Abstract
There are plenty of topological indices used in chemistry to study the chemical behavior and physical properties of molecular graphs. In the literature, several results are computed for degree based topological indices like “first Zagreb index, second Zagreb index, modified second Zagreb index, generalized Randić index, inverse Randić index, symmetric sum division index, harmonic index, inverse sum index, augmented Zagreb index”. In this paper, we have investigated the aforesaid degree based topological indices for Hanoi graph and generalized wheel graph with the help of $M$-polynomial.

Keywords
Topological indices, $M$-polynomial, Hanoi graph $H_n$, generalized wheel graph $W^m$.

AMS Subject Classification
05C07, 05C09, 05C31, 05C76, 05C92.

In 1972, Gutman & Trinajstić [7] proposed the “first Zagreb index” $M_1(G)$ and they found “second Zagreb index” $M_2(G)$ in 1975 (see [8]). These were named Zagreb group indices which are defined by $M_1(G) = \sum_{v \in V} d^2(v_i) = \sum_{v_i \in V} d(v_i) d(v_j)$ and $M_2(G) = \sum_{v_i \in E} d(v_i) d(v_j)$. The theoretical description and properties of two Zagreb indices can be obtained from several papers and articles (see [9], [10] and [11]). $M_1(G)$ and $M_2(G)$ are the “modified first and second Zagreb index” of a graph $G$ respectively, which are defined by

$$M_1(G) = \sum_{v \in V} \frac{1}{d^2(v_i)}$$ and $$M_2(G) = \sum_{v_i \in E} \frac{1}{d(v_i) d(v_j)}.$$

In 2015, Deutsch & Klavžar [4] proposed $M$-polynomial, by which several topological indices based on degree, were determined in [5] and [6]. The $M$-polynomial is a recent polynomial containing vast information about degree based topological indices of a graph. The $M$-polynomial of a graph $G$ is denoted as $M(G : p, q)$ and defined as $M(G : p, q) = \sum_{\delta \leq i \leq \Delta} m_{i,p}(G) p^i q^{\delta-i}$, where $\delta = \min\{d_v : v \in V\}$, $\Delta = \max\{d_v : v \in V\}$ and $m_{i,p}(G)$ is the number of edges $uv$ such that $d(u) = i$ and $d(v) = j$.

1. Introduction
In chemical graph theory, atoms are represented by vertices and chemical bonds are represented by edges. We assume $G = (V, E)$ be a graph, where $V$ is the set of objects called vertices and $E$ is the set of unordered pair of elements of $V$ called edges of the graph $G$. $d(v)$ denotes the degree of vertex $v$ is the number of edges incident on $v$ in a graph $G$. A topological index is a graph invariant which is mostly applicable in chemistry. There are many degree based graph invariants such as Zagreb index, Randić index, SSD index, inverse sum index, ABC index, harmonic index etc., have been studied in the literature (see [1], [2] and [3]).
connectivity index” which is defined by

$$ABC(G) = \sum_{v_i,v_j \in E} \sqrt{\frac{d(v_i) + d(v_j) - 2}{d(v_i)d(v_j)}}.$$  

It is useful to find the thermodynamical properties of alkanes. It has been revised by Furtula et al. in [13] and they put forward a new index “augmented Zagreb index”. It is defined as

$$A(G) = \sum_{v_i,v_j \in E} \left(\frac{d(v_i) + d(v_j) - 2}{d(v_i)d(v_j)}\right)^3.$$  

In 1975, Randić [14] unveiled the degree based graph invariant “Randić index”. It is defined by

$$R(G) = \sum_{v_i,v_j \in E} \frac{1}{\sqrt{d(v_i)d(v_j)}}.$$  

In 1998, Bollobás & Erdős [15] and Amic et al. [16] introduced the generalized Randić index. It is defined as

$$R_{\alpha}(G) = \sum_{v_i,v_j \in E}(d(v_i)d(v_j))^{\alpha},$$  

where \(\alpha\) is a non zero real number. For more details about generalized randić index, we refer the book [17]. The inverse Randić index of a graph G is defined by

