Some properties for subclass of analytic functions with nonzero coefficients

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Abstract
In the given article, we introduced new subclass of normalized analytic function namely \(R^t(1, A, B, \alpha)\). Coefficient inequality, necessary and sufficient condition for the functions in this class are given. The inclusion property and condition for univalency with linear \(n^{th}\) derivative operator is also pointed out.

Keywords
Univalent, Analytic, Starlike.

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1. Introduction

Let \(N\) denotes subclass of normalized analytical function in open unit disc \(U = \{z : |z| < 1\}\) given by

\[
\psi(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]

For the function \(f \in N, [1]\) has introduced the class namely \(R^1(A, B, \alpha)\) as

\[
\left| \frac{\psi'(z) - 1}{f(A-B) - B(\psi'(z) - 1)} \right| < \infty, \quad (z \in U).
\]

We note that \(F(a, b, c, 1) = \frac{\tau(v+n)}{\tau(v)}\)

for different values of \(t, A, B\) and \(\alpha\) The Gaussian hypergeometric function \(F(a, b, c, z)\) is given by

\[
F(a, b, c, z) = \sum_{k=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^k
\]

\((a,b,c \in \mathbb{C}; c \neq 0, -1, -2, \ldots; z \in U)\), where \((v)_n\) is defined in terms of gamma function as given bellow

\[
(v)_n = \frac{\tau(v+n)}{\tau(v)}
\]

\[
= \begin{cases} 
1 & \text{if } n = 0 \text{ and } v \in \mathbb{C}\setminus\{0\} \\
v(v+1) \ldots (v+n-1) & \text{if } n \in \mathbb{N} \text{ and } v \in \mathbb{C}.
\end{cases}
\]

The geometric properties of \(zF(a, b, c, z)\) like univalence, star likeness and convexity have been studied by Vuorinen and Ponnusamy [3] and Ruscheweyh and Singh [4]. For \(\psi \in N, [1]\) has defined the operator \(I_{a,b,c,\psi}\) by

\[
I_{a,b,c,\psi}(z) = zF(a, b, c, z) \ast \psi(z)
\]

\((\Re\{c-a-b\} > 0)\).

2. Introduction

Let \(A\) and \(B\) are complex numbers with \(A \neq B, t \in \mathbb{C}\setminus\{0\}\) and \(\alpha \in \mathbb{R}^+\). Dixit and Pal [5] studied the class \(R^1(A, B, 1)\). Moreover, the researchers in ([6, 7, 8]) studied the class \(R^1(A, B, \alpha)\).

Here * denotes the usual Hamadard product.
3. Linear $n$th differential operator and class $R^n(t, A, B\alpha)$

[1] has used the linear operator as defined in (3). We introduced Linear $n$th differential operator which is obtained by taking $n$th derivative of (3) as given below.

**Definition 3.1.** For $\psi \in N$, we define the operator $(I_{a,b,c} \psi(z))^t$ by

$$(I_{a,b,c} \psi(z))^t = t!L_t + \sum_{k=2}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t},$$

where

$$L_t = \frac{(a)_{t-1}(b)_{t-1}}{(c)_{t-1}} a_t \quad a_1 = 1, t \geq 1.$$

**Definition 3.2.** A function $\psi \in N$ is said to be in class $R^n(t, A, B\alpha)$ if

$$\left| \frac{\psi'(z) - t!a_1}{l(A-B) - B(\psi'(z) - t!a_1)} \right| < \infty, \quad (z \in U, a_1 = 1, t \geq 1),$$

where $A$ and $B$ are complex numbers with $A \neq B, t \in \mathbb{C}\setminus\{0\}$ and $\alpha \in \mathbb{R}^+$. For $t = 1$, we get the subclass defined by [1].

4. Main results

In this theorem, we found necessary condition for the class $R^n(t, A, B\alpha)$.

