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A note on edge domination in fuzzy graphs

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Abstract

An edge dominating set $D^{'}$ of a fuzzy graph $G=(\sigma,\mu)$ is an effective edge dominating set if every edge in D' is effective. In the present work, the concept of effective edge domination in fuzzy graphs is introduced. We determine the effective edge domination number $\gamma'(G)$ of some kind of fuzzy graphs. We present some general bounds related to the effective edge domination number. In addition, this new domination is being discussed in the join of fuzzy graphs.

Keywords

Fuzzy graph, Effective edge, Edge dominating set, Effective edge domination number, Fuzzy cycle, Fuzzy path, Complete fuzzy graph.

AMS Subject Classification

05C72.

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Contents

1. Introduction

A mathematical framework to describe uncertainty in the real - life situation was first suggested by Zadeh [1, 3]. Rosenfeld [7] first introduced the idea of fuzzy graphs. Then, the theory of fuzzy graphs became a huge field of research. Applications of fuzzy graph include data mining, image segmentation, clustering, image capturing, networking, communication, planning, scheduling, and more. Crisp graph and fuzzy graphs have the same structural features. But when there is uncertainty about the vertices and edges, then the fuzzy graphs have a distinctive meaning. When the world is so full of uncertainty, fuzzy graphs appear in many real life situations. Bhutani and Rosenfeld have worked on strong arcs in fuzzy graphs [10]. Mathew and Sunitha have classifies the types of arcs in fuzzy graphs and investigated their properties [9]. Domination in fuzzy graphs is one of the widest fields which have witness a tremendous growth recently. The study of domination set in graphs begun by Ore [6] and Berge [2]. A. Somasundaram and S. Somasundaram present the concept of domination in fuzzy graphs [5].

In recent years, some kinds of domination in fuzzy graphs have been studied. Most of those belong to the vertex domination of fuzzy graphs [4]. The above observation motivates the researcher in the study of edge domination in fuzzy graphs. The concept of edge domination in graphs was introduced by Mitchell and Hedetniemi [11]. Characteristics of edge domination in graphs are obtained by Arumugam and Velammal [12]. This article introduces the concept of effective edge domination in fuzzy graphs and this parameter is being discussed in the join of two fuzzy graphs. For graph-theoretic terminology we refer to Harary,1969[8].

2. Preliminaries

Fuzzy graph $G=(\sigma,\mu)$ is a pair of function $\sigma: V\to[0,1]$ and $\mu: V\times V\to[0,1]$ such that $\mu(u,v)\leq\sigma(u)\wedge\sigma(v)$ for all u, v in V. The fuzzy graph $H(\tau,\rho)$ is called a fuzzy subgraph of $G=(\sigma,\mu)$ if $\tau(u)\leq\sigma(u)$ for all u in V and $\rho(u,v)\leq\mu(u,v)$ for all u,v in V. The underlying crisp graph of a fuzzy graph $G=(\sigma,\mu)$ is denoted by $G^*=(\sigma^*,\mu^*)$, where, $\sigma^*=\{u\in V\mid\sigma(u)>0\}$ and $\mu^*=\{(u,v)\in V\times V\mid \mu(u,v)>0\}$. A fuzzy graph $G=(\sigma,\mu)$ is called complete fuzzy graph if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all u,v in σ^*. A fuzzy graph $G=(\sigma,\mu)$ is said to be bipartite if the node-set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1,v_2)=0$ if $v_1,v_2 \varepsilon V_1$ or $v_1,v_2 \varepsilon V_2$. Further if $\mu(v_1,v_2)>0$ for all $v_1 \in V_1$ and $v_2 \varepsilon V_2$ then G is called a complete bipartite graph. A complete bipartite graph on 'n' vertices is denoted by F(K_2). An arc (u, v) of a fuzzy graph $G=(\sigma,\mu)$ is called an effective edge if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ and effective edge neighborhood of $u \in V$ is $N_e(u)=\{v \in V : edge(u,v)$ is effective $\}$. A vertex u in a fuzzy graph $G=(\sigma,\mu)$ is said to be isolated if $d(u)=0$. Let $G=(\sigma,\mu)$ be a fuzzy graph and u,v be two nodes of G. We say that u dominates v if the arc (u, v) is an effective edge. A subset D of V is called an effective dominating set of $G=(\sigma,\mu)$ if for every $v \in V-D$, there exists $u \in D$ such that u dominates v. An effective dominating set D is called a minimal effective dominating set if no proper subset of D is an effective dominating set. The minimum cardinality taken over all minimal effective dominating set is called effective domination number, denoted by $\gamma(G)$ and the corresponding dominating set is called minimum effective dominating set. An independent edge set of a fuzzy graph $G=(\sigma,\mu)$ is a subset of the edges of G such that no two edges in the subset share a vertex in G. A maximum independent edge set is an independent set of largest possible size for a given graph.

