t₁ + p₁
(ii) [t₁t₁(t₁ + s₁) + p₁ = t₁ + s₁ + p₁.

Proof. Presume that T is a completely p-regular ternary semiring. If for any t₁ ∈ T there exists s₁ ∈ T and p₁ ∈ P such that t₁ + s₁ + t₁ + p₁ = t₁ + p₁. Therefore (i) is hold. We need
only verify that (ii) condition
For any \( t_\alpha \in T \) there exists \( s_\alpha \in T \) and \( p_1 \in P \)
we have,
\[
[t_\alpha a t_\alpha] + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 + [t_\alpha a t_\alpha] = [t_\alpha a t_\alpha] + p_1.
\]
Then
\[
[t_\alpha a s_\alpha] + ([t_\alpha a t_\alpha] + [t_\alpha a(t_\alpha + s_\alpha)] + p_1) = [t_\alpha a s_\alpha] + [t_\alpha a t_\alpha] + p_1
\]
\[
([t_\alpha a s_\alpha] + [t_\alpha a t_\alpha]) + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = [t_\alpha a(s_\alpha + t_\alpha)] + p_1
\]
\[
[t_\alpha + s_\alpha + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = s_\alpha + t_\alpha + p_1. \tag{3.1}
\]
Since
\[
[t_\alpha a(t_\alpha + s_\alpha)] + p_1 + [t_\alpha a t_\alpha] = [t_\alpha a(t_\alpha + s_\alpha + t_\alpha)] + p_1 = [t_\alpha a t_\alpha] + p_1.
\]
we get \( (t_\alpha + s_\alpha) + p_1 + [t_\alpha a t_\alpha] = t_\alpha + p_1 \)
\[
[t_\alpha + s_\alpha + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = t_\alpha + s_\alpha + [t_\alpha a t_\alpha] + p_1 = [t_\alpha a t_\alpha] + [t_\alpha a s_\alpha] + p_1 = [t_\alpha a t_\alpha] + [t_\alpha a s_\alpha] + p_1 = [t_\alpha a(s_\alpha + t_\alpha)] + p_1
\]
\[
[t_\alpha + s_\alpha + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 \tag{3.2}
\]
From (3.1) and (3.2) , we get
\[
[t_\alpha a(t_\alpha + s_\alpha)] + p_1 = s_\alpha + t_\alpha + p_1.
\]
Similar Proof of the converse part.

**Theorem 3.2.** If \( T \) is a completely \( P \)-regular ternary semiring
if and only if for any \( t_\alpha \in T \) there exists \( s_\alpha \in T \) and \( p_1 \in P \)
such that the following condition
(i) \( t_\alpha + s_\alpha + t_\alpha + p_1 = t_\alpha + p_1 \)
(ii) \( t_\alpha a(t_\alpha + s_\alpha) + p_1 = t_\alpha + s_\alpha + p_1 \)
(iii) \( t_\alpha a(t_\alpha + s_\alpha) + p_1 = t_\alpha + s_\alpha + p_1 \)
(iv) \( [t_\alpha a + [t_\alpha a t_\alpha]] + p_1 = t_\alpha + s_\alpha + p_1 \)
(v) \( t_\alpha + [t_\alpha a + [t_\alpha a t_\alpha]] + p_1 = t_\alpha + s_\alpha + p_1 \)
(vi) \( t_\alpha a + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = t_\alpha + p_1 \)
(vii) \( t_\alpha + [t_\alpha a + [t_\alpha a t_\alpha]] + p_1 = t_\alpha + p_1 \)
(viii) \( [t_\alpha a + [t_\alpha a t_\alpha]] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 \)
\[
\text{From (3.3) and (3.4), we get}
\]
\[
[t_\alpha a + s_\alpha + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 \tag{3.4}
\]
From (3.3) and (3.4), we get
\[
[t_\alpha a + s_\alpha + [t_\alpha a(t_\alpha + s_\alpha)] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 \tag{3.4}
\]
Similar Proof of the converse part.
As well as For any \( t_\alpha \in T \) there exists \( s_\alpha \in T \) and \( p_1 \in P \)
we have,
\[
[t_\alpha a t_\alpha] + [t_\alpha a + s_\alpha + s_\alpha] + p_1 = [t_\alpha a(t_\alpha + s_\alpha)] + p_1 + [t_\alpha a t_\alpha] = [t_\alpha a t_\alpha] + p_1
\]
Theorem 3.3. The following statements are equivalent for any element $t_\alpha \in T$.