$$RR_{\alpha}(G) = \sum_{v_i,v_j \in E}(d(v_i)d(v_j))^{-\alpha}.$$  

In 1987, Fajtlowicz [18] introduced the harmonic index. It is defined as

$$H(G) = \sum_{v_i,v_j \in E} \frac{2}{d(v_i) + d(v_j)}.$$  

The relationship between the graph’s harmonic index and the eigen values was studied by Favaron et al. in [19]. The maximum and minimum values of harmonic index on simple connected graphs were determined by Zhong in [20]. The inverse sum index of a graph G is defined as

$$I(G) = \sum_{v_i,v_j \in E} \frac{d(v_i)d(v_j)}{(d(v_i)+ d(v_j))}.$$  

The values of inverse sum index across connected graphs, chemical graphs, trees and chemical trees were determined in [21]. Vukicevic [22] defined the symmetric sum division index of a graph G as

$$SSD(G) = \sum_{v_i,v_j \in E} \frac{d^2(v_i) + d^2(v_j)}{d(v_i)d(v_j)}.$$  

2. Preliminaries

The aforesaid topological indices are also determined using M-polynomial of a graph. Let \(F(p,q)\) be a M-polynomial of the graph \(G\) then

(i) first Zagreb index

$$M_1(G) = \left\{[D_p + D_q]F(p,q)\right\}_{p=q=1},$$  

(ii) second Zagreb index

$$M_2(G) = \left\{[D_qD_pF(p,q)]\right\}_{p=q=1},$$  

(iii) modified second Zagreb index

$$mM_2(G) = \left\{S_pS_qF(p,q)\right\}_{p=q=1},$$  

(iv) generalized Randić index

$$R_{\alpha}(G) = \left\{[D_p^\alpha D_q^\alpha F(p,q)]\right\}_{p=q=1},$$  

(v) inverse Randić index

$$RR_{\alpha}(G) = \left\{[S_p^\alpha S_q^\alpha F(p,q)]\right\}_{p=q=1},$$  

(vi) symmetric sum division index

$$SSD(G) = \left\{|(S_pD_p + S_qD_q)F(p,q)|\right\}_{p=q=1},$$  

(vii) harmonic index

$$H(G) = 2[S_pJF(p,q)]_{p=q=1},$$  

(viii) inverse sum index

$$I(G) = \left\{|S_pJ^pD_qF(p,q)|\right\}_{p=q=1},$$  

(ix) augmented Zagreb index

$$A(G) = \left\{|S_p^3Q - 2J^2D_q^3F(p,q)|\right\}_{p=q=1}. $$  

3. Main Results

3.1 Topological indices of Hanoi graph

The Hanoi graph \(H_n\) can be created by taking the vertices as odd binomial coefficients of Pascal’s triangle calculated on the integers from 0 to \(2^n - 1\) and drawing an edge when coefficients are together diagonally or horizontally. The Hanoi graphs \(H_2\) and \(H_3\) are shown in Figure 1. A Hanoi graphs \(H_n\) has \(3^n\) number of vertices and \(\frac{3}{2}(3^n - 1)\) number of edges.

**Theorem 3.1.** Let \(H_n\) be a Hanoi graph then the M-polynomial of \(H_n\) is given by

$$M(H_n : p,q) = 6p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3.$$
The edge set of $H_n = (V, E)$ can be partitioned as:

$E_{[2,3]} = \{uv \in E : d(u) = 2, d(v) = 3\}$,

$E_{[3,3]} = \{uv \in E : d(u) = 3, d(v) = 3\}$.

Now,

$|E_{[2,3]}| = 6 = m_{2,3}$,

$|E_{[3,3]}| = \frac{3}{2}(3^n - 5) = m_{3,3}$.

