On Kasaj topological spaces
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Abstract
In year 2013, L. Thivagar et al. introduced nano topological space and he analysed some properties of weak open sets. In this paper we shall introduce Kasaj-topological space. We shall introduce some new classes of weak open sets namely Kasaj-pre-open sets and Kasaj-semi-open sets in Kasaj topological spaces and analyze their basic properties. We shall also define new types of continuous functions namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function in Kasaj topological space.

Keywords
Kasaj topological space, Kasaj-pre-open set, Kasaj-semi-open set.

AMS Subject Classification
54A05, 54B05.

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1. Introduction
In recent times many people have introduced new topological space and it is studied very well. For example, nano topological space was introduced by L. Thivagar et al. \textsuperscript{[3]}, S. Chandrasekar \textsuperscript{[5]} introduced Micro topological spaces which are extension of nano topological spaces. He has used Levine’s simple extension concepts in nano topological spaces. The notations of Semi-open sets and Pre-open sets were introduced by Levine \textsuperscript{[4]}, Mashhour et al. \textsuperscript{[1]}, respectively. In this paper, we shall define new topological space namely Kasaj topological spcae. We shall also define Kasaj-pre-open set and Kasaj-semi-open set, investigate basic properties and find the relation between these new classes. We shall also define new types of continuous function namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function.

2. Preliminary

Definition 2.1. A subset $\mathcal{P}$ of a topological space $(\mathfrak{X}, \mathcal{T})$ is called

\begin{itemize}
  \item a semi-open set \textsuperscript{[4]} if $\mathcal{P} \subseteq \text{cl}(\text{int}(\mathcal{P}))$.
  \item a pre-open set \textsuperscript{[1]} if $\mathcal{P} \subseteq \text{int}(\text{cl}(\mathcal{P}))$.
\end{itemize}

The complement of a semi-open set (pre-open set) in a space $\mathfrak{X}$ is called semi-closed set (pre-closed set) in $\mathfrak{X}$.

2.1 Nano Topological Spaces

Definition 2.2. \textsuperscript{[3]} Let $\mathfrak{A}$ be a non-empty Universal set and $\mathcal{R}$ be an equivalence relation on $\mathfrak{A}$ and it is named as the indiscernibility relation. The pair $(\mathfrak{A}, \mathcal{R})$ is called as approximation space. Let $\mathfrak{X} \subseteq \mathfrak{A}$.

1. The lower approximation of $\mathfrak{X}$ with respect to $\mathcal{R}$ is denoted by $\mathcal{L}_\mathcal{R}(\mathfrak{X})$ and is defined by $\mathcal{L}_\mathcal{R}(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} \{P(x) : P(x) \subseteq \mathfrak{X}\}$

where $P(x)$ denotes the equivalence relation which contains $x \in \mathfrak{A}$.
2. The upper approximation of $X$ with respect to $R$ is denoted by $\mathcal{V}_R(X)$ and is defined by

$$\mathcal{V}_R(X) = \bigcup_{x \in A} \{ P(x) : P(x) \cap X \neq \emptyset \}$$

where $P(x)$ denotes the equivalence relation which contains $x \in A$.

3. The boundary region of $X$ with respect to $R$ is denoted by $\Omega_R(X)$ and is defined by

$$\Omega_R(X) = \mathcal{V}_R(X) \setminus \mathcal{F}_R(X).$$

Definition 3.3. [3] Let $A$ be an universal set. $R$ be an equivalence relation on $A$, $X \subseteq A$ and $\mathcal{V}_R(X) = \{A, \emptyset, \mathcal{V}_R(X), \mathcal{F}_R(X), \Omega_R(X)\}$ which satisfies the following axioms.

1. $\emptyset, X \in \mathcal{V}_R(X)$.
2. The union of elements of any subcollection of $\mathcal{V}_R(X)$ is in $\mathcal{V}_R(X)$.
3. The intersection of any finite subcollection of elements of $\mathcal{V}_R(X)$ is in $\mathcal{V}_R(X)$.

