1-Harmonious coloring of triangular snakes

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Abstract
In this article, we discuss the 1-harmonious coloring and investigate the 1-harmonious chromatic number of triangular snakes and alternate triangular snakes. We also find some relations between the 1-harmonious chromatic number of triangular snakes and alternate triangular snakes.

Keywords
1-harmonious coloring, triangular snakes, alternate triangular snakes.

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05C15, 05C75.

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1. Introduction
Throughout this paper, we considered only finite and undirected graphs without any loops or multiple edges. A proper vertex coloring of a graph $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ in which if $u, v \in V(G)$ are adjacent, then $c(u) \neq c(v)$ and if this coloring uses at most $k$ colors is known as $k$-coloring. The minimum number of colors are required for this coloring is called its chromatic number, and is generally denoted by $\chi(G)$. The 1-harmonious coloring [4] is a kind of vertex coloring such that the color pairs of end vertices of every edge are different only for adjacent edges and a minimum number of colors are required for this coloring is called the 1-harmonious chromatic number, denoted by $h_1(G)$. A triangular snake [2, 5–7, 9, 10] is a triangular cactus whose block-cutpoint graph is a path (a triangular snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, \ldots, n - 1$). A double triangular snake graph $D(T_n)$ consists of two triangular snakes that have a common path, a triple triangular snake consists of three triangular snakes with a common path and consequently $k$-triangular snake graph $k(T_n)$ consists of $k$ triangular snakes with a common path. A double alternate triangular snake graph $D(AT_n)$ consists of two alternate triangular snakes with a common path, a triple alternate triangular snake consists of three alternate triangular snakes with a common path and consequently $k$-alternate triangular snake graph $k(AT_n)$ consists of $k$ alternate triangular snakes with a common path. In this paper, we study the 1-harmonious coloring with the chromatic number of above mentioned triangular snakes and find some relations between the 1-harmonious chromatic number of these snakes.
That is, a triple triangular snake is obtained from a path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to a new vertex $v_i$ for $i = 1, 2, ..., n - 1$ and to a new vertex $w_i$ for $i = 1, 2, ..., n - 1$.

**Definition 2.4** ([2, 3, 5–8]). An alternate triangular snake $A_{T_n}$ is the graph obtained from a path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to new vertex $v_i$ for $i = 1, 2, ..., n - 1$ (that is, every alternate edge of a path is replaced by cycle $C_5$).

**Definition 2.5** ([2, 3, 5–8]). A double alternate triangular snake $D(A_{T_n})$ is obtained from a path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to new vertex $v_i$ for $i = 1, 2, ..., n - 1$ and $w_i$ for $i = 1, 2, ..., n - 1$.

**Definition 2.6** ([2, 5, 9]). A triple alternate quadrilateral snake $T(A_{T_n})$ is obtained from a path $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to new vertices $v_i$ for $i = 1, 2, ..., n - 1$ and $x_i$ for $i = 1, 2, ..., n - 1$.

Throughout the paper we consider $n$ as the number of vertices of path $P_n$.

### 3. 1-Harmonious Coloring of Triangular Snakes

**Theorem 3.1.** For $n \geq 3$, triangular snake $T_n$, the 1-harmonious chromatic number, $h_1(T_n) = \Delta(T_n) + 1$.

**Proof.** Let us consider the path graph $P_n$ with $n$ vertices $u_1, u_2, ..., u_n$ and $T_n$ as the triangular snake with maximum degree, $\Delta(T_n) = 4$. Let the vertices of $T_n$, $V(T_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n - 1\}$ and the edges of $T_n$, $E(T_n) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_i, u_iw_i : 1 \leq i \leq n - 1\}$. The number of vertices and edges in $T_n$ are $2n - 1$ and $3n - 3$ respectively. Now we split the proof into three cases.

**Case 1:** Suppose $n = 3k$. Define coloring $c : V(T_n) \rightarrow \{1, 2, 3, 4, 5\}$ for $n \geq 3$ by $c(u_i) = 1$ ($i = 1, 4, 7, ..., n - 2$), $c(u_i) = 2$ ($i = 2, 5, 8, ..., n - 1$), and $c(u_i) = 3$ ($i = 3, 6, 9, ..., n$). Two sub cases arise here for even $n$ and odd $n$.