Thus, the $M$-polynomial of $H_n$ is

$$M(H_n : p, q) = \sum_{0 \leq i, j \leq 3} m_{i,j}(H_n)p^i q^j$$

$$= m_{2,3}(H_n)p^2 q^3 + m_{3,3}(H_n)p^3 q^3$$

$$= 6p^2 q^3 + \frac{3}{2}(3^n - 5)p^3 q^3.$$  \hfill (3.1)

We denote $M$-polynomial of a Hanoi graph $H_n$ by $F(p, q)$. i.e.,

$$M(H_n : p, q) = 6p^2 q^3 + \frac{3}{2}(3^n - 5)p^3 q^3 = F(p, q).$$  \hfill (3.2)

Now, we compute the following expressions:

$$D_q F(p, q) = \frac{\partial}{\partial q} \{6p^2 q^3 + \frac{3}{2}(3^n - 5)p^3 q^3\}$$

$$= 12p^2 q^3 + \frac{9}{2}(3^n - 5)p^3 q^3,$$  \hfill (3.3)

$$S_p F(p, q) = \int_0^p \frac{1}{t} \{6p^2 q^3 + \frac{3}{2}(3^n - 5)p^3 q^3\} dt$$

$$= 3p^2 q^3 + \frac{1}{2}(3^n - 5)p^3 q^3,$$  \hfill (3.4)

$$S_q F(p, q) = \int_0^q \frac{1}{t} \{6p^2 q^3 + \frac{3}{2}(3^n - 5)p^3 q^3\} dt$$

$$= 2p^2 q^3 + \frac{1}{2}(3^n - 5)p^3 q^3,$$  \hfill (3.5)

and

$$JF(p, q) = F(p, p)$$

$$= 6p^5 + \frac{3}{2}(3^n - 5)p^6.$$  \hfill (3.6)

**Theorem 3.2.** Let $H_n$ be a Hanoi graph, then

(i) first Zagreb index $M_1(H_n) = 3(3^{n+1} - 5)$,

(ii) second Zagreb index $M_2(H_n) = \frac{9}{2}(3^{n+1} - 1)$,

(iii) modified second Zagreb index $mM_2(H_n) = \frac{1}{6}(3^n + 1)$.

**Proof.** Adding equation (3.2) and (3.3),

$$(D_q + D_q) F(p, q) = 30p^2 q^3 + 9(3^n - 5)p^3 q^3,$$

using equation (2.1), the first Zagreb index

$$M_1(H_n) = \{D_q + D_q\} F(p, q)|_{p=q=1}$$

$$= 3(3^{n+1} - 5).$$

From the equation (3.2),

$$D_q D_q F(p, q) = \frac{\partial}{\partial q} \{12p^2 q^3 + \frac{9}{2}(3^n - 5)p^3 q^3\}$$

$$= 36p^2 q^3 + \frac{27}{2}(3^n - 5)p^3 q^3,$$

using equation (2.2), the second Zagreb index

$$M_2(H_n) = [D_q D_q F(p, q)]|_{p=q=1}$$

$$= \frac{9}{2}(3^{n+1} - 1).$$

From the equation (3.5),

$$S_p S_q F(p, q) = \int_0^p \{2r^2 q^3 + \frac{1}{2}(3^n - 5)r^3 q^3\} dr$$

$$= p^2 q^3 + \frac{1}{6}(3^n - 5)p^3 q^3,$$
using equation (2.3), the modified second Zagreb index
\[ mM_2(H_n) = [S_p S_q F(p, q)]_{p=q=1} \]
\[ = \frac{1}{6}(3^n + 1). \]

**Theorem 3.3.** Let \( H_n \) be a Hanoi graph, then
(i) generalized Randić index
\[ R_\alpha(H_n) = \frac{3^{\alpha+1}}{2}(2^{\alpha+2} + 3^n + 3\alpha - 5.3^\alpha), \]
(ii) inverse Randić index
\[ RR_\alpha(H_n) = \frac{1}{2^{-\alpha-1,3\alpha-1}} + \frac{3^n - 5}{2,32^{\alpha-1}}, \]
where \( \alpha > 0. \)