3. EDGE DOMINATION IN FUZZY GRAPHS

Definition 3.1. *Let $G=(\sigma,\mu)$ be a fuzzy graph. Let e_i and e_j be two edges of G. We say that e_i dominates e_j if e_i is an effective edge and e_i is adjacent to e_j. A subset D' of* E(G) *is called an effective edge dominating set if for every $e_j \in E(G)-D'$ there exist $e_i \in D'$ such that e_i dominates e_j.*

Definition 3.2. *An effective edge dominating set D' of a fuzzy graph $G=(\sigma,\mu)$ is called a minimal effective edge dominating set if no proper subset of D' is an edge dominating set of G.*

Definition 3.3. *The smallest number of edges in any effective edge dominating set is called effective edge domination number, denoted by $\gamma'(G)$ and the corresponding edge dominating set is called minimum effective edge dominating set.*

Remark 3.4. *(1) If D' is an minimal effective edge dominating set then* E(G) $-D'$ *need not be an edge dominating set.*

For the fuzzy graph represented in above figure, $D'=\{(u,y),(v,x)\}$ is an effective edge dominating set. But E(G) $-D'$ *is not an effective edge dominating set*

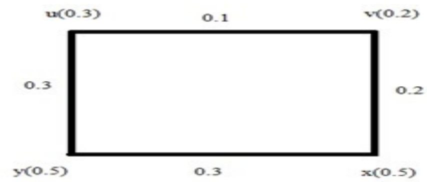

figure 3.1

(2) The set of all effective edges of a fuzzy graph $G=(\sigma,\mu)$ need not form an effective edge dominating set of G.

Here the set of all effective edges $D=\{(u_1,u_2),(u_2,u_3)\}$. But D is not an effective edge dominating set.

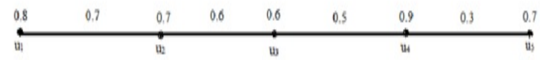

figure 3.2

(3) A fuzzy graph $G=(\sigma,\mu)$ may or may not have an effective edge dominating set.

The fuzzy graph represented in above figure has no effective edge and therefore it has no effective edge dominating set.

Definition 3.5. *A fuzzy graph $G=(\sigma,\mu)$ is said to be effective fuzzy graph if all of its edges are effective.*

Definition 3.6. *The effective degree of a vertex 'v' in a fuzzy graph $G=(\sigma,\mu)$ is defined by the number of effective edges incident at 'v' and it is denoted by $d_E(v)$.*

Definition 3.7. *A fuzzy graph $G=(\sigma,\mu)$ is said to be trivial fuzzy graph if it has no effective edge dominating set.*

Theorem 3.8. *Let $G=(\sigma,\mu)$ be a fuzzy graph such that $d_E(v)>0 \forall v \in G$ then G has an effective edge dominating set.*