**Sub case 1:** If $n$ is odd, $c(v_i) = 4$ ($i = 1, 3, 5, ..., n - 2$), $c(v_i) = 5$ ($i = 2, 4, 6, ..., n - 1$).

**Sub case 2:** If $n$ is even, $c(v_i) = 4$ ($i = 1, 3, 5, ..., n - 1$), $c(v_i) = 5$ ($i = 2, 4, 6, ..., n - 2$).

Vertices $u_2, u_3, ..., u_{n-1}$ are of maximum degree 4 whereas the degree of $u_1, u_n$ is 2, $u_i$ is adjacent to $u_{i+1}$ ($1 \leq i \leq n - 1$) and vertices $u_i$ ($1 \leq i \leq n$) are adjacent to $v_j$ ($1 \leq j \leq n - 1$). Therefore 5 colors are to be needed to color $T_n$. From figure 1, clearly we find that for each vertex, adjacent vertices are colored with different color. Therefore, $h_1(T_n) = 5$.

**Case 2:** Suppose $n = 3k + 1$. Define coloring $c : V(T_n) \rightarrow \{1, 2, 3, 4, 5\}$ for $n \geq 3$ by $c(u_i) = 1$ ($i = 1, 4, 7, ..., n$), $c(u_i) = 2$ ($i = 2, 5, 8, ..., n - 2$), $c(u_i) = 3$ ($i = 3, 6, 9, ..., n - 1$). Again two sub cases arises for even $n$ and odd $n$, sub-cases and remaining procedure can be done as describe in case 1.

**Case 3:** Suppose $n = 3k + 2$. Define coloring $c : V(T_n) \rightarrow \{1, 2, 3, 4, 5\}$, for $n \geq 3$ by $c(u_i) = 1$ ($i = 1, 4, 7, ..., n - 1$), $c(u_i) = 2$ ($i = 2, 5, 8, ..., n - 2$), $c(u_i) = 3$ ($i = 3, 6, 9, ..., n - 1$), $c(v_i) = 4$ ($i = 1, 3, 5, ..., n - 1$), $c(v_i) = 5$ ($i = 2, 4, 6, ..., n - 2$), and $c(w_i) = 6$ for ($i = 1, 3, 5, ..., n - 1$), $c(w_i) = 7$ ($i = 2, 4, 6, ..., n - 2$).

...
we follow the procedure as described in case 1. In all three

Again two sub cases arises for even

Case 2: Suppose $n = 3k + 1$ Define coloring $c : V(DT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7\} \text{ for } n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 2)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 1)$. Again two sub cases arises for even $n$ and odd $n$, these subcases and remaining procedure can be done as described in case 1. Figure 5 shows the coloring for $DT_4$.

Case 3: Suppose $n = 3k + 2$. Define coloring $c : V(DT_k) \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$, for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 1)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 2)$. Here again two sub cases arises for even $n$ and odd $n$, for that we follow the procedure as described in case 1. In all three cases, 1-harmonious chromatic number, $h_1(DT_n) = 7$. Figure 6 shows the coloring for $T_5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{1-harmonious coloring of $DT_6$, $h_1(DT_6) = 7$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{1-harmonious coloring of $DT_4$, $h_1(DT_4) = 7$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{1-harmonious coloring of $DT_5$, $h_1(DT_5) = 7$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{1-harmonious coloring of $TT_6$, $h_1(TT_6) = 9$}
\end{figure}

**Theorem 3.3.** For $n \geq 3$, triple triangular snake $TT_n$, the 1-harmonious chromatic number, $h_1(TT_n) = \Delta(TT_n) + 1$.

**Proof.** Let us consider the path graph $P_n$ with $n$ vertices $u_1, u_2, \ldots, u_n$ and $TT_n$ as the triangular snake with maximum degree, $\Delta(TT_n) = 8$. Let the vertices of $TT_n$, $V(TT_n) = \{u_i : 1 \leq i \leq n\} \cup \{w_i, v_i : 1 \leq i \leq n - 1\}$ the edges of $TT_n$ $E(TT_n) = \{u_iu_{i+1} : 1 \leq i \leq n\} \cup \{u_iw_i, v_iu_{i+1}, u_iw_i, w_iu_{i+1}, u_iw_i, x_iu_{i+1} : 1 \leq i \leq n - 1\}$. The number of vertices and edges in $TT_n$ are $4n - 3$ and $7n - 7$ respectively. Now split the proof into following three cases.

**Case 1:** Suppose $n = 3k$. Define coloring $c : V(TT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n - 2)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 1)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n)$. Two sub cases arise here for even $n$ and odd $n$.