**Proof.** From equation (3.3),
\[ D_p F(p, q) = 18p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3, \]
\[ D_q F(p, q) = 6.3^\alpha p^2q^3 + \frac{3.3^\alpha}{2}(3^n - 5)p^3q^3, \]
\[ D_p D_q F(p, q) = \frac{p}{\alpha 3^\alpha p^2q^3 + \frac{3.3^\alpha}{2}(3^n - 5)p^3q^3}, \]
\[ = 6.2.3^\alpha p^2q^3 + \frac{3.3^\alpha}{2}(3^n - 5)p^3q^3, \]
\[ D_p^\alpha D_q^\alpha F(p, q) = 6.2^\alpha 3^\alpha p^2q^3 + \frac{3.3^\alpha 3^\alpha}{2}(3^n - 5)p^3q^3, \]
\[ = 2^{\alpha+1}3^{\alpha+1}p^2q^3 + \frac{32^{\alpha+1}}{2}(3^n - 5)p^3q^3, \]
using equation (2.4), the generalized Randić index
\[ R_\alpha(H_n) = [D_p^\alpha D_q^\alpha F(p, q)]_{p=q=1} \]
\[ = \frac{3^{\alpha+1}}{2}(2^{\alpha+2} + 3^n + 3\alpha - 5.3^\alpha). \]

From equation (3.5),
\[ S_q F(p, q) = \frac{6}{3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n - 5)p^3q^3, \]
\[ S_p F(p, q) = \frac{6}{3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n - 5)p^3q^3, \]
\[ S_p S_q F(p, q) = \int_0^1 \left[ \frac{6}{3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n - 5)p^3q^3 \right] dt \]
\[ = \frac{6}{2.3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n - 5)p^3q^3, \]
\[ S_p^\alpha S_q^\alpha F(p, q) = \frac{6}{2.3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n - 5)p^3q^3, \]
using equation (2.5), the inverse Randić index
\[ RR_\alpha(H_n) = [S_p^\alpha S_q^\alpha F(p, q)]_{p=q=1} \]
\[ = \frac{1}{2^{-\alpha-1,3\alpha-1}} + \frac{3^n - 5}{2,32^{\alpha-1}}. \]

**Theorem 3.4.** Let \( H_n \) be a Hanoi graph, then the symmetric sum division index is given by
\[ SSD(H_n) = 3^n + 2. \]

**Proof.** From equation (3.2),
\[ D_p F(p, q) = 12p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3, \]
\[ S_q D_p F(p, q) = \int_0^{\frac{1}{2}} \left[ 12p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3 \right] dt \]
\[ = 4p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3, \]
from equation (3.3),
\[ D_p F(p, q) = 18p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3, \]
\[ S_q D_p F(p, q) = \int_0^{\frac{1}{2}} \left[ 18p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3 \right] dt \]
\[ = 9p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3, \]
using equation (2.6), the symmetric sum division index
\[ SSD(H_n) = [(S_q D_p + S_p D_q) F(p, q)]_{p=q=1} \]
\[ = 13p^2q^3 + 3(3^n - 5)p^3q^3 \]
\[ = 3^n + 2. \]

**Theorem 3.5.** Let \( H_n \) be a Hanoi graph then
(i) harmonic index \( H(H_n) = \frac{1}{10}(5.3^n - 1), \)
(ii) inverse sum index \( I(H_n) = \frac{9}{20}(5.3^n - 9), \)
(iii) augmented Zagreb index \( A(H_n) = \frac{1}{27}(3^{n+7} - 8887). \)

**Proof.** From equation (3.7),
\[ JF(p, q) = 6p^5 + \frac{3}{2}(3^n - 5)p^6, \]
\[ S_pJF(p, q) = \int_0^p \left[ \frac{1}{7} (6r^4 + \frac{3}{2} (3^n - 5)r^3) \right] dr = \frac{6}{5} p^5 + \frac{1}{4} (3^n - 5) p^6, \]

using equation (2.7), the harmonic index

\[ H(H_n) = 2 |S_pJF(p, q)|_{p=1} = \frac{1}{10} [5, 3^n - 1]. \]

