On some contra $\mathcal{H}$-continuous functions in topological spaces

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Abstract
In this article, we first introduce contra $\mathcal{H}$-continuous functions, contra $\mathcal{H}_g$-continuous functions and study their relations with various contra continuous functions. A notions of contra $\mathcal{H}_g$-continuous function have been introduced and study the relation between contra $\mathcal{H}_g$-continuous function and contra $\mathcal{H}$-continuous functions.

Keywords
$\mathcal{H}$-closed, contra $\mathcal{H}$-continuous, contra $\mathcal{H}$-continuous and contra $\mathcal{H}_g$-continuous.

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1. Introduction

J. Jeyanthi et al. [13–15] introduced $\Lambda_r$-closed, $\Lambda_r$-continuous and Caldas et al. introduced $\lambda$-closed and $\lambda$-continuous. Levin introduced generalized closed sets developed by more generalized sets.

In the article, we first introduce contra $\mathcal{H}$-continuous functions, contra $\mathcal{H}$-continuous functions and study their relations with various contra continuous functions. A notions of contra $\mathcal{H}_g$-continuous function have been introduced and investigate the relation between contra $\mathcal{H}_g$-continuous function and contra $\mathcal{H}$-continuous functions.

2. Preliminaries

Through out paper obtained in the Topological space $(X, \tau)$ (resp. $(X, \sigma)$ and $(X, \eta)$) is denoted by TS $(X, \tau)$ (resp. TS $(X, \sigma)$ and TS $(X, \eta)$).

For a subset $C$ of a TS $X$, int$(C)$, cl$(C)$ denoted the interior, closure of $C$ respectively. And $\lambda$ symbol use this thesis $\mathcal{A}$.

For so many author introduced various definitions pre-open set [19], semi-open set [17], $\lambda$-closed set [1], $\mathcal{S}$-set and $\mathcal{H}$-closed [22], $\lambda$-kernal [5], $\Lambda_r$-closed [14], $\lambda$-Urysohn [10], Ultra Hausdroff [23] and some various continuous g-continuous [16], $\lambda$-continuous [1], $\lambda$-irresolute [8], $\lambda$-closed [6], $\Lambda_r$-continuous [15], $\Lambda_r$-open [15], $\Lambda_r$-closed [15]. Contra-continuous [9], contra-precontinuous [12], contra $\lambda$-continuous [4], contra $\Lambda_r$-continuous [13].

Definition 2.1. [22] Consent to $S$ be a subset of a TS $X$, then we define a $SS = \cap \{Q/Q \supset S, Q \in \mathcal{A} O(X, \tau)\}$.

Lemma 2.2. [21] Every $\Lambda_r$-set is $\Lambda$-set.


1. $x \in \text{Ker}(C) \iff C \cap F \neq \emptyset$ for any $F \in C(X, x)$;
2. $C \subseteq \text{Ker}(C)$ and $C = \text{Ker}(C)$ if $C$ is open;
3. If $C \subseteq B \implies \text{Ker}(C) \subseteq \text{Ker}(B)$.

Theorem 2.4. [6] If $C$ is $\mathcal{A}$-open & $\Lambda_r$-closed in TS $X \implies C$ is closed.

Theorem 2.5. [9] For a set $C \subseteq (X, \tau)$ the next conditions are equivalent:

1. $C$ is clopen
2. \( C \) is \( \alpha \)-open and closed.
3. \( C \) is nearly open & closed.

**Proposition 2.6.** [22] Consent to \( C \) be a subset of a TS \( X \). For each \( x \in X \), \( x \in \text{Hcl}(C) \) if \( fU \cap C \neq \emptyset \) for any \( \text{H-open} \) \( U \) containing \( x \).