Proof. Let D' be the set of all effective edges of G. If D' is not an edge dominating set, then there exist an edge (x, y) $\in E(G)-D'$ such that (x, y) is not adjacent to any edge in D'. since (x, y) is not effective and $d_E(x)>0$, the vertex x has a effective neighbor say x' and hence the edge $(x,x') \in D'$. Therefore the edge (x,y) is adjacent to the edge $(x,x') \in D'$. That is the set D' dominates the edge (x,y). This is a contradiction. □

Remark 3.9. *Converse of the theorem need not be true. For example, consider the fuzzy graph $G=(\sigma,\mu)$ represented in the following figure.*

The fuzzy graph $G=(\sigma,\mu)$ has an effective edge dominating set $D'=\{(u,v)\}$. But $d_E(x)=d_E(y)=0$

Theorem 3.10. *Let $G=(\sigma,\mu)$ be a fuzzy graph such that $d_E(v)>0 \forall v \in G$ then $\gamma(G) \leq 2\gamma'(G)$*

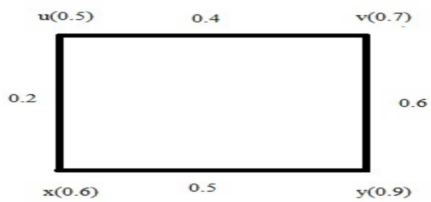

figure 3.3

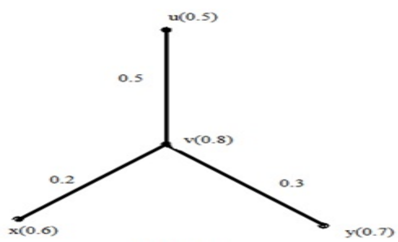

figure 3.4

Proof. Let D' be an minimum effective edge dominating set of G. Let $D = \{u_i \mid u_i$ is an end vertex of any edge in $D'\}$. Let $x \in V - D$. since $d_E(x) > 0$, there exist an effective edge say $(x, y) \in E(G)$. Suppose $y \in V - D$ then y is not an end vertex of any edge in D'. Consequently the edge (x, y) is not adjacent to any edge in D'. This is a contradiction to D' is an effective edge dominating set. Therefore $y \in D$. Therefore D is an effective vertex dominating set. Hence $\gamma(G) \leq |D| \leq 2|D'| = 2\gamma'(G)$. □

Theorem 3.11. *Let $G = (\sigma, \mu)$ be a non trivial fuzzy graph such that $\Delta(G) \leq 2$, then $\gamma'(G) \leq \gamma(G)$*

Proof. Let $D = \{u_1, u_2, u_3, \ldots\ldots u_n\}$ be an minimum effective vertex dominating set of G. Let $d_s(u_i) = \{v \in V - D$ such that (u_i, v) is an effective edge $\}$. Arrange the elements of D such that $|d_s(u_i)| \leq |d_s(u_{i+1})|$ for all $u_i \in D$. Let u_m be the first vertex such that a vertex in $d_s(u_m)$ has a neighbor in V – D. Define $v_m = \{w \in d_s(u_m) \mid w$ is adjacent to some vertex in $V - D\}$. For each $u_j, j > m$ define $v_j = \{w \in d_s(u_j) \mid w$ is not adjacent to $v_{j-k}, 1 \leq k \leq j - m\}$.In defining this v_j if tie occurs it can be broken arbitrarily. Since D is a minimum vertex dominating set, $d_s(u_i) \neq \emptyset$. For each $u_i, 1 \leq i < m$ choose any one of the vertex of $d_s(u_i)$ as v_i. Let $D' = \{(u_i, v_i), 1 \leq i \leq n\}$. It is clear that each edge in D' is effective. Let (x, y) be any edge in E(G). If the vertex x or y in D then $x = u_k$ or $y = u_k$ for some $1 \leq k \leq n$. Therefore the edge (x, y) is adjacent to the edge $(u_k, v_k) \in D'$. Therefore we assume $x \in V - D$ and $y \in V - D$. since D is an effective vertex dominating set, there exist vertices u_k and u_s in D such that (u_k, x) and (u_s, y) are effective edges. since $(x, y) \in V - D, k \geq m$ and $s \geq m$. Suppose $y \neq v_s$ then by the definition of v_s, y is adjacent to some $v_{s-i}, 1 \leq i \leq s - m$. since $\Delta(G) \leq 2, x = v_{s-i}, 1 \leq i \leq s - m$. Therefore the edge (x, y) is adjacent to the edge $(u_{s-i}, v_{s-i}) \in D'$. If the vertices x and y are dominated by the same vertex say u_k in D, then since $\Delta(G) \leq 2, x = v_k$ or $y = v_k$ Therefore the edge (x, y) is dominated by an edge in D'. Consequently $\gamma'(G) \leq n = \gamma(G)$. □