From equation (3.2),

\[ D_pF(p, q) = 12 p^2 q^3 + \frac{9}{2} (3^n - 5) p^3 q^3, \]
\[ D_qD_pF(p, q) = q \frac{\partial}{\partial q} \{ 12 p^2 q^3 + \frac{9}{2} (3^n - 5) p^3 q^3 \} = 36 p^2 q^3 + \frac{27}{2} (3^n - 5) p^3 q^3, \]
\[ JD_qD_pF(p, q) = 36 p^5 + \frac{27}{2} (3^n - 5) p^6, \]
\[ S_pJD_qD_pF(p, q) = \int_0^p \left[ \frac{1}{7} \{ 36 r^5 + \frac{27}{2} (3^n - 5) r^3 \} \right] dr = \frac{36}{5} p^5 + \frac{27}{12} (3^n - 5) p^6, \]

using equation (2.8), the inverse sum index

\[ I(H_n) = |S_pJD_qD_pF(p, q)|_{p=q=1} = \frac{9}{20} (5, 3^n - 9). \]

From equation (3.3),

\[ D_qF(p, q) = 18 p^2 q^3 + \frac{9}{2} (3^n - 5) p^3 q^3, \]
\[ D_q^2F(p, q) = 6.3^3 p^2 q^3 + \frac{3.3^3}{2} (3^n - 5) p^3 q^3, \]
\[ D_pD_q^2F(p, q) = p \frac{\partial}{\partial p} \{ 6.3^3 p^2 q^3 + \frac{3.3^3}{2} (3^n - 5) p^3 q^3 \} = 6.2.3^3 p^2 q^3 + \frac{3.3.3^3}{2} (3^n - 5) p^3 q^3, \]
\[ D_p^3D_q^2F(p, q) = 6.2.3^3 p^2 q^3 + \frac{3.3.3^3}{2} (3^n - 5) p^3 q^3 = 2^4 3^4 p^2 q^3 + \frac{3^7}{2} (3^n - 5) p^3 q^3, \]
\[ JD_p^3D_q^2F(p, q) = J \{ 2^4 3^4 p^2 q^3 + \frac{3^7}{2} (3^n - 5) p^3 q^3 \} = 2^4 3^4 p^5 + \frac{3^7}{2} (3^n - 5) p^6, \]
\[ Q-2JD_p^3D_q^2F(p, q) = Q-2 \{ 2^4 3^4 p^5 + \frac{3^7}{2} (3^n - 5) p^6 \} = 2^4 3^4 p^3 + \frac{3^7}{2} (3^n - 5) p^4, \]
\[ S_pQ-2JD_p^3D_q^2F(p, q) = \int_0^p \left[ \frac{1}{7} \{ 2^4 3^4 r^3 + \frac{3^7}{2} (3^n - 5) r^4 \} \right] dr = \frac{2^4 3^4}{3} p^3 + \frac{3^7}{2.4^3} (3^n - 5) p^4, \]
\[ S_p^3Q-2JD_p^3D_q^2F(p, q) = \frac{2^4 3^4}{3} p^3 + \frac{3^7}{2.4^3} (3^n - 5) p^4 = 2^4 3^3 p^3 + \frac{3^7}{2^2} (3^n - 5) p^4, \]

using equation (2.9), the augmented Zagreb index

\[ A(H_n) = |S_p^3Q-2JD_p^3D_q^2F(p, q)|_{p=1} = \frac{1}{2^7} (3^n + 7 - 8887). \]

### 3.2 Topological indices of generalized wheel graph

A generalized wheel graph \( W^m_n = mC_n + K_1 \) is defined as: take \( m \) copies of cycle \( C_n \); \( n \geq 3 \) and then, join every vertex of \( m \) copies of cycle \( C_n \) with a vertex of a complete graph \( K_1 \). The generalized wheel graph \( W^3_n \) is shown in Figure 2. For \( m = 1 \), the generalized wheel graph is turn out as wheel graph. A generalized wheel graph \( W^m_n \) has \( mn + 1 \) number of vertices and \( 2mn \) number of edges.