(i) $t_\alpha$ is completely P-regular.

(ii) There exists a unique element $w \in V^+(t_\alpha)$ such that $[t_\alpha(t_\alpha + s_\alpha)] + p_1 = [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = (t_\alpha + s_\alpha)t_\alpha + p_1$.

(iii) There exists an element $w \in V^+(t_\alpha)$ such that $[t_\alpha(t_\alpha + w)] + p_1 = t_\alpha + w + p_1$.

(iv) $H^+_{{\alpha}(t_\alpha)}$ is a ternary subring of $T$, where $H^+_{{\alpha}(t_\alpha)}$ is the H-class on $(T, +)$ containing $t_\alpha \in T$.

Proof. Let $t_\alpha \in T$ be completely p-regular. There exists an element $s_\alpha \in T$ satisfying the following conditions:

$\quad t_\alpha + s_\alpha + t_\alpha + p_1 = t_\alpha + p_1$

$\quad [t_\alpha(t_\alpha + s_\alpha)] + p_1 = t_\alpha + s_\alpha + p_1$

$\quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + s_\alpha + p_1$

$\quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + s_\alpha + p_1$

$\quad [t_\alpha(t_\alpha + w)] + p_1 = t_\alpha + w + p_1$

$\quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = (t_\alpha + s_\alpha)t_\alpha + p_1$

Then

$\quad [t_\alpha s_\alpha t_\alpha] + ([t_\alpha t_\alpha t_\alpha] + [t_\alpha t_\alpha(t_\alpha + s_\alpha)] + p_1) \quad = \quad [t_\alpha s_\alpha t_\alpha] + [t_\alpha t_\alpha t_\alpha] + p_1$.

It exhibit,

$\quad ([t_\alpha s_\alpha t_\alpha] + [t_\alpha t_\alpha t_\alpha]) + [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 \quad = \quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1$.

Now

$s_\alpha + t_\alpha + [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + s_\alpha + p_1$. (3.5)

Again,

$s_\alpha + t_\alpha + [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = s_\alpha + t_\alpha + [t_\alpha s_\alpha t_\alpha] + [t_\alpha t_\alpha t_\alpha] + p_1 \quad = \quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1$. (3.6)

From (3.3) and (3.4), we get

$\quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + s_\alpha + p_1$ (3.7)

Similar Proof of the converse part. In similar way, we get

$\quad [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + s_\alpha + p_1$. (3.8)

Using (3.5) we have, adding $t_\alpha$ on both sides,

$t_\alpha + [(t_\alpha + s_\alpha)t_\alpha t_\alpha] + p_1 = t_\alpha + s_\alpha + t_\alpha + p_1 \quad = \quad t_\alpha + p_1$

Similar way, we get

$t_\alpha + [t_\alpha(t_\alpha + s_\alpha)t_\alpha] + p_1 = t_\alpha + p_1$

and

$t_\alpha + [t_\alpha t_\alpha(t_\alpha + s_\alpha)] + p_1 = t_\alpha + p_1$.

Hence

$\quad [t_\alpha(t_\alpha + s_\alpha)] + p_1 = [t_\alpha t_\alpha + s_\alpha t_\alpha] + p_1 = [(t_\alpha + s_\alpha)t_\alpha t_\alpha] + p_1$. (3.9)

$\quad \Box$
\[(t_{α} + w + p_1)t_{α}t_{α} + t_{α} = t_{α} + p_1\]

and \[t_{α}(t_{α} + w + p_1)t_{α} + t_{α} = t_{α} + p_1.\]

Hence
\[t_{α}t_{α}(t_{α} + w + p_1) = t_{α}(t_{α} + w + t_{α})t_{α} = [(t_{α} + w + p_1)t_{α}t_{α}]\]

and
\[t_{α} + [t_{α}t_{α}(t_{α} + w + p_1)] = t_{α} + [t_{α}t_{α}(w + t_{α})]
\]
\[= t_{α} + w + t_{α} + p_1
\]
\[= 2t_{α} + w + p_1.\]

Uniqueness:
Let \(x \in V^+(t)\) be another element satisfying the conditions, Hence
\[w = w + t_{α} + w
\]
\[= 2w + t_{α}
\]
\[= 2w + t_{α} + x + t_{α}
\]
\[= 2w + 2t_{α} + x
\]
\[= 2t_{α} + 2x
\]
\[= 2t_{α} + w + t_{α} + w + 2x
\]
\[= 2t_{α} + w + 2x
\]
\[= t_{α} + 2x
\]
\[= x.\]

Therefore \(w = x\).