Then $(A, \mathcal{V}_R(X))$ is called nano topological space. The members of $\mathcal{V}_R(X)$ are called nano open sets.

### 3. Kasaj Topological Space

**Definition 3.1.** Let $(A, \mathcal{V}_R(X))$ be a nano topological space and Kasaj topology is defined by $KS_R(X) = \{ (K \cap S) \cup (K' \cap S') : K, K' \in \mathcal{V}_R(X), \text{fixed } S, S' \in \mathcal{V}_R(X), S \cup S' = A \}$ and is called Kasaj topological space.

**Definition 3.2.** The Kasaj topology $KS_R(X)$ satisfies the following postulates:

1. $\emptyset, X \in KS_R(X)$.
2. The union of elements of any subcollection of $KS_R(X)$ is in $KS_R(X)$.
3. The intersection of any finite subcollection of elements of $KS_R(X)$ is in $KS_R(X)$.

Then $(A, \mathcal{V}_R(X), KS_R(X))$ is called Kasaj topological spaces and the members of $KS_R(X)$ are called Kasaj open sets ($KS$-open sets) and the complement of a Kasaj-open set is called a Kasaj-closed(KS-closed) set and the collection of all Kasaj-closed sets is denoted by $KSCL(X)$.

**Definition 3.3.** The Kasaj closure and the Kasaj interior of a set $\mathcal{P}$ is denoted by $KS_cl(\mathcal{P})$ and $KS_int(\mathcal{P})$, respectively. It is defined by

$$KS_cl(\mathcal{P}) = \bigcap \{O : \mathcal{P} \subseteq O, O \in KS - \text{closed}\}$$

$$KS_int(\mathcal{P}) = \bigcup \{O : O \subseteq \mathcal{P}, O \in KS - \text{open}\}.$$

**Remark 3.4.**

1. $KS_cl(\mathcal{P})$ is the largest $KS$-open set contained in $\mathcal{P}$.
2. $KS_cl(\mathcal{P})$ is the smallest $KS$-closed set containing $\mathcal{P}$.

**Definition 3.5.** For any two subsets $\mathcal{P}, \Omega$ of $A$ in a Kasaj topological space $(A, \mathcal{V}_R(X), KS_R(X))$.

1. $\mathcal{P}$ is a Kasaj-closed set if and only if $KS_cl(\mathcal{P}) = \mathcal{P}$.
2. $\mathcal{P}$ is a Kasaj-open set if and only if $KS_int(\mathcal{P}) = \mathcal{P}$.
3. If $\mathcal{P} \subseteq \Omega$, then $KS_int(\mathcal{P}) \subseteq KS_int(\Omega)$ and $KS_cl(\mathcal{P}) \subseteq KS_cl(\Omega)$.
4. $KS_cl(KS_cl(\mathcal{P})) = KS_cl(\mathcal{P})$ and $KS_int(KS_int(\mathcal{P})) = KS_int(\mathcal{P})$.
5. $KS_cl(\mathcal{P} \cup \Omega) \supseteq KS_cl(\mathcal{P}) \cup KS_cl(\Omega)$.
6. $KS_int(\mathcal{P} \cup \Omega) \supseteq KS_int(\mathcal{P}) \cup KS_int(\Omega)$.
7. $KS_cl(\mathcal{P} \cap \Omega) \subseteq KS_cl(\mathcal{P}) \cap KS_cl(\Omega)$.
8. $KS_int(\mathcal{P} \cap \Omega) \subseteq KS_int(\mathcal{P}) \cap KS_int(\Omega)$.
9. $KS_cl(\mathcal{P}) = [KS_int(\mathcal{P})]^c$.
10. $KS_int(\mathcal{P}) = [KS_cl(\mathcal{P})]^c$.