![W_3_n](image)

**Figure 2.** A representation of a generalized wheel graph \( W^3_n \)

**Theorem 3.6.** The \( M \)-polynomial of a generalized wheel graph \( W^m_n \) is given by

\[ M(W^m_n : p, q) = mn[p^3 q^3 + p^3 q^{mn}]. \]

**Proof.** We have,

\[ |V(W^m_n)| = mn + 1 \]

and

\[ |E(W^m_n)| = 2mn. \]
The edge set of a graph $W_n^m = (V,E)$ has the following two partitions:

$$E_{[3,mm]} = \{uv \in E : d(u) = 3, d(v) = mn\},$$

$$E_{[3,3]} = \{uv \in E : d(u) = 3, d(v) = 3\}.$$ 

Now,

$$|E_{[3,mm]}| = mn = m_{3,mm},$$

$$|E_{[3,3]}| = mn = m_{3,3}.$$ 

Thus, the $M$-polynomial of $W_n^m$

$$M(W_n^m : p, q) = \sum_{0 \leq i, j \leq mn} m_{i,j}(W_n^m) p^i q^j$$

$$= m_{3,mm}(W_n^m) p^{3} q^{mn} + m_{3,3}(W_n^m) p^{3} q^{3}$$

$$= mn \left(p^{3} q^{mn} + p^{3} q^{3}\right).$$

Now, we consider $M$-polynomial of a generalized wheel graph $W_n^m$ as $G(p,q)$, i.e.,

$$M(W_n^m : p, q) = mn(p^{3} q^{mn} + p^{3} q^{3}) = G(p,q). \quad (3.8)$$

Then, we compute the following expressions:

$$D_p G(p,q) = p \frac{\partial}{\partial p} \{mn(p^{3} q^{mn} + p^{3} q^{3})\}$$

$$= 3mpn^{2} q^{mn} + 3mpn^{2} q^{3}, \quad (3.9)$$

$$D_q G(p,q) = q \frac{\partial}{\partial q} \{mn(p^{3} q^{mn} + p^{3} q^{3})\}$$

$$= m^{2}n^{2} q^{mn} + 3mnp^{3} q^{3}, \quad (3.10)$$

$$S_p G(p,q) = \int_{0}^{p} \left[\frac{1}{4} \{mn(3^{m} q^{mn} + 3^{m} q^{3})\}\right] dt$$

$$= \frac{mn}{3} p^{3} q^{mn} + \frac{mn}{3} p^{3} q^{3}, \quad (3.11)$$

$$S_q G(p,q) = \int_{0}^{q} \left[\frac{1}{4} \{mn(3^{m} q^{mn} + 3^{m} q^{3})\}\right] dt$$

$$= p^{3} q^{mn} + \frac{mn}{3} p^{3} q^{3}, \quad (3.12)$$

and

$$JG(p,q) = G(p,p)$$

$$= mn(p^{mn+3} + p^{6}). \quad (3.13)$$

Theorem 3.7. Let $W_n^m$ be a generalized wheel graph, then

(i) first Zagreb index $M_1(W_n^m) = mn(mn + 9)$,

(ii) second Zagreb index $M_2(W_n^m) = 3mn(mn + 3)$,

(iii) modified second Zagreb index $mM_2(W_n^m) = mn(\frac{1}{3mn} + \frac{1}{9})$.

Proof. Adding equations (3.9) and (3.10),

$$(D_p + D_q) G(p,q) = mn\{6p^3 q^3 + (mn+3)p^3 q^{mn}\},$$

using equation (2.1), the first Zagreb index

$$M_1(W_n^m) = [(D_p + D_q) G(p,q)]_{p=q=1} = mn(mn + 9).$$

From equation (3.9)

$$D_p G(p,q) = 3mpn^{2} q^{mn} + 3mpn^{2} q^{3},$$

$$D_q D_p G(p,q) = p \frac{\partial}{\partial q} (3mpn^{2} q^{mn} + 3mpn^{2} q^{3})$$

$$= 3mnp^{3} q^{mn} + 3mnp^{3} q^{3},$$

using equation (2.2), the second Zagreb index

$$M_2(W_n^m) = [D_q D_p G(p,q)]_{p=q=1} = 3mn(mn + 3).$$

From equation (3.12)