**3. On contra \( \text{H} \)-continuous functions**

**Definition 3.1.** A function \( f : X \to Y \) is called a

1. \( \text{contra} \text{H}-\text{continuous} \) if \( f^{-1}(L) \) is \( \text{H}-\text{closed} \) in \( TS \) \( X \) for each open \( L \) of \( TS \) \( Y \).
2. \( \text{contra} \text{H}_{\text{a}} \)-continuous if \( f^{-1}(L) \) is \( \text{H}_{\text{a}} \)-closed in \( TS \) \( X \) for each \( \text{a} \)-open \( L \) of \( TS \) \( Y \).

**Example 3.2.** Consent to \( X = \{1, 2, 3, 4\} = Y \), \( \tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\} \) and \( \sigma = \{\emptyset, Y, \{1\}, \{2\}, \{1, 3\}, \{2, 3, 4\}, \{1, 2, 3\}\} \). Define a \( f : X \to Y \) by \( f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2 \). In that case \( f \) is contra \( \text{H} \)-continuous.

**Example 3.3.** Consent to \( X = \{1, 2, 3, 4\} = Y \), \( \tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 2, 3\}\} \) and \( \sigma = \{\emptyset, Y, \{1\}, \{2\}, \{1, 3\}, \{2, 3, 4\}\} \). Define a \( f : X \to Y \) by \( f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2 \). In that case \( f \) is contra \( \text{H}_{\text{a}} \)-continuous.

**Lemma 3.4.** Condition \( f : X \to Y \) is a contra \( \text{a} \)-continuous \( \iff \) contra \( \text{H} \)-continuous.

**Proof.** Consent to \( L \) be a open in \( TS \) \( Y \). In the case \( f^{-1}(L) \) is \( \text{a} \)-closed in \( TS \) \( X \) and hence \( f^{-1}(L) \) is \( \text{H} \)-closed in \( TS \) \( X \). For that reason \( f \) is contra \( \text{H} \)-continuous.

**Remark 3.5.** The reverse of above Lemma 3.4 is need not be a true.

**Example 3.6.** Consent to \( X = \{1, 2, 3, 4\} = Y \), \( \tau = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\} \) and \( \sigma = \{\emptyset, Y, \{4\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\} \). Define a \( f : X \to Y \) by \( f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3 \). In the case \( f \) is contra \( \text{H} \)-continuous but not contra \( \text{a} \)-continuous, since \( f^{-1}(\{1, 3, 4\}) = \{1, 2, 4\} \) is not \( \text{a} \)-closed.

**Lemma 3.7.** Condition \( f : X \to Y \) is a contra \( \Lambda_{\alpha} \)-continuous \( \iff \) contra \( \text{H} \)-continuous.

**Proof.** Consent to \( L \) be an open in \( TS \) \( Y \). In the case \( f^{-1}(L) \) is \( \Lambda_{\alpha} \)-closed in \( TS \) \( X \) and hence \( f^{-1}(L) \) is \( \text{H} \)-closed in \( TS \) \( X \). Therefore \( f \) is contra \( \text{H} \)-continuous.

**Remark 3.8.** The reverse of Lemma 3.7 is but not a true.

**Example 3.9.** Consent to \( X = \{1, 2, 3, 4\} = Y \), \( \tau = \{\emptyset, X, \{3\}, \{3, 4\}, \{1, 2\}, \{1, 2, 3\}\} \) and \( \sigma = \{\emptyset, Y, \{2, 4\}, \{1, 2, 4\}\} \). Define a function \( f : X \to Y \) by \( f(1) = 1, f(2) = 1, f(3) = 4 \) and \( f(4) = 2 \). Here \( f \) is contra \( \text{a} \)-continuous and thus contra \( \text{H} \)-continuous but not \( \Lambda_{\alpha} \)-continuous, since of \( f^{-1}(\{1, 2, 4\}) = \{1, 2, 3\} \) is not \( \Lambda_{\alpha} \)-closed in \( X \).

![Tree](tree.png)
Theorem 3.13. Consent to L be closed in TS Y. Because

1. $f$ is contra continuous

Consent to L be closed in Z. Since

2. L be closed in TS Z. Since

3. $f$ is contra continuous & $f$ is $H$-irresolute.