**Sub case 1:** If $n$ is odd, $c(v_i) = 4 \ (i = 1, 3, 5, \ldots, n - 2)$, $c(v_i) = 5 \ (i = 2, 4, 6, \ldots, n - 1)$, $c(w_i) = 6 \ (i = 1, 3, 5, \ldots, n - 2)$, $c(w_i) = 7 \ (i = 2, 4, 6, \ldots, n - 1)$, $c(x_i) = 8 \ (i = 1, 3, 5, \ldots, n - 2)$, $c(x_i) = 9 \ (i = 2, 4, 6, \ldots, n - 1)$.

**Sub case 2:** If $n$ is even, $c(v_i) = 4 \ (i = 1, 3, 5, \ldots, n - 1)$, $c(v_i) = 5 \ (i = 2, 4, 6, \ldots, n - 2)$, $c(w_i) = 6 \ (i = 1, 3, 5, \ldots, n - 1)$, $c(w_i) = 7 \ (i = 2, 4, 6, \ldots, n - 2)$, $c(x_i) = 8 \ (i = 1, 3, 5, \ldots, n - 1)$, $c(x_i) = 9 \ (i = 2, 4, 6, \ldots, n - 2)$.

Vertices $u_2, u_3, \ldots, u_{n-1}$ are of maximum degree 8 whereas the degree of $u_1, u_n$ is 4, $u_i$ is adjacent to $u_{i+1}$ $(1 \leq i \leq n - 1)$ and vertices $u_i$ $(1 \leq i \leq n)$ are adjacent to $v_j$, $w_j$ $(1 \leq j \leq n - 1)$. Therefore 7 colors are to be needed to color $DT_n$. From figure 4, clearly we find that for each vertex, the adjacent vertices are colored with different color. Therefore, $h_1(DT_n) = 7$.

**Case 2:** Suppose $n = 3k + 1$ Define coloring $c : V(TT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 2)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 1)$.

**Case 3:** Suppose $n = 3k + 2$. Define coloring $c : V(TT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n - 2)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 1)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 2)$. Again two sub cases arises for even $n$ and odd $n$, these subcases and remaining procedure can be done as described in case 1. From figure 7, clearly we find that for each vertex, the adjacent vertices are colored with different color. Therefore, $h_1(TT_n) = 9$.

**Case 2:** Suppose $n = 3k + 1$ Define coloring $c : V(TT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 2)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 1)$.

Again two sub cases arises for even $n$ and odd $n$, these subcases and remaining procedure can be done as described in case 1. Figure 8 shows the coloring for $TT_4$.

**Case 3:** Suppose $n = 3k + 2$. Define coloring $c : V(TT_n) \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for $n \geq 3$ by $c(u_i) = 1 \ (i = 1, 4, 7, \ldots, n - 2)$, $c(u_i) = 2 \ (i = 2, 5, 8, \ldots, n - 1)$, $c(u_i) = 3 \ (i = 3, 6, 9, \ldots, n - 2)$. Here again two sub cases arises for even $n$ and odd $n$, for that we follow the procedure as describe in case 1. In all three
Theorem 3.4. For $n \geq 3$, $k$-triangular snake $kT_n$, the 1-harmonious chromatic number, $h_1(kT_n) = \triangle(kT_n) + 1$.

Proof. Consequently, it is obvious from above theorems. □

4. 1-Harmonious Coloring of Alternate Triangular Snakes

Theorem 4.1. For $n \geq 4$, alternate triangular snake $AT_n$, the 1-harmonious chromatic number, $h_1(AT_n) = \triangle(AT_n) + 1$.

Proof. Let us consider the path graph $P_n$ with $n$ vertices $u_1, u_2, \ldots, u_n$ and $AT_n$ as the alternate triangular snake with maximum degree, $\triangle(AT_n) = 3$.

Let the vertices of $AT_n$, $V(AT_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq \frac{n}{2}\}$ and the edges of $AT_n$, $E(AT_n) = \{u_iu_{i+1} : 1 \leq i \leq n\} \cup \{u_iv_i, v_iu_{i+1} : 1 \leq i \leq n-1\}$. The number of vertices and edges in $AT_n$ are $\frac{3n}{2}$ and $2n - 1$ respectively. Define coloring $c : V(AT_n) \rightarrow \{1, 2, 3, 4\}$. Three case are arises here; for $n = 6k$, $n = 6k - 2$ and $n = 6k + 2$. Remaining proof and coloring process may be followed as discussed in the section 3.

\[ \triangle(D(AT_n)) = 4 \]

Figure 10, 11 and 12 shows the coloring for $n = 6k$, $n = 6k - 2$ and $n = 6k + 2$ respectively. Hence the result. □

Theorem 4.2. For $n \geq 4$, double alternate triangular snake $D(AT_n)$, the 1-harmonious chromatic number, $h_1(D(AT_n)) = \triangle(D(AT_n)) + 1$.