Thus (i) \(⇒\) (ii) and (ii) \(⇒\) (iii) is obviously true. Let us prove that (iii) \(⇒\) (iv) . Assume that there exist an unique element \(w \in V^+(t_{α})\) such that \(t_{α}(t_{α} + w + t_{α}) = t_{α} + w + t_{α}.

To prove \(H^+_t\) is a ternary subring of \(T\), where \(H^+_t\) is the H-class on \((T,+)\) containing \(t_{α} \in T.\n
We have \([t_{α}t_{α}(w + p_1)] = t_{α} + w + 1\). Adding \(t_{α}\) on both sides, and we get
\[t_{α}(w + t_{α} + p_1) = (t_{α} + w + p_1) + t_{α}.

Therefore \(t_{α}H^+_t(t_{α} + w + p_1).\) Hence \(H^+_t\) contains an additive idempotent element \(t_{α} + w + p_1 (= w + t_{α} + p_1)\). Therefore \(H^+_t\) is a group. Now
\[t_{α} + p_1 = (t_{α} + w + p_1) + t_{α}
\]
\[= [t_{α}t_{α} + [t_{α}t_{α}(w + p_1)] + t_{α}
\]
\[= t_{α}t_{α}(t_{α} + w + p_1) + t_{α}
\]
\[= [t_{α}t_{α} + [t_{α}t_{α}(w + p_1) + t_{α}]
\]

Also
\[t_{α} = [t_{α}(w + t_{α})]
\]
\[= t_{α}(t_{α} + w) + [t_{α}t_{α}]
\]
\[= t_{α} + (w + [t_{α}t_{α}]).\n
This implies that \(t_{α}R^+t_{α}^3\). Similarly \(t_{α}L^+t_{α}^3\). Therefore \(t_{α}H^+t_{α}^3\). Let \(m, n, o \in H^+_t\). Therefore \(m, n, o \in L^+_t\) and \(m, n, o \in R^+_t\). Hence, there exists \(a, b, c, d, g, f \in T\) such that \(t_{α} = a + m, m = d + t_{α}, t_{α} = b + n, n = g + t_{α}, t_{α} = c + o, o = f + t_{α}.

Now
\[\begin{align*}
\{mno\} &= \{(d + t_{α})(g + t_{α})(f + t_{α})\}
\end{align*}\]
\[= ((d + t_{α})(g + t_{α})(f + t_{α}))
\]
\[= (d + t_{α})(g + t_{α})(f + t_{α})
\]
\[= [d + t_{α}] + [g + t_{α}] + [t_{α}g + t_{α}f] + [t_{α}f + t_{α}g] + [d + t_{α}] + [g + t_{α}]
\]
\[+ [d + t_{α}] + [g + t_{α}] + [t_{α}g + t_{α}f].\]

Also
\[\begin{align*}
[t_{α}t_{α}t_{α}] &= [(a + m)(b + n)(c + o)]
\end{align*}\]
\[= [(a + m)(b + n) + [(a + m)(b + n)]]
\]
\[= [abc] + [anc] + [mbc] + [mnc] + [abo]
\]
\[+ [ano] + [mba] + [mno].\]

Therefore \(\{mno\}^L \in [t_{α}t_{α}t_{α}] \Rightarrow \{mno\} \in L^+_t\) is \(L^+_t\). Similarly \(\{mno\} \in R^+_t\) is \(R^+_t\).

Therefore \(\{mno\} \in H^+_t\) is an arbitrary multiplication identity then \(t_{α} = s_{α} = p_1\) (= e).

Now \([t_{α}t_{α}(t_{α} + s_{α})] + p_1 = t_{α} + s_{α} + p_1\) where \(p_1 \in P\) is an arbitrary multiplication identity then \(t_{α} = s_{α} = p_1\) (= e).

\[\begin{align*}
\text{Proof. Let } t_{α} = s_{α} = p_1
\end{align*}\]

Take \(t_{α} = s_{α} = p_1\)

Now \([t_{α}t_{α}(t_{α} + s_{α})] + p_1 = t_{α} + s_{α} + p_1\)

\(\Rightarrow \{t_{α}t_{α}t_{α}\} + t_{α} = t_{α} \Rightarrow t_{α} \subseteq t_{α}.

Hence \(t_{α} \in T\) is an completely P-regular.


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