**Theorem 4.1.** Let function of the form (1) be in class $R^n(t, A, B\alpha)$ with

$$a_n = |a_n| e^{i(\lambda n + \sigma)},$$

then

$$\sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1) \sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t} \leq \alpha |l| |A - B|.$$

**Proof.** Suppose $f \in R^n(t, A, B\alpha)$

$$|\psi'(z) - t!a_1| < \alpha |l(A-B) - B(\psi'(z) - t!a_1)|$$

$$\psi'(z) = a_t + \sum_{k=2}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t}$$

$$\left| \sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t} \right|$$

$$\leq \alpha |l(A-B) - B(\psi'(z) - t!a_1)|$$

$$= \alpha |l(A-B) - B\sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t}.$$

Put $z = r e^{\theta}$.

$$a_n z^{k-t} = |a_n| e^{(\lambda n + \sigma)k-t} e^{i(\lambda n + \sigma)k-t}$$

$$= |a_n| r^{k-t} e^{i\lambda k-t}$$

4.1

$$\sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)A_k z^{k-t}$$

$$\leq \alpha \sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)|a_k| r^{k-t}$$

$$\leq \alpha |l(A-B)| r^{k-t} \sum_{k=t+1}^{\infty} k(k-1) \ldots (k-t+1)|a_k|.$$
Theorem 4.3. Let a function of the form (1) be in class $N$. If
\[ \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)(1+\alpha|B|)|a_k| \leq \alpha|l||A-B|. \]
(4.4)

Then, $\psi \in R^1(t,A,B,\alpha)$.

Proof. Given that
\[ \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)(1+\alpha|B|)|a_k| \leq \alpha|l||A-B| \]
r $< 1 \Rightarrow r^{k-t} < 1$.
\[ \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)(1+\alpha|B|)|a_k|r^{k-t} \]
\[ \leq \alpha|l||A-B| \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)|a_k|r^{k-t} \]
\[ \leq \alpha|l||A-B| - |B| \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)|a_k|r^{k-t}. \]

Hence,
\[ |\psi(z) - t!a_t| \leq \alpha|l(A-B) - B(\psi(z) - t!a_t)| \]
\[ \frac{|\psi(z) - t!a_t|}{|l(A-B) - B(\psi(z) - t!a_t)|} < \alpha \]

Therefore, $\psi \in R^1(t,A,B,\alpha)$. Now with operator $(I_{a,b,c}\psi(z))^t$ we obtained inclusion relationship for the class $R^1(t,A,B,\alpha)$.

\[ I_{a,b,c}\psi \in R^1(t,A,B,\alpha). \]

Therefore, $I_{a,b,c}\psi \in R^1(t,A,B,\alpha)$.

\[ I_{a,b,c}\psi \left( R^1(t,A,B,\alpha) \right) \subseteq R^1(t,A,B,\alpha) \]

Then, $I_{a,b,c}\psi$ is defined in (3).

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\[ I_{a,b,c}\psi \left( R^1(t,A,B,\alpha) \right) \subseteq R^1(t,A,B,\alpha) \]

Then, $I_{a,b,c}\psi$ is defined in (3).

Proof. Given that $\psi \in R^1(t,A,B,\alpha)$. From theorem 3, we have
\[ \sum_{k=1}^{\infty} k(k-1)\ldots(k-t+1)(1+\alpha|B|)|L_k| \leq \alpha|l||A-B|, \]
where,
\[ L_k = \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}}a_k. \]

From corollary 2, we have
\[ |a_k| \leq \frac{\alpha|l||A-B|}{k(k-1)\ldots(k-t+1)(1-\alpha|B|)}, \quad k \geq 2. \]
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\[
\because |w(z)|^\beta \left| \frac{zw'(z)}{w(z)} \right|^\gamma \geq \left| \left( \frac{(I_{a,b,c} \psi(z))^\gamma - t!L_\gamma}{1 - \alpha} \right) \right|^\beta
\]

\[
\left| \frac{(I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z}{1 - \alpha} \right| < 1.
\]

This shows that

\[
\left| (I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z \