**Example 3.6.** Let $A = \{\{\Gamma, \Omega, \Psi, \Phi, \Gamma\}\}$ with $A/R = \{\{\Gamma\}, \{\Omega, \Psi, \Phi\}, \{\Gamma\}\}$. Then $\mathcal{V}_R(X) = \{\emptyset, A\}$. If we consider $S = \{\Gamma, \Omega, \Gamma\}$ and $S' = \{\Psi, \Phi, \Gamma\}$, then $KS_R(X) = \{\emptyset, \{\Gamma\}, \{\Gamma, \Omega, \Gamma\}, \{\Omega, \Gamma\}, \{\Psi, \Phi, \Gamma\}, \{\Gamma, \Psi, \Phi, \Gamma\}, A\}$.

### 4. Kasaj-pre-open sets

**Definition 4.1.** Let $(A, \mathcal{V}_R(X), KS_R(X))$ be a Kasaj topological space and $\mathcal{P} \subseteq A$. Then $\mathcal{P}$ is called Kasaj-pre-open(KS-pre-open) set if $\mathcal{P} \subseteq KS_int(KS_cl(\mathcal{P}))$ and Kasaj-pre-closed(KS-pre-closed) set if $KS_cl(KS_int(\mathcal{P})) \subseteq \mathcal{P}$. The set of all Kasaj-pre-open and Kasaj-pre-closed sets are denoted by $KSP(O)(A, X)$ and $KSP(C)(A, X)$, respectively.

**Theorem 4.2.** $KS_R(X) \subseteq KSP(O)(A, X)$.

**Proof.** Let $\mathcal{P} \in KS_R(X)$, i.e., $\mathcal{P} = KS_int(\mathcal{P})$. Since $\mathcal{P} \subseteq KS_cl(\mathcal{P})$ for all subset $\mathcal{P}$ of $A$, therefore, $\mathcal{P} = KS_int(\mathcal{P}) \subseteq KS_int(KS_cl(\mathcal{P}))$, which implies that $\mathcal{P} \subseteq KS_int(KS_cl(\mathcal{P}))$. Therefore $\mathcal{P} \in KSP(O)(A, X)$. 

**Remark 4.3.** In general, $KSP(O)(A, X) \subset KS_R(X)$ (See Example 4.4).

**Example 4.4.** Let $A = \{\Gamma, \Omega, \Psi, \Phi, \Gamma\}$ with $A/R = \{\{\Gamma\}, \{\Omega, \Psi, \Phi\}, \{\Gamma\}\}$. Then $\mathcal{V}_R(X) = \{\emptyset, A\}$. If we consider $S = \{\Gamma, \Omega, \Gamma\}$ and $S' = \{\Psi, \Phi, \Gamma\}$, then

- $KS_R(X) = \{\emptyset, \{\Gamma\}, \{\Gamma, \Omega, \Gamma\}, \{\Omega, \Gamma\}, \{\Psi, \Phi, \Gamma\}, \{\Gamma, \Omega, \Gamma\}, \{\Psi, \Phi, \Gamma\}, A\}$. 

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One can easily see that \{Ω, Ψ\} ∈ KSPO(𝔸, ℳ) but not in KSₚ(ℳ).

**Theorem 4.5.** \(KSCL(ℳ) \subseteq KSPLCL(𝔸, ℳ)\).

**Proof.** Let \(ℙ \in KSCL(ℳ)\), i.e., \(KS_c(ℙ) = ℙ\). Then we have \(KS_c(KS_c(ℙ)) \subseteq ℙ\). Since

\[
KS_c(ℙ) \subseteq ℙ
\]

and

\[
KS_c(KS_c(ℙ)) \subseteq KS_c(KS_c(ℙ))
\]

so, it follows that

\[
(KS_c(KS_c(ℙ))) \subseteq KS_c(KS_c(ℙ)) \subseteq ℙ.
\]

Hence \(ℙ \in KSPLCL(𝔸, ℳ)\).

**Remark 4.6.** In general, \(KSPLCL(𝔸, ℳ) \not\subseteq KSCL(ℳ)\). Consider Example 4.4, one can see that \(\{Γ, Φ, Γ\}\) is in \(KSPLCL(𝔸, ℳ)\) but not in \(KSCL(ℳ)\).