Corollary 3.12. *If $C_n(\sigma, \mu)$ be a non trivial fuzzy cycle, then $\gamma'[C_n(\sigma, \mu)] \leq \gamma[C_n(\sigma, \mu)]$*

Corollary 3.13. *If $P_n(\sigma, \mu)$ be a non trivial fuzzy path, then $\gamma'[P_n(\sigma, \mu)] \leq \gamma[P_n(\sigma, \mu)]$.*

Corollary 3.14. *If $G = (\sigma, \mu)$ be a non trivial fuzzy graph such that $\Delta(G) \leq 2$, then $\gamma'(G) \leq \gamma(G) \leq 2\gamma'(G)$.*

Proof. Proof follows by theorem 3.10 and theorem 3.11. □

Remark 3.15.

1. *If $C_n(\sigma, \mu)$ is a fuzzy cycle such that all of its edges are effective then it is known that $\gamma[C_n(\sigma, \mu)] = \lceil \frac{n}{3} \rceil$.*
2. *If $P_n(\sigma, \mu)$ be a non trivial fuzzy path such that all of its edges are effective then it is known that $\gamma[P_n(\sigma, \mu)] = \lceil \frac{n}{3} \rceil$.*
3. *If $G = (\sigma, \mu)$ is a non trivial fuzzy cycle or fuzzy path on 'n' vertices then $\gamma(G) \geq \lceil \frac{n}{3} \rceil$.*

Theorem 3.16. *Let $G = (\sigma, \mu)$ be a non trivial bipartite graph then $\gamma'(G) \leq \min\{|V_1|, |V_2|\}$, where V_1, V_2 be the partition of $V(G)$.*

Proof. Let the vertices $u_1, u_2, u_3, \ldots, u_m \in V_1$ $v_1, v_2, v_3, \ldots, v_n \in V_2$. Without loss of generality assume $m < n$. Let $d_s(u_i) = \{v_j \in V_2 \mid (u_i, v_j)$ is an effective edge $\}$. For each $u_i \in V_1$ choose a vertex v_i such that v_i is any of the vertex in $d_s(u_i)$. Let $D' = \{(u_i, v_i), 1 \leq i \leq m\}$. Let (x, y) be any edge in G. It is clear that all the edges of D' are effective. since G is bipartite, $x = u_i$ or $y = u_i$ for some i. Therefore (x, y) is adjacent to the edge $(u_i, v_i) \in D'$. Therefore D' is an effective edge dominating set. Therefore $\gamma'(G) \leq |D'| = m\cdot$ Therefore, $y'(G) \leq \min\{|V_1|, |V_2|\}$. □

Theorem 3.17. *Let $G = (\sigma, \mu)$ be a non trivial complete bipartite graph then $\gamma'(G) = \min\{|V_1|, |V_2|\}$, where V_1, V_2 be the partition of* V(G)

Proof. Let the vertices $u_1, u_2, u_3, \ldots, u_m \in V_1$ and $v_1, v_2, v_3, \ldots, v_n \in V_2$. Without loss of generality assume $m < n$. Let D' be any set of $m - 1$ effective edges of G. since $|D'| < m$, there exist vertices $u_i \in V_1$ and $v_j \in V_2$ such that u_i and v_j are not the end vertices of any edge in D'. since G is complete there is an edge (u_i, v_j) in G. But this edge (u_i, v_j) is not dominated by any edge in D'. Consequently $\gamma'(G) > m - 1$. Therefore $\gamma'(G) \geq m$. But by theorem 3.16$\gamma'(G) \leq m$ Therefore $\gamma'(G) = \min\{|V_1|, |V_2|\}$. □

Definition 3.18. *Let D be any subset of effective edges of G. The solo set of D is defined by $\chi(D) = \{u \in G \mid u$ is not an end vertex of any edge in $D\}$*

Theorem 3.19. *If D be the subset of effective edges of G such that $\chi(D) = \emptyset$, then D is an effective edge dominating set of G.*

Proof. Let $(x, y) \in E(G) - D$. since, $\chi(D) = \emptyset$, the vertex x is an end vertex of some edge in D Consequently (x, y) is adjacent to some edge in D. Therefore D is an effective edge dominating set of G. □

Remark 3.20. *1. Converse of the above theorem need not be true. For example, the fuzzy graph represented in figure 3.4 has effective edge dominating set $D' = \{(u,v)\}$. But $\chi(D') = \{x,y\} \neq \emptyset$.*

2. If $G = (\sigma, \mu)$ be a complete fuzzy graph on 'n' vertices having minimal edge dominating set D', then $\chi(D') = \emptyset$, when $n > 3$.

Theorem 3.21. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph on 'n'vertices then $\gamma'(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let D' be the maximum independent edge set of G. since G is complete all the edges of D' are effective. Let $(x,y) \in E(G) - D'$. By the choice of D', the edge (x, y) is adjacent to some edge in D'. Consequently D' is an effective edge dominating set. Therefore $\gamma'(G) \leq |D'| = \lfloor \frac{n}{2} \rfloor$ □

Theorem 3.22. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph on 'n'vertices then $\gamma'(G) \geq \lfloor \frac{n}{2} \rfloor$, provided $n > 3$.*

Proof. Let D be any set of $\lfloor \frac{n}{2} \rfloor - 1$ effective edges of G. since G contains 'n'vertices there exist at least two vertices x and y which are not the end vertices of any edges of D. since G is complete there is an edge $(x,y) \in E(G) - D$ and it is not adjacent to any edge of D. Therefore D is not an edge dominating set. Therefore $\gamma'(G) > |D| = \lfloor \frac{n}{2} \rfloor - 1$ and hence $\gamma'(G) \geq \lfloor \frac{n}{2} \rfloor$. □

Theorem 3.23. *Let $G = (\sigma, \mu)$ be a complete fuzzy graph on 'n'vertices then $\gamma'(G) = \lfloor \frac{n}{2} \rfloor$, provided $n > 3$.*

Proof. Proof follows by theorem - 3.21 and theorem - 3.22. □

4. Edge domination in Join of fuzzy graphs

Definition 4.1. *Let $G_1(\sigma_1, \mu_1)$, and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively, with $V_1 \cap V_2 = \emptyset$. The join of $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$, denoted by $G_1 + G_2$ is the fuzzy graph $G(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ on $V_1 \cup V_2$, where,*

$$(\sigma_1 + \sigma_2)u = \begin{cases} \sigma_1(u) \text{ if } u \in V_1 \\ \sigma_2(u) \text{ if } u \in V_2 \end{cases} \tag{4.1}$$

and

$$(\mu_1 + \mu_2)(uv) = \begin{cases} \mu_1(uv) \text{ if } u, v \in V_1 \\ \mu_2(uv) \text{ if } u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v) \text{ if } u \in V_1 \text{ and } v \in V_2 \end{cases} \tag{4.2}$$

Example 4.2.

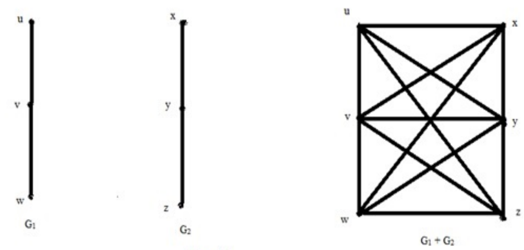

figure 4.1

Let $\sigma_1 = (u|0.6, v|0.9, w \mid 0.7)$ *and* $\mu_1 = (uv|0.5, vw|0.6)$.
Let $\sigma_2 = (x|0.8, y|0.6, z \mid 0.7)$ *and* $\mu_2 = (xy|0.4, yz|0.6)$.
$\sigma_1 + \sigma_2 = (u|0.6, v|0.9, w|0.7, x|0.8, y|0.6, z|0.7)$.
$\mu_1 + \mu_2 = (uv|0.5, vw|0.6, xy|0.4, yz|0.6, ux|0.6, uy|0.6, uz|0.6, vx|0.8, vy|0.6, vz|0.7, wx \mid 0.7, wy \mid 0.6, wz \mid 0.7)$.

Theorem 4.3. *Let $G_1(\sigma_1, \mu_1)$, and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively, with $V_1 \cap V_2 = \emptyset$. Then $\gamma'(G_1 + G_2) \leq \max\{|V(G_1)|, |V(G_2)|\}$.*

Proof. Let $x_1, x_2, \ldots . x_m$ be the vertices of G_1 and $y_1, y_2, \ldots, y_n$ be the vertices of G_2. Without loss of generality assume $m < n$. Let $D_1 = \{(x_i, y_i) \mid 1 \leq i \leq m\}$ and $D_2 = \{(x_1, y_k \mid m < k \leq n\}$. Let $D' = D_1 \cup D_2$. By the definition of $G_1 + G_2$ any edge of the form $(x_i, y_j), 1 \leq i \leq m \& 1 \leq j \leq n$ is an effective edge. Therefore all the edges of D' are effective. It is clear that $\chi(D') = \phi$ Therefore, D' is an effective edge dominating set and $\gamma'(G_1 + G_2) \leq |D'| = n$. Therefore, $\gamma'(G_1 + G_2) \leq \max\{|V(G_1)|, |V(G_2)|\}$. □

Remark 4.4. *1. Any graph $G_1 + G_2$ has an effective edge dominating set.*

2. If D_1 and D_2 are effective edge dominating sets of G_1 and G_2 respectively, then $D_1 \cup D_2$ need not be an effective edge dominating set of $G_1 + G_2$.

Theorem 4.5. *Let $G_1(\sigma_1, \mu_1)$, and $G_2(\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \emptyset$. If $|V(G_1)| = m > 3$ and $|V(G_2)| = n > 3$, then $\gamma'(G_1 + G_2) \leq \gamma'(G_1) + \gamma'(G_2)$.*

Proof. Let D_1 and D_2 be minimum effective edge dominating sets of G_1 and G_2 respectively. Let $D' = D_1 \cup D_2$. since G_1 and G_2 are complete, $\chi(D_1) = \phi$ and $\chi(D_2) = \phi \cdot \chi(D') = \chi(D_1 \cup D_2) = \chi(D_1) \cup \chi(D_2) = \phi$. Therefore, D' is an effective edge dominating set of $G_1 + G_2$. Therefore, $\gamma'(G_1 + G_2) \leq |D'| = \gamma'(G_1) + \gamma'(G_2)$ □

5. Conclusion

In this paper, relation between vertex dominating set and effective edge dominating set of a fuzzy graph is given. The condition for the existence of effective edge dominating set of a fuzzy graph is discussed. The effective edge domination number of complete fuzzy graphs and complete fuzzy bipartite graphs are obtained. We have found some bounds for $\gamma'(G)$. Also, we explored this domination in the join of fuzzy graphs.

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