Proof. Let us consider the path graph $P_n$ with $n$ vertices $u_1, u_2, \ldots, u_n$ and $D(AT_n)$ as the double alternate triangular snake with maximum degree, $\triangle(D(AT_n)) = 5$.

\[ \triangle(D(AT_n)) = 5 \]

Figure 13, 14 shows the coloring for $n = 6k$, $n = 6k - 2$ and $n = 6k + 2$ respectively. Hence the result. □
respectively. Define coloring $c: V(D(AT_n)) \to \{1, 2, 3, 4, 5\}$. Three case are arises here: for $n = 3k, n = 6k - 2$ and $n = 6k + 2$. Remaining proof and coloring process may be followed as discussed in the section 3. Figure 13, 14 and 15 show the coloring for $n = 3k$, $n = 6k - 2$ and $n = 6k + 2$ respectively. Hence the result.

**Theorem 4.3.** For $n \geq 4$, triple alternate triangular snake $T(AT_n)$, the 1-harmonious chromatic number, $h_1(T(AT_n)) = \Delta(T(AT_n)) + 1$.

**Proof.** Let $P_n$ Let us consider the path graph $P_n$ with $n$ vertices $u_1, u_2, \ldots, u_n$ and $T(AT_n)$ as the triple alternate triangular snake with maximum degree, $\Delta(T(AT_n)) = 5$.

**Figure 15.** 1-harmonious coloring of $D(AT_5)$, $h_1(D(AT_5)) = 5$.

Let the vertices of $T(AT_n)$,

$$V(T(AT_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i : 1 \leq i \leq \frac{n}{2}\}$$

and the edges of $T(AT_n)$, $E(T(AT_n)) = \{u_iu_{i+1} : 1 \leq i \leq n\} \cup \{u_iw_i, v_iw_{i+1}, w_ix_{i+1}, u_ix_i, x_{i}u_{i+1} : 1 \leq i \leq n - 1\}$. The number of vertices and edges in $T(AT_n)$ are $\frac{3n}{2}$ and $4n - 1$ respectively. Define coloring $c: V(T(AT_n)) \to \{1, 2, 3, 4, 5, 6\}$ Three case are arises here: for $n = 3k, n = 6k - 2$ and $n = 6k + 2$. Remaining proof and coloring process may be followed as discussed in the section 3. Figure 16, 17 and 18 show the coloring for $n = 3k$, $n = 6k - 2$ and $n = 6k + 2$ respectively. Hence the result.

**Theorem 4.4.** For $n \geq 4$, $k$-alternate triangular snake $kAT_n$, the 1-harmonious chromatic number, $h_1(kAT_n) = \Delta(kT_n) + 1$.

**Proof.** Consequently, it is obvious from above theorems.

**5. Relations Between the 1-Harmonious Chromatic Number of Triangular and Alternate Triangular Snakes**

From section 3 and 4, we observed the following relations between the 1-harmonious chromatic number of these triangular and alternate triangular snakes:

- $h_1(T_n) = h_1(AT_n) + 1$.
- $h_1(DT_n) = h_1(D(AT_n)) + 2$.
- $h_1(TT_n) = h_1(T(AT_n)) + 3$ and so on... consequently.
- $h_1(kT_n) = h_1(kAT_n) + k$.

**6. Conclusions**

In this article, we discuss the 1-harmonious coloring and find the 1-harmonious chromatic number of triangular and
alternate triangular snakes i.e.

\[ h_1(T_n) = \triangle (T_n) + 1, \]
\[ h_1(DT_n) = \triangle (DT_n) + 1, \]
\[ h_1(TT_n) = \triangle (TT_n) + 1, \]
\[ h_1(kT_n) = 2k + 3, \]
\[ h_1(AT_n) = \triangle (AT_n) + 1, \]
\[ h_1(D(AT_n)) = \triangle (D(AT_n)) + 1, \]
\[ h_1(T(AT_n)) = \triangle (T(AT_n)) + 1 \]
\[ h_1(kAT_n) = \triangle (kT_n) + 1. \]

We also find the relations between 1-harmonious chromatic number of these snakes i.e. \( h_1(T_n) = h_1(AT_n) + 1, h_1(DT_n) = h_1(D(AT_n)) + 2, h_1(TT_n) = h_1(T(AT_n)) + 3 \) and so on. Consequently, \( h_1(kT_n) = h_1(kAT_n) + k. \)

References