**Definition 4.7.** The Kasaj-pre-closure and the Kasaj-pre-interior of a set ℙ is denoted by \(KS_{\text{pre}}c(ℙ)\) and \(KS_{\text{pre}}\text{int}(ℙ)\), respectively. It is defined by

\[
KS_{\text{pre}}c(ℙ) = \bigcap\{Ω : ℙ \subseteq Ω, Ω \text{ is } KS_{\text{pre}}\text{-closed}\}
\]

\[
KS_{\text{pre}}\text{int}(ℙ) = \bigcup\{Ω : ℙ \subseteq Ω, Ω \text{ is } KS_{\text{pre}}\text{-open}\}.
\]

**Remark 4.8.**

1. \(KS_{\text{pre}}\text{int}(ℙ)\) is the largest \(KS_{\text{pre}}\text{-open}\) set contained in \(ℙ\).

2. \(KS_{\text{pre}}c(ℙ)\) is the smallest \(KS_{\text{pre}}\text{-closed}\) set containing \(ℙ\).

**Theorem 4.9.**

1. \(\bigcup_{α ∈ Λ} ℙ_α \in KSPO(𝔸, ℳ)\) whenever \(ℙ_α \in KSPO(𝔸, ℳ)\) and \(Λ\) is an index set.

2. \(\bigcap_{α ∈ Λ} ℙ_α \in KSPLCL(𝔸, ℳ)\) whenever \(ℙ_α \in KSPLCL(𝔸, ℳ)\) and \(Λ\) is an index set.

**Proof.** (1) Let \(\{ℙ_α : α ∈ I\} \subseteq KSPO(𝔸, ℳ)\). By definition of \(KS_{\text{pre}}\text{-open}\) set, for each \(α\), \(ℙ_α \subseteq KS_{\text{pre}}\text{int}(KS_c(ℙ_α))\), which implies that

\[
\bigcup_{α} ℙ_α \subseteq \bigcup_{α} KS_{\text{pre}}\text{int}(KS_c(ℙ_α))
\]

\[
\subseteq KS_{\text{pre}}\text{int}(\bigcup_{α} KS_c(ℙ_α))
\]

\[
\subseteq KS_{\text{pre}}\text{int}(KS_c(\bigcup_{α} ℙ_α))
\]

Hence \(\bigcup_{α} ℙ_α \in KSPO(𝔸, ℳ)\).

(2) Let \(\{ℙ_α : α ∈ I\} \subseteq KSPLCL(𝔸, ℳ)\). By definition of \(KS_{\text{pre}}\text{-closed}\) set, for each \(α\), \(KS_c(ℙ_{\text{pre}}\text{int}(ℙ_α)) \subseteq ℙ_α\).

\[
KS_c(ℙ_{\text{pre}}\text{int}(ℙ_α)) \subseteq ℙ_α
\]

Now

\[
KS_c(KS_{\text{pre}}\text{int}(ℙ_α)) \subseteq ℙ_α
\]

\[
\bigcap_{α} ℙ_α \in KSPLCL(𝔸, ℳ).
\]

Hence, \(\bigcap_{α} ℙ_α \in KSPLCL(𝔸, ℳ)\).

**5. Kasaj-semi-open sets**

**Definition 5.1.** Let \((𝔸, Sₚ(ℳ), KSₚ(ℳ))\) be a Kasaj topological space and \(ℙ ⊆ ℳ\). Then \(ℙ\) is called Kasaj-semi-open(\(KS_{\text{semi}}\)-open) set if \(ℙ \subseteq KS_c(KS_{\text{semi}}\text{-open}(ℙ))\) and Kasaj-semi-closed(\(KS_{\text{semi}}\)-closed) set if \(KS_{\text{semi}}\text{int}(KS_c(ℙ)) \subseteq ℙ\). The set of all Kasaj-semi-open sets is denoted by \(KSPO(𝔸, ℳ)\) and similarly, the set of all Kasaj-semi-closed sets is denoted by \(KSCL(𝔸, ℳ)\).