$$S_q G(p,q) = p^{3} q^{mn} + \frac{mn}{3} p^{3} q^{3}$$

$$S_p S_q G(p,q) = \int_{0}^{p} \left[\frac{1}{4} (3^{m} q^{mn} + 3^{m} q^{3})\right] dt$$

$$= \frac{1}{3} p^{3} q^{mn} + \frac{mn}{9} p^{3} q^{3},$$

using equation (2.3), the modified second Zagreb index

$$mM_2(W_n^m) = [S_p S_q G(p,q)]_{p=q=1} = \frac{1}{3} + \frac{mn}{9}.$$

Theorem 3.8. Let $W_n^m$ be a generalized wheel graph, then

(i) generalized Randić index $R_q(W_n^m) = mn\{(3mn)^{\alpha} + 9^{\alpha}\}$,

(ii) inverse Randić index $RR_\alpha(W_n^m) = mn\left\{\frac{1}{(3mn)^{\alpha}} + \frac{1}{9^{\alpha}}\right\},$ where $\alpha > 0$.

Proof. From equation (3.10),

$$D_q G(p,q) = mn(mnp^{3} q^{mn} + 3p^{3} q^{3}),$$

$$D_q^{\alpha} G(p,q) = mn(m^{\alpha} n^{\alpha} p^{3} q^{mn} + 3^{\alpha} p^{3} q^{3}),$$

$$D_p D_q^{\alpha} G(p,q) = p \frac{\partial}{\partial p} [mn(m^{\alpha} n^{\alpha} p^{3} q^{mn} + 3^{\alpha} p^{3} q^{3})]$$

$$= mn(m^{\alpha} n^{\alpha} p^{3} q^{mn} + 3^{\alpha} p^{3} q^{3}),$$

$$D_p^{\alpha} D_q^{\alpha} G(p,q) = mn(3^{\alpha} m^{\alpha} n^{\alpha} p^{3} q^{mn} + 3^{2\alpha} p^{3} q^{3}).$$
Theorem 3.9. Let $W_n^m$ be a generalized wheel graph, then the symmetric sum division index

$$SSD(W_n^m) = 3 + 2mn + \frac{(mn)^2}{3}.$$ 

Proof. From equation (3.9),

$$D_pG(p, q) = 3mp^3q^{mn} + 3mp^3q^3,$$

$$S_qD_qG(p, q) = \int_0^p \{ \frac{1}{6} \left( 3mp^3q^{mn} + 3mp^3q^3 \right) \} dt$$

$$= 3p^3q^{mn} + mnp^3q^3,$$

from equation (3.10),

$$D_qG(p, q) = (mn)^2p^3q^{mn} + mnp^3q^3,$$

$$S_pD_qG(p, q) = \int_0^q \left\{ \frac{1}{6} \left( (mn)^2p^3q^{mn} + mnp^3q^3 \right) \right\} dt$$

$$= \frac{(mn)^2}{3}p^3q^{mn} + mnp^3q^3,$$

using equation (2.6), the symmetric sum division index

$$SSD(W_n^m) = \{ (S_qD_p + S_pD_q)G(p, q) \}_{p=q=1}$$

$$= 3 + 2mn + \frac{(mn)^2}{3}.$$ 

\[\square\]

Theorem 3.10. Let $W_n^m$ be a generalized wheel graph, then

(i) harmonic index $H(W_n^m) = \frac{2mn}{mn + 3} + \frac{1}{6},$

(ii) inverse sum index $I(W_n^m) = \frac{3mn}{mn + 3} + \frac{1}{2},$

(iii) augmented Zagreb index $A(W_n^m) = mn\left( \frac{3m^3n^3}{(mn + 1)^3} + \frac{3^6}{4^3} \right).$

Proof. From equation (3.14),

$$JG(p, q) = mn[p^{mn+3} + p^6],$$

$$S_pJG(p, q) = \int_0^p \left\{ \frac{1}{6} \left( mn(p^{mn+3} + p^6) \right) \right\} dt$$

$$= mn\left( p^{mn+3} + p^6 \right),$$

using equation (2.7), the harmonic index

$$H(W_n^m) = 2[S_pJG(p, q)]_{p=1}$$

$$= 2mn\left( \frac{p^{mn+3}}{mn + 3} + \frac{p^6}{6} \right)_{p=1}$$

$$= 2mn\left( \frac{1}{mn + 3} + \frac{1}{6} \right).$$

From equation (3.9),

$$D_pG(p, q) = 3mp^3q^{mn} + 3mp^3q^3,$$

$$D_qD_pG(p, q) = \frac{\partial}{\partial q} (3mp^3q^{mn} + 3mp^3q^3)$$

$$= 3mn \left( mnp^3q^{mn} + 3p^3q^3 \right),$$

$$JD_qD_pG(p, q) = J\{ 3mn(mnp^3q^{mn} + 3p^3q^3) \}$$

$$= 3mn \left( mnp^{mn+3} + 3p^6 \right),$$

$$S_pJD_qD_pG(p, q) = \int_0^p \left\{ \frac{1}{6} \left( 3mn(mnp^{mn+3} + 3p^6) \right) \right\} dt$$

$$= 3mn\left( \frac{mn}{mn + 3} + \frac{1}{2}p^6 \right),$$

using equation (2.8), the inverse sum index

$$I(W_n^m) = [S_pJD_qD_pG(p, q)]_{p=1}$$

$$= 3mn\left( \frac{mn}{mn + 3} + \frac{1}{2} \right).$$

From equation (3.10),

$$D_qG(p, q) = m^2n^2p^3q^{mn} + 3mp^3q^3,$$

$$D_q^2G(p, q) = mn \left( m^3n^3p^3q^{mn} + 3p^3q^3 \right).$$

\[\square\]
The formulas given in the introduction.

The generalized Randić Zagreb group indices, modified second Zagreb index, have derived nine degree based topological indices namely graph and generalized wheel graph. Using \( M \)-polynomial, the augmented Zagreb index

\[
D_p D_q G(p, q) = p \frac{\partial}{\partial q} \{mn(m^3 n^3 p^m q^n + 3p^3 q^3)\} = mn\{3m^3 n^3 p^m + 34 p^3 q^3\},
\]

\[
D_p D_q G(p, q) = mn\{3m^3 n^3 p^m + 36 p^3 q^3\},
\]

\[
JD_p D_q G(p, q) = J\{mn(3m^3 n^3 p^m q^n + 36 p^3 q^3)\} = mn(3m^3 n^3 p^{m+3} + 36 p^6),
\]

\[
Q_{-2} JD_p D_q G(p, q) = Q_{-2}\{mn(3m^3 n^3 p^{m+3} + 36 p^6)\} = mn(3m^3 n^3 p^{m+1} + 36 p^4),
\]

\[
S_p Q_{-2} JD_p D_q G(p, q) = \int_0^1 [\{mn(3m^3 n^3 p^{m+1} + 36 p^4)\}] dt = mn\{3m^3 n^3 p^{m+1} + \frac{36}{4} p^4\},
\]

\[
S_p Q_{-2} JD_p D_q G(p, q) = mn\{3m^3 n^3 (mn+1)^3 p^{m+1} + \frac{36}{4} p^4\},
\]

Using equation (2.9), the augmented Zagreb index

\[
A(W_{nm}) = [S_p Q_{-2} JD_p D_q G(p, q)]_{p=1} = mn\{3m^3 n^3 (mn+1)^3 p^{m+1} + \frac{36}{4} p^4\}_{p=1} = mn\{3m^3 n^3 (mn+1)^3 + \frac{36}{4} p^4\}.
\]

4. Conclusion

In this paper, we have computed \(M\)-polynomial for Hanoi graph and generalized wheel graph. Using \(M\)-polynomial, we have derived nine degree based topological indices namely Zagreb group indices, modified second Zagreb index, generalized Randić index, inverse Randić index, symmetric sum division index, harmonic index, inverse sum index, augmented Zagreb index of graphs \(H_p\) and \(W_{nm}\). It is important to note that such topological indices may be determined directly via the formulas given in the introduction.

References