4. $g \circ f$ is contra $H$-continuous & $f$ is $H$-irresolute.

5. $g \circ f$ is $H$-continuous $\implies g$ is contra $\alpha$-continuous & $f$ is $\alpha$-continuous.

Proof.

1. Consent to L a closed set in TS Z. Because $g$ is contra continuous, $g^{-1}(L)$ is open in TS Y. Since $f$ is $H$-continuous, $f^{-1}(g^{-1}(L))$ is $H$-open in TS X. That is $(g \circ f)^{-1}(L)$ is $H$-open in TS X. Hence $g \circ f$ is contra $H$-continuous.

2. Consent to L be an open set. Because $g$ is $H$-continuous, $g^{-1}(L)$ is $H$-open in TS Y. Because Y is locally $H$-indiscrete, $g^{-1}(L)$ is $\alpha$-closed. Since $f$ is $\alpha$-continuous, $f^{-1}(g^{-1}(L))$ is $H$-closed in TS X. That is $(g \circ f)^{-1}(L)$ is $H$-closed in TS X. Thus $g \circ f$ is contra $H$-continuous.

3. Consent to L be a closed set in TS Z. Since $g$ is contra $H$-continuous, $g^{-1}(L)$ is $H$-open in TS Y. Because $f$ is $H$-irresolute, $f^{-1}(g^{-1}(L))$ is $H$-open in TS X. That is $(g \circ f)^{-1}(L)$ is $H$-open in TS X. Thus $g \circ f$ is contra $H$-continuous.

4. Consent to L be closed in TS Z. Since $g$ is contra $\alpha$-continuous, $g^{-1}(L)$ is $\alpha$-open in TS Y. Because $f$ is $\alpha$-continuous, $f^{-1}(g^{-1}(L))$ is $H$-open in TS X. That is $(g \circ f)^{-1}(L)$ is $H$-open in TS X. Thus $g \circ f$ is contra $H$-continuous.

5. Consent to L be closed in Z. Since $g$ is contra $\alpha$-continuous, $g^{-1}(L)$ is $\alpha$-open in TS Y. Since $f$ is contra $H$-continuous, $f^{-1}(g^{-1}(L))$ is $H$-closed in TS X. That is $(g \circ f)^{-1}(L)$ is $H$-closed in TS X. Hence $g \circ f$ is $H$-continuous.

3. Consent to L be closed in TS Y. Because $f$ is contra $H$-continuous, $f^{-1}(L)$ is $H$-open in TS X. Since X is a locally $H$-indiscrete, $f^{-1}(L)$ is closed in TS X. Hence $f$ is continuous.

4. Consent to L be closed in TS Y. Because $f$ is contra $H$-continuous, $f^{-1}(L)$ is $H$-open in TS X. Because X is a locally $H$-indiscrete, $f^{-1}(L)$ is $\alpha$-closed in TS X. Thus $f$ is $\alpha$-continuous.

Theorem 3.14. If a function $f : X \to Y$ is contra $\alpha$-continuous & $g : Y \to Z$ is continuous In the case $g \circ f : X \to Z$ is contra-semi continuous, where X is a $\alpha$-$S_\delta$-space.

Proof. Consent to L be a closed set in Z. Because $g$ is continuous, $g^{-1}(L)$ is closed in Y. Because $f$ is contra $\alpha$-continuous, $f^{-1}(g^{-1}(L))$ is $H$-open in X. But X is a $\alpha$-$S_\delta$-space, so we get $f^{-1}(g^{-1}(L))$ is semi-open in X. That is $(g \circ f)^{-1}(L)$ is semi-open in TS X. Hence $g \circ f$ is contra-semi continuous.

Theorem 3.15. If X is a TS and for each pair of disjoint points $p_1$ and $p_2$ in X, there exists a function $f$ of $X$ in to a $\alpha$-$Urysohn$ TS $Y : f(p_1) \neq f(p_2)$ and $f$ is contra $H$-continuous at $p_1$ and $p_2 \implies X$ is $H$-$T_2$.