**Theorem 5.2.** \(KS_{ₚ}(ℳ) \subseteq KSPO(𝔸, ℳ)\).

**Proof.** Let \(ℙ \in KS_{ₚ}(ℳ)\), i.e., \(ℙ = KS_{\text{semi}}\text{int}(ℙ)\). Since \(ℙ \subseteq KS_c(ℙ)\) for all subset \(ℙ\) of \(ℳ\), therefore, \(ℙ = KS_{\text{semi}}\text{int}(ℙ) \subseteq KS_c(KS_{\text{semi}}\text{int}(ℙ))\), which implies that \(ℙ \subseteq KS_{\text{semi}}\text{int}(KS_c(ℙ))\). Therefore \(ℙ \in KSPO(𝔸, ℳ)\).

**Remark 5.3.** In general \(KSPO(𝔸, ℳ) \not\subseteq KS_{ₚ}(ℳ)\) (See Example 5.4).

**Example 5.4.** Let \(𝕄 = \{Γ, Ω, Ψ, Φ, Γ\}\) with \(𝕄/ℙ = \{Γ\}\), \(\{Ω, Ψ\}\), \(\{Φ, Γ\}\) and \(ℳ = \{Γ, Ψ\}\) \(⊆ ℳ\). Then \(Sₚ(ℳ) = \{∅, ℳ, \{Γ\}, \{Ω, Ψ\}, \{Φ, Γ\}\}\). If we consider \(S = \{Γ, Ω, Φ\}\) and \(S' = \{Ψ, Γ\}\), then
Theorem 5.5. \( KSCL(\mathcal{A}, \mathcal{X}) \subseteq \text{KSSCL}(\mathcal{A}, \mathcal{X}) \).

Proof. Let \( \mathcal{Y} \in KSCL(\mathcal{A}, \mathcal{X}) \), i.e., \( KS_cl(\mathcal{Y}) = \mathcal{Y} \). Then we have
\[
KS_cl(KS_cl(\mathcal{Y})) \subseteq \mathcal{Y}.
\]
Since
\[
KS_cl(\mathcal{Y}) \subseteq \mathcal{Y},
\]
and
\[
KS_cl(KS_cl(\mathcal{Y})) \subseteq KS_cl(KS_cl(\mathcal{Y})).
\]
So, it follows that
\[
(KS_cl(KS_cl(\mathcal{Y}))) \subseteq KS_cl(KS_cl(\mathcal{Y})) \subseteq \mathcal{Y}.
\]
Hence \( \mathcal{Y} \in \text{KSSCL}(\mathcal{A}, \mathcal{X}) \).

Remark 5.6. In general, \( \text{KSSCL}(\mathcal{A}, \mathcal{X}) \not\subseteq \text{KSCL}(\mathcal{A}, \mathcal{X}) \). Consider Example 5.4. One can see that \( \{\Omega\} \) is in \( \text{KSSCL}(\mathcal{A}, \mathcal{X}) \) but not in \( \text{KSCL}(\mathcal{A}, \mathcal{X}) \).

Definition 5.7. The Kasaj-semi-closure and the Kasaj-semi-interior of a set \( \mathcal{Y} \) are denoted by \( KS_{semi}(\mathcal{Y}) \) and \( KS_{semi-int}(\mathcal{Y}) \), respectively. They are defined by
\[
KS_{semi}(\mathcal{Y}) = \bigcap\{\Omega : \mathcal{Y} \subseteq \Omega, \Omega \text{ is } KS-semi-closed\}
\]
\[
KS_{semi-int}(\mathcal{Y}) = \bigcup\{\Omega : \mathcal{Y} \subseteq \Omega, \Omega \text{ is } KS-semi-open\}.
\]

Remark 5.8.
1. \( KS_{semi-int}(\mathcal{Y}) \) is the largest \( KS-semi-open \) set contained in \( \mathcal{Y} \).
2. \( KS_{semi}(\mathcal{Y}) \) is the smallest \( KS-semi-closed \) set containing \( \mathcal{Y} \).

