Neighbourhood $V_4$–magic labeling of some subdivision graphs

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**Abstract**
Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be Neighbourhood $V_4$-magic if there exists a labeling $f : V(G) \rightarrow V_4 \setminus \{0\}$ such that the sum $N^+_f(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is $p$, where $p$ is any non-zero element in $V_4$, then we say that $f$ is a $p$-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a $p$-neighbourhood $V_4$-magic graph. If this constant is 0, then we say that $f$ is a 0-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a 0-neighbourhood $V_4$-magic graph.

**Keywords**
Klein-4-group, $\alpha$-neighbourhood $V_4$-magic graphs and 0-neighbourhood $V_4$-magic graphs.

**AMS Subject Classification**
05C78, 05C25.

1. Introduction

In this paper we consider graphs that are connected, finite, simple and undirected. For graph theory notations and terminology not directly defined in this paper, we refer readers to [1]. The Klein 4-group, denoted by $V_4$, is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, with $a + a = b + b = c + c = 0$ and $a + b = b + c = c + a = a + c = a = b$. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$, is said to be Neighbourhood $V_4$-magic if there exists a labeling $f : V(G) \rightarrow V_4 \setminus \{0\}$ such that the sum $N^+_f(v) = \sum_{u \in N(v)} f(u)$ is a constant map. If this constant is $p$, where $p$ is any non-zero element in $V_4$, then we say that $f$ is a $p$-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a $p$-neighbourhood $V_4$-magic graph. If this constant is 0, then we say that $f$ is a 0-neighbourhood $V_4$-magic labeling of $G$ and $G$ is said to be a 0-neighbourhood $V_4$-magic graph.

The subdivision graph of a graph $G$ is denoted by $S(G)$ and is obtained by inserting an additional vertex to each edge of $G$. In this paper, we investigate Neighbourhood $V_4$-magic labeling of subdivision graph of some graphs and we classify them into the following three categories:

(i) $\Omega_{\alpha} := \text{the class of all } \alpha\text{-neighbourhood } V_4\text{-magic graphs}$,
(ii) $\Omega_0 := \text{the class of all } 0\text{-neighbourhood } V_4\text{-magic graphs}$, and
(iii) $\Omega_{\alpha,0} := \Omega_{\alpha} \cap \Omega_0$.

2. Main Results

**Theorem 2.1.** [2] $C_n \in \Omega_{\alpha}$ if and only if $n \equiv 0\pmod{4}$.

**Theorem 2.2.** For $n \geq 3$, $S(C_n) \in \Omega_{\alpha}$ if and only if $n \equiv 0\pmod{2}$.

**Proof.** We have $S(C_n) \simeq C_{2n}$. Then by Theorem 2.1, $S(C_n) \in \Omega_{\alpha}$ if and only if $2n \equiv 0\pmod{4}$, i.e., if and only if $n \equiv 0\pmod{2}$. This completes the proof.

**Theorem 2.3.** $S(C_n) \in \Omega_0$ for all $n \geq 3$.

**Proof.** By labeling all the vertices of $S(C_n)$ by $a$, we get $S(C_n) \in \Omega_0$.

**Corollary 2.4.** For $n \geq 3$, $S(C_n) \in \Omega_{\alpha,0}$ if and only if $n \equiv 0\pmod{2}$. 

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**Proof.** Proof directly follows from Theorem 2.2 and Theorem 2.3.

**Theorem 2.7.**

**Definition 2.10.** [6] The flower graph $F_m$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the central vertex of the helm.

**Theorem 2.11.** $S(F_m) \notin \Omega_n$ for any $n \geq 3$.

**Diagram:**

Figure 1. An $a$-neighbourhood $V_4$-magic labeling of $S(C_4)$.

**Definition 2.14.** [7] The Sunflower $SF_n$ is obtained from a wheel $W_n$ with the central vertex $v_0$ and cycle $C_n = v_1v_2v_3 \cdots v_n$ and additional vertices $v_1, v_2, v_3, \ldots, v_n$ where $v_i$ is joined by edges to $v_i$ and $v_{i+1}$, where $i+1$ is taken over modulo $n$.

**Theorem 2.15.** $S(SF_n) \notin \Omega_n$ for any $n \geq 3$.

**Proof.** Consider the sunflower $SF_n$ with vertex set $V = \{v_0, v_1, v_2, v_3, \ldots, v_n\}$ where $v_0$ is the central vertex, $v_1, v_2, v_3, \ldots, v_n$ are vertices of the cycle and $v_i$ is the vertex joined by edges to $v_i$ and $v_{i+1}$ where $i+1$ is taken over modulo $n$. For $1 \leq i \leq n$, let $w_i, v_i, v_i, u_i$ be the vertices in $S(SF_n)$ corresponding to the edges $w_i, v_i, v_i, v_i, v_{i+1}, w_{i+1}, v_{i+1}$ of $SF_n$, where $i+1$ is taken over modulo $n$. Suppose that $S(SF_n) \in \Omega_n$ for some $n$ with a labeling $f$. Then $N_f^+(v_i) = a = N_f^+(v_{i+1})$, implies that $f(w_i) = f(v_i)$ and consequently $N_f^+(v_i) = 0$, which is a contradiction. Hence $S(SF_n) \notin \Omega_n$ for any $n$.

**Theorem 2.16.** For $n \geq 3$, $S(SF_n) \in \Omega_n$ if $n \equiv 0 \pmod 2$.

**Proof.** Consider the sunflower $SF_n$ with vertex set $V = \{v_0, v_1, v_2, v_3, \ldots, v_n\}$ where $v_0$ is the central vertex, $v_1, v_2, v_3, \ldots, v_n$ are vertices of the cycle and $v_i$ is the vertex joined by edges to $v_i$ and $v_{i+1}$ where $i+1$ is taken over modulo $n$. For $1 \leq i \leq n$, let $w_i, v_i, v_i, u_i$ be the vertices in $S(SF_n)$ corresponding to the edges $w_i, v_i, v_i, v_i, v_{i+1}, w_{i+1}, v_{i+1}$ of $SF_n$, where $i+1$ is taken over modulo $n$. Suppose that $n \equiv 0 \pmod 2$. We define $f : V[S(SF_n)] \to V_4 \setminus \{0\}$ as :

\[
   f(w_i) = f(v_i) = f(v_i) = f(v_i) = f(v_i)^n = a \quad \text{for} \quad 1 \leq i \leq n,
\]

\[
   f(u_i) = \begin{cases} 
   b & \text{if } i \equiv 1 \pmod 2 \\
   c & \text{if } i \equiv 0 \pmod 2
   \end{cases}
\]

Clearly $f$ is an $a$-neighbourhood $V_4$-magic labeling of $S(SF_n)$. This completes the proof of the theorem.

**Corollary 2.17.** $S(SF_n) \notin \Omega_n$ for any $n \geq 3$. 

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Theorem 2.21. S(B_{m,n}) \in \Omega_a if and only if both m and n are even.

Proof. Consider the bistar B_{m,n} with vertex set \{u,v,u_i,v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} where u_i(1 \leq i \leq m) and v_j(1 \leq j \leq n) are pendant vertices adjacent to u and v respectively. Let w,u_i and v_j be vertices in S(B_{m,n}) corresponding to the edges uw,u_iw and v_jw respectively. Suppose that both m and n are even. Define f : V[S(B_{m,n})] \rightarrow V_4 \setminus \{0\} as:

f(u) = f(v) = c \text{ for } 1 \leq j \leq n
f(v) = f(u_i) = b \text{ for } 1 \leq i \leq m
f(w) = f(u'_i) = f(v'_j) = a \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.

Then f is an a-neighbourhood V_4-magic labeling of S(B_{m,n}). Hence S(B_{m,n}) \in \Omega_a. Conversely, suppose that not both m and n are even. Then either m or n is odd. Without loss of generality take m is odd. If possible, let S(B_{m,n}) \in \Omega_a with a labeling f. Then N_i^+(u) = a implies that f(u_i) = a for 1 \leq i \leq m. Then N_j^+(u) = a implies that ma + f(u) = a, consequently f(a) = 0. This is a contradiction. Hence S(B_{m,n}) \notin \Omega_a. Which completes the proof of the theorem.

Theorem 2.22. S(B_{m,n}) \notin \Omega_a for any m and n.

Proof. Proof is obvious, due to the presence of pendant vertices in S(B_{m,n}).

Theorem 2.23. S(B_{m,n}) \notin \Omega_{a,0} for any m and n.

Proof. Proof directly follows from Theorem 2.21.

Definition 2.24. Theorem 2.25. S(J(m,n)) \notin \Omega_a for any m and n.

Proof. Proof is obvious, since S(J(m,n)) has pendant vertex in it.

Corollary 2.26. S(J(m,n)) \notin \Omega_{a,0} for any m and n.

Proof. Proof directly follows from Theorem 2.24.

Definition 2.27. The graph P_2 \times P_n is called a Ladder. It is denoted by L_n.

Theorem 2.28. S(L_n) \in \Omega_a for all n \geq 1.

Proof. Consider the ladder L_n with vertex set V = \{u_i,v_i : 1 \leq i \leq n\} and edge set E = \{u_0u_1,v_0v_1 : 1 \leq i \leq n-1\} \cup \{u_nv_i : 1 \leq i \leq n\}. Let u_i,v_i be the new vertices corresponding to the edges uu_{i+1}, vv_{i+1} for 1 \leq i \leq n-1 and w_i be the vertices corresponding to the edges uu_i for 1 \leq i \leq n in S(L_n). Now define f : V[S(L_n)] \rightarrow V_4 \setminus \{0\} as:

f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \text{ (mod 2)} \\ c & \text{if } i \equiv 0 \text{ (mod 2)} \end{cases}

f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \text{ (mod 2)} \\ b & \text{if } i \equiv 0 \text{ (mod 2)} \end{cases}

f(w_i') = f(v'_i) = c \text{ for } 1 \leq i \leq n-1
f(w_i) = f(w_n) = b.

Then N_i^+(a) \equiv a. Hence S(L_n) \in \Omega_a.

Definition 2.29. The open ladder O(L_n) is the graph obtained from two paths of lengths n - 1 with V(G) = \{u_i,v_i : 1 \leq i \leq n\} and E(G) = \{u_iu_{i+1},v_iv_{i+1} : 1 \leq i < n\} \cup \{u_nv_i : 1 < i < n\}.

Theorem 2.30. S(O(L_n)) \in \Omega_a for all n \geq 2.

Proof. Consider the open ladder O(L_n) with vertex set V(G) = \{u_i,v_i : 1 \leq i \leq n\} and E(G) = \{uu_{i+1},vv_{i+1} : 1 \leq i < n\} \cup \{uu_i : 1 < i < n\}. Let u'_i, v'_i be the new vertices corresponding to the edges uu_{i+1}, vv_{i+1} for 1 \leq i < n and w_i be the vertices corresponding to uu_i for 1 < i < n in S(O(L_n)). Now define f : V[S(O(L_n))] \rightarrow V_4 \setminus \{0\} as:

f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \text{ (mod 2)} \\ c & \text{if } i \equiv 0 \text{ (mod 2)} \end{cases}

f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \text{ (mod 2)} \\ b & \text{if } i \equiv 0 \text{ (mod 2)} \end{cases}

f(w_i') = f(v'_i) = c \text{ for } 1 \leq i \leq n-1
f(w_i) = f(w_n) = b.

Then N_i^+(a) \equiv a. Hence S(O(L_n)) \in \Omega_a.
Theorem 2.31. \( S(O(L_n)) \not\in \Omega_o \) for any \( n > 2 \).

Proof. Proof is obvious, since \( S(O(L_n)) \) has pendant vertices in it. \( \square \)

Corollary 2.32. \( S(O(L_n)) \not\in \Omega_{a,0} \) for any \( n > 2 \).

Proof. Proof directly follows from Theorem 2.31. \( \square \)

Definition 2.33. The Corona \( G_1 \odot G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph obtained by taking one copy of \( G_1 \), which has \( p_1 \) vertices, and \( p_1 \) copies of \( G_2 \) and then joining the \( i \)th vertex of \( G_1 \) by an edge to every vertex in the \( i \)th copy of \( G_2 \).

Definition 2.34. The Corona \( P_n \odot K_1 \) is called the comb graph \( CB_n \).

Theorem 2.35. \( S(CB_n) \notin \Omega_o \) for any \( n \geq 2 \).

Proof. Consider \( CB_n \) with vertex set \( \{u_i, v_i : 1 \leq i \leq n\} \) and edge set \( \{u_i, v_i : 1 \leq i \leq n\} \cup \{u_i, u_{i+1} : 1 \leq i \leq n-1\} \). Let \( v_i \) and \( u_i \) be vertices in \( S(CB_n) \) corresponding to the edges \( u_i, u_{i+1} \) for \( 1 \leq i \leq n \) and \( 1 \leq i \leq n-1 \) of \( CB_n \). Suppose that \( S(CB_n) \in \Omega_o \) for some \( n \) with a labeling \( f \). Then \( N_{iw}^j(v_i) = a \) implies that \( f(v^i_j) = a \) for \( 1 \leq i \leq n \). Also, \( N_{iw}^j(u_i) = a \) implies that \( f(v^i_j) + f(u^i_j) = a \) implies that \( f(u^i_j) = 0 \). This is a contradiction. Hence \( S(CB_n) \notin \Omega_o \) for any \( n \). \( \square \)

Theorem 2.36. \( S(CB_n) \notin \Omega_o \) for any \( n \).

Proof. Proof is obvious, since \( S(CB_n) \) has pendant vertices in it. \( \square \)

Corollary 2.37. \( S(CB_n) \notin \Omega_{a,0} \) for any \( n \).

Proof. Proof directly follows from Theorem 2.36. \( \square \)

Definition 2.38. [11] A Crown graph \( C_n \) is obtained from \( C_n \) by attaching a pendant edge at each vertex of the cycle \( C_n \).

Theorem 2.39. For \( n \geq 3 \), \( S(C_n) \in \Omega_o \) if and only if \( n \equiv 0 \pmod{2} \).

Proof. Consider the crown \( C_n \) with vertex set \( \{u_i, v_i : 1 \leq i \leq n\} \), where \( u_1, u_2, u_3, \ldots, u_n \) are vertices on the cycle \( C_n \) and \( v_1, v_2, v_3, \ldots, v_n \) are pendant vertices adjacent to the vertices \( u_1, u_2, u_3, \ldots, u_n \) respectively. Let \( v_i (1 \leq i \leq n) \) be the vertices in \( S(C_n) \) corresponding to the edges \( u_i, v_i (1 \leq i \leq n) \) and \( u_i \) be the vertices corresponding to the edges \( u_i, u_{i+1} (1 \leq i \leq n) \), where \( i + 1 \) is taken over modulo \( n \). Suppose \( S(C_n) \in \Omega_o \) with a labeling \( f \). Then \( N^j_{iw}(v_i) = a \) gives either \( f(u_i) = b \) or \( f(u_i) = c \). Without loss of generality, take \( f(u_i) = b \). Then \( N^j_{iw}(u_i) = a \) implies that \( f(u_i) = c \). Also \( N^j_{iw}(u_i) = a \) implies that \( f(u_i) = b \). Proceeding like this, we should have

\[
f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}
\]

Now, \( N^j_{iw}(u_i) = a \) implies that \( f(u_i) = c \). Therefore, \( n \equiv 0 \pmod{2} \). Conversely suppose that \( n \equiv 0 \pmod{2} \). Define \( f : V[S(C_n)] \rightarrow V_4 \setminus \{0\} \) as:

\[
f(u_i) = \begin{cases} b & \text{if } i \equiv 1 \pmod{2} \\ c & \text{if } i \equiv 0 \pmod{2} \end{cases}, f(v_i) = \begin{cases} c & \text{if } i \equiv 1 \pmod{2} \\ b & \text{if } i \equiv 0 \pmod{2} \end{cases}
\]

\[
f(u_i) = f(v_i) + a \text{ for } 1 \leq i \leq n.
\]

Then \( f \) is an \( a \)-neighbourhood \( V_4 \)-magic labeling of \( S(C_n) \). This completes the proof of the theorem. \( \square \)

Theorem 2.40. \( S(C_n) \notin \Omega_o \) for any \( n \geq 3 \).

Proof. Proof is obvious, since \( S(C_n) \) has pendant vertices in it. \( \square \)

Corollary 2.41. \( S(C_n) \notin \Omega_{a,0} \) for any \( n \geq 3 \).

Proof. Proof directly follows from Theorem 2.40. \( \square \)

References


