$r_c$-operator on topological spaces

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Abstract
In this paper a new operator called $r_c$-operator on topological spaces is introduced. Conditions for the operator to be an expansive, shrinking and invariant operator is determined. It is also shown that regular closed sets are fixed points of this operator.

Keywords
$r_c$-operator, Closure, Interior, regular closed sets.

AMS Subject Classification
54C10.

1. Introduction

M.H. Stone introduced the concept of regular open set[4] in 1937. R.C. Jain[1] in 1980, worked on regularly open sets in Topology on his thesis. In this paper an attempt is done to find an operator for which complement of regular open set called regular closed set is a fixed point. In section 2 , preliminary ideas are given. In section 3 , $r_c$-operator is defined. Section 4 , discusses about properties of $r_c$-operator and finds its fixed points.

2. Preliminary Ideas

Let $(X, \tau)$ be a topological space. $(X, \tau)$ is abbreviated as $X$ . For a set $A$, $\bar{A}$ denotes the closure of $A$ and $A^\circ$ denotes its interior.

2.1 Definition[4]

A subset $A$ of $X$ is

(i.) regular open, if $A = A^\circ$.

(ii.) regular closed, if $A = \bar{A}$.

2.2 Properties of regular closed sets

(i.) Every regular closed set is closed.

(ii.) If $A$ and $B$ are regular closed sets, then $A \cup B$ is regular closed.

(iii.) If $A$ and $B$ are regular closed sets, then $A \cap B$ need not be regular closed.

3. $r_c$-operator

Definition 3.1. Let $(X, \tau)$ be a topological space. The operator $r_c$ defined on $P(X)$ by $r_c(A) = \bar{A}^\circ$ is known as $r_c$-operator.

Example 3.2. Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $r_c(\{a\}) = \{a, c\}, r_c(\{b\}) = \{b, c\}, r_c(\{a, b\}) = X, r_c(\{c\}) = \phi, r_c(\{b, c\}) = \{b, c\}

Example 3.3. Consider $(R, \tau)$, where $R$ is the set of real numbers and $\tau$ is the usual topology. Then,

1. $r_c(\{(a, b)\}) = [a, b]$ for any open interval $(a, b)$ in $R$.

2. $r_c(\{[a, b]\}) = [a, b]$ for any closed interval $[a, b]$ in $R$.

3. $r_c(\{(a, b)\}) = [a, b] = r_c(\{(a, b)\})$ for any half open intervals in $R$. 

References

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4. Properties of $r_c$-operator

Theorem 4.1.

1. For any subset $A$ of $X$, $A^c \subseteq r_c(A)$. 
2. If $A$ is an open set, then $r_c$ is an expansive operator. That is $A \subseteq r_c(A)$ for any open set $A$. 
3. For any subset $A$ of $X$, $r_c(A) \subseteq \overline{A}$. 
4. If $A$ is a closed set, then $r_c$ is a shrinking operator. That is $r_c(A) \subseteq A$, for any closed set $A$. 
5. The operator $r_c$ is Idempotent. That is $r_c(r_c(A)) = r_c(A)$

Proof. 1. $A^c \subseteq \overline{A}$ by definition of Closure of a set.

2. $A^c \subseteq r_c(A)$. (by 1).

3. $A^c \subseteq \overline{A}$. (by 2).

4. $r_c(A) \subseteq \overline{A}$. (by 3).

5. $r_c(r_c(A)) = \overline{A^c}$

Theorem 4.2.

1. For any subset $A$ of $X$, $r_c(A^c) = r_c(A)$. 
2. If $A \subseteq B$, then $r_c(A) \subseteq r_c(B)$. 
3. $r_c(A \cup B) \supseteq r_c(A) \cup r_c(B)$, where $A, B \subseteq X$ 
4. $r_c(A \cap B) \subseteq r_c(A) \cap r_c(B)$, where $A, B \subseteq X$

Proof.

1. $r_c(A^c) = \overline{A^c}$

2. $A \subseteq B \Rightarrow A^c \subseteq B^c$

3. $A \subseteq A \cap B \Rightarrow r_c(A) \subseteq r_c(A \cap B)$

Theorem 4.3.

1. Regular closed sets are fixed points of $r_c$-operator. That is, $r_c(A) = A$. 
2. $\emptyset$ and $X$ are fixed points of $r_c$-operator. That is, $r_c(\emptyset) = \emptyset$, $r_c(X) = X$. 

Proof. 1. If $A$ is a regular closed set $\overline{A} = A$.

2. Trivial.

Theorem 4.4.

1. If $A$ and $B$ are non-empty regular closed sets, then $r_c(A \cup B) = A \cup B$. 
2. If $A$ and $B$ are non-empty regular closed sets, then $r_c(A \cap B) \neq A \cap B$. 

Proof. 1. Union of regular closed sets is regular closed. $r_c(A \cup B) = A \cup B$. 

2. Intersection of regular closed sets need not be regular closed.

Example 4.5. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. 

$A = \{b, c\}$ and $B = \{a, c\}$ are regular closed sets. 

$A \cap B = \{c\}$

$r_c(A \cap B) = r_c(\{c\}) = \emptyset$

$r_c(A) = r_c(\{b, c\}) = \{b, c\}$ $r_c(B) = r_c(\{a, c\}) = \{a, c\}$

$r_c(A) \cap r_c(B) = \{c\}$

Hence $r_c(A \cap B) \neq r_c(A) \cap r_c(B)$. 

References