Theorem 5.9.
1. \( \bigcup_{\alpha \in \Lambda} \mathcal{Y}_\alpha \in \text{KSSO}(\mathcal{A}, \mathcal{X}) \) whenever \( \mathcal{Y}_\alpha \in \text{KSSO}(\mathcal{A}, \mathcal{X}) \) and \( \Lambda \) is an index set.
2. \( \bigcap_{\alpha \in \Lambda} \mathcal{Y}_\alpha \in \text{KSSCL}(\mathcal{A}, \mathcal{X}) \) whenever \( \mathcal{Y}_\alpha \in \text{KSSCL}(\mathcal{A}, \mathcal{X}) \) and \( \Lambda \) is an index set.

Proof. (1.) Let \( \{\mathcal{Y}_\alpha : \alpha \in I\} \subseteq \text{KSSO}(\mathcal{A}, \mathcal{X}) \). By definition of \( KS-semi-open \) set, for each \( \alpha \), \( \mathcal{Y}_\alpha \subseteq KS_{semi}(KS_{semi-int}(\mathcal{Y}_\alpha)) \), which implies that
\[
\mathcal{Y} \subseteq KS_{semi}(KS_{semi-int}(\mathcal{Y})) \subseteq KS_{semi}(KS_{semi-int}(\mathcal{Y}_\alpha)) \subseteq KS_{semi-int}(\mathcal{Y}_\alpha).
\]
Hence \( \bigcup_{\alpha \in \Lambda} \mathcal{Y}_\alpha \in \text{KSSO}(\mathcal{A}, \mathcal{X}) \).

(2.) Let \( \{\mathcal{Y}_\alpha : \alpha \in I\} \subseteq \text{KSSCL}(\mathcal{A}, \mathcal{X}) \). By definition of \( KS-semi-closed \) set, for each \( \alpha \),
\[
KS_{semi}(\mathcal{Y}_\alpha) \subseteq \mathcal{Y}_\alpha.
\]
So, it follows that
\[
KS_{semi}(\mathcal{Y}_\alpha) \subseteq \mathcal{Y}_\alpha \subseteq \bigcap_{\alpha \in \Lambda} \mathcal{Y}_\alpha.
\]
Hence, \( \bigcap_{\alpha \in \Lambda} \mathcal{Y}_\alpha \in \text{KSSCL}(\mathcal{A}, \mathcal{X}) \).

Example 5.12. Let \( \mathcal{A} = \{\{\gamma\}, \{\Omega, \Psi, \Phi, \Gamma\}\} \) with \( A/\mathcal{K} = \{\{\gamma\}, \{\Omega, \Psi, \Phi, \Gamma\}\} \) and \( \mathcal{X} = \{\Phi, \Gamma\} \). Then \( S(\mathcal{X}) = \{\emptyset, \{\gamma\}, \{\Omega, \Psi, \Phi, \Gamma\}\} \). If we consider \( S = \{\{\gamma, \Omega, \Gamma\} \} \) and \( S' = \{\Psi, \Phi\} \), then
\[
KS_{semi}(\mathcal{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Psi, \Phi, \Omega, \Gamma\}, \{\Psi, \Phi, \Omega, \Gamma\}, \{\Psi, \Phi, \Omega, \Gamma\}, \{\Psi, \Phi, \Omega, \Gamma\}, \{\Omega, \Psi, \Phi, \Gamma\}\}.
\]
KSSO(𝒜, ℳ) = {∅, {Ω}, {Γ}, {Γ, Ω}, {Ψ, Φ}, {Ω, Γ}, {Γ, Ψ, Φ}, {Ψ, Φ}, {Ω, Φ, Γ}, {Γ, Ψ, Φ}, {Ψ, Φ}, {Ω, Ψ, Φ, Γ}, {Γ, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Γ, Ω, Ψ, Φ, Γ}, {Ω, Ψ, Φ, Γ}, {Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Γ, Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Γ, Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Γ, Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}}.