Proof. Consent to $p_1$ and $p_2$ be any disjoint points in X. In the case by hypothesis, there is a $\alpha$-$Urysohn$ space and a function $f : X \to Y$, Consent to $y_1 = f(p_1)$ for $i = 1$ to 2. In the case $y_1 \neq y_2$. Since $Y$ is $\alpha$-$Urysohn$, there exists $\alpha$-open neighborhoods $T_{y_1}$ and $T_{y_2}$ of $y_1$ and $y_2$, respectively, in Y such that $\alpha cl(T_{y_1}) \cap \alpha cl(U_{y_2}) = \emptyset$. Since $f$ is contra $H$-continuous at $p_i$, there exist a $H$-open neighborhood $K_{p_i}$ of $p_i$ in X such that $f(K_{p_i}) \subset \alpha cl(U_{y_2})$, for $i = 1$ to 2. Hence, we get $K_{p_1} \cap K_{p_2} = \emptyset$. Since $\alpha cl(U_{y_1}) \cap \alpha cl(U_{y_2}) = \emptyset$. Thus X is $H$-$T_2$.

Corollary 3.16. If $f$ is contra $H$-continuous injection of a TS X in to a $\alpha$-$Urysohn$ $\implies X$ is $H$-$T_2$.

Proof. For each pair of disjoint points $p_1$ and $p_2$, f is contra $H$-continuous of X in to a $\alpha$-$Urysohn$ Y such that $f(p_1) \neq f(p_2)$, because f is injection. Thus by Theorem 3.15, X is $H$-$T_2$.

Corollary 3.17. If $f$ is contra $H$-continuous injection of a space X in to ultra Hausdorff space $Y \implies X$ is $H$-$T_2$.

Proof. Consent to $p_1$ and $p_2$ be any disjoint points in X. Since $f$ is injective and Y is ultra Hausdorff $f(p_1) \neq f(p_2)$ and there exists $L_1, L_2 \in CO(Y)$ such that $f(p_1) \in L_1, f(p_2) \in L_2$ and $L_1 \cap L_2 = \emptyset$. In the case $p_i \in f^{-1}(L_i) \in \bar{H}(X)$ for $i = 1, 2$ and $f^{-1}(L_1) \cap f^{-1}(L_2) = \emptyset$. Hence X is $H$-$T_2$.

Theorem 3.18. If $f : X \to Y$ is contra pre continuous and continuous $\implies f$ is $H$-continuous.
Proof. Consent to L be any closed set in Y. Because f is contra pre continuous and continuous, \( f^{-1}(L) \) is pre open set in X and closed in X. By Theorem 2.5, \( f^{-1}(L) \) is clopen and hence \( f^{-1}(L) \) is \( \mathcal{H} \)-open. Thus f is contra \( \mathcal{H} \)-continuous. Hence f is contra \( \mathcal{H} \)-continuous.

Lemma 3.19. If a subset C of a TS X is regular closed and open then C is \( \mathcal{H} \)-closed.

Proof. Since regular closed is closed and open is \( \Lambda \)-set, C is \( \mathcal{H} \)-closed.

Theorem 3.20. If \( f : X \to Y \) is contra-continuous and super continuous and \( g : Y \to Z \) is contra-continuous \( \implies \) their composition \( g \circ f : X \to Z \) is contra \( \mathcal{H} \)-continuous.

Proof. Consent to T be any open in TS Z. Because g is contra continuous, in the case \( g^{-1}(T) \) is closed in TS Y and f is contra-continuous and super continuous, then \( f^{-1}(g^{-1}(T)) \) is twice open and regular closed in TS X. By Lemma 3.19, \( (g \circ f)^{-1}(T) \) is \( \mathcal{H} \)-closed in TS X and hence it is a \( \mathcal{H} \)-closed set. \( \therefore, g \circ f \) is contra \( \mathcal{H} \)-continuous.

Theorem 3.21. If \( f : X \to Y \) is \( \mathcal{H} \)-continuous and \( g : Y \to Z \) is contra-precontinuous \( \implies \) their composition \( g \circ f : X \to (Z, \eta) \) is contra \( \mathcal{H} \)-continuous where TS Y is submaximal.

Proof. Consent to T be any closed set in Z. Because g is contra pre continuous, \( g^{-1}(T) \) is pre open in TS Y. Since TS Y is submaximal, \( g^{-1}(T) \) is open in TS Y. Because f is \( \mathcal{H} \)-continuous, \( f^{-1}(g^{-1}(T)) \) is \( \mathcal{H} \)-open in TS X. Therefore \( (g \circ f)^{-1}(T) \) is \( \mathcal{H} \)-open in TS X and \( g \circ f \) is contra \( \mathcal{H} \)-continuous.

Theorem 3.22. If \( f : X \to Y \) is \( \mathcal{H} \)-continuous and \( g : Y \to Z \) is contra \( \mathcal{H} \)-continuous \( \implies \) their composition \( g \circ f : X \to Z \) is contra \( \mathcal{H} \)-continuous.

Proof. Consent to T be each open set in TS Z. Because g is contra \( \mathcal{H} \)-continuous. In the case \( g^{-1}(T) \) is a \( \mathcal{H} \)-closed set in TS Y and since f is \( \mathcal{H} \)-continuous, then \( f^{-1}(g^{-1}(T)) = (g \circ f)^{-1}(T) \) is \( \mathcal{H} \)-closed in TS X. Therefore \( g \circ f \) is contra \( \mathcal{H} \)-continuous.

Theorem 3.23. If \( f : X \to Y \) and \( g : X \to Y \) be contra \( \mathcal{H} \)-continuous functions and Y is Urysohn \( \implies \) S = \{ p \in X | f(p) = g(p) \} \( \subseteq \mathcal{H} \)-closed in TS X.

Proof. Consent to p \( \in X \setminus S \). Then \( f(p) \neq g(p) \). Because Y is Urysohn, \( \exists \) a open sets L and Q : \( f(p) \in L \) and \( g(p) \in Q \) and \( cl(L) \cap cl(Q) = \emptyset \). Because f and g are contra \( \mathcal{H} \)-continuous, In the case \( f^{-1}(cl(L)) \) and \( g^{-1}(cl(Q)) \) are \( \mathcal{H} \)-open sets in TS X. Consent to T = \( f^{-1}(cl(L)) \) and \( G = g^{-1}(cl(Q)) \). In the case T and G are \( \mathcal{H} \)-open sets containing p. Set A = \( T \cap G \). In the case A is \( \mathcal{H} \)-open in TS X. Thus \( f(C) \cap g(C) = f(T \cap G) \cap g(T \cap G) = f(T) \cap g(T) \cap cl(G) = cl(L) \cap cl(Q) = \emptyset \), \( \therefore, C \cap S = \emptyset \) and so \( x \notin \mathcal{H} cl(S) \) by Proposition 2.6. Hence S is \( \mathcal{H} \)-closed in TS X.

Definition 3.24. A subset C of a TS X is called a \( \mathcal{H} \)-dense in TS X if \( \mathcal{H} cl(C) = X \).

Theorem 3.25. Consent to F : X \( \to Y \) and g : X \( \to Y \). If TS Y is Urysohn, f & g are contra \( \mathcal{H} \)-continuous & f = g on \( \mathcal{H} \)-dense set C \( \subseteq X \implies f = g \) on TS X.

Proof. Because f & g are contra \( \mathcal{H} \)-continuous and Y is Urysohn, by Theorem 3.23, \( S = \{ p \in X | f(p) = g(p) \} \) is \( \mathcal{H} \)-closed in TS X. By suppose, we've f = g on \( \mathcal{H} \)-dense set C \( \subseteq X \). Since C \( \subseteq S \) and C is \( \mathcal{H} \)-dense. \( X = \mathcal{H} cl(C) \subseteq \mathcal{H} cl(S) = S \). Thus f = g on TS X.

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