KSPO(𝒜, ℳ) = {∅, {Ω}, {Γ}, {Ψ}, {Ψ, Φ}, {Ω, Γ}, {Ω, Ψ, Φ}, {Ω, Φ, Γ}, {Γ, Ω, Γ, Ψ, Φ}, {Γ, Ω, Ψ, Φ}, {Ψ, Φ}, {Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Γ, Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}, {Ω, Ω, Ψ, Φ, Γ}}.

Remark 5.13.

KSPO(𝒜, ℳ) ⊈ KSSO(𝒜, ℳ). In example 5.12 {Φ} is in KSPO(𝒜, ℳ) but not in KSSO(𝒜, ℳ).

KSPO(𝒜, ℳ) ⊈ KSSO(𝒜, ℳ). In example 5.12 {Γ, Ω} is in KSSO(𝒜, ℳ) but not in KSPO(𝒜, ℳ).

### 6. Kasaj-continuous functions

We first define Kasaj-continuous (KS-continuous) functions.

**Definition 6.1.** Let (𝒜, ℳ, KS(ℳ)) and (𝒜, ℳ, KS(ℳ)) be two Kasaj topological spaces and ℳ ⊆ ℳ and ℳ ⊆ ℳ. Then f : ℳ → ℳ is Kasaj-continuous (KS-continuous) function if f⁻¹(D) ∈ KS(ℳ) whenever D ∈ KS(ℳ).

**Theorem 6.2.** Let (𝒜, ℳ, KS(ℳ)) and (𝒜, ℳ, KS(ℳ)) be two Kasaj topological spaces and ℳ ⊆ ℳ and ℳ ⊆ ℳ. Then f : ℳ → ℳ is KS-continuous function if and only if f⁻¹(D) ∈ KSCL(ℳ) whenever D ∈ KSCL(ℳ).

**Proof.** Let f : ℳ → ℳ be a KS-continuous function and D ∈ KSCL(ℳ). Then D" ∈ KSCL(ℳ). By hypothesis f⁻¹(D") ∈ KSCL(ℳ). Hence f⁻¹(D) ∈ KSCL(ℳ) whenever D ∈ KSCL(ℳ).

Conversely suppose [f⁻¹(D)]⁺ ∈ KSCL(ℳ) whenever D ∈ KSCL(ℳ). Let D ∈ KSCL(ℳ) then D" ∈ KSCL(ℳ). By assumption then f⁻¹(D") ∈ KSCL(ℳ). Hence f⁻¹(D) ∈ KSCL(ℳ).

**Definition 6.3.** Let (𝒜, ℳ, KS(ℳ)) and (𝒜, ℳ, KS(ℳ)) be two Kasaj topological spaces and ℳ ⊆ ℳ and ℳ ⊆ ℳ. Then f : ℳ → ℳ is pre-KS-continuous function if f⁻¹(D) ∈ KSPCL(ℳ) whenever D ∈ KSPCL(ℳ).

**Theorem 6.4.** Every KS-continuous function is KS-pre-continuous function.

**Definition 6.5.** Let (𝒜, ℳ, KS(ℳ)) and (𝒜, ℳ, KS(ℳ)) be two Kasaj topological spaces and ℳ ⊆ ℳ and ℳ ⊆ ℳ. Then f : ℳ → ℳ is KS-semi-continuous function if f⁻¹(D) ∈ KSCL(ℳ) whenever D ∈ KSCL(ℳ).

**Theorem 6.6.** Every KS-continuous function is KS-semi-continuous function.

### Conclusion

In this paper, some of the properties of these new classes are discussed and we get the following inversion:

<table>
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<tr>
<th>KSPO(𝒜, ℳ)</th>
<th>KS(ℳ)</th>
<th>KSSO(𝒜, ℳ)</th>
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we have shown that none of implication is reversible. This shall be extended in future research with some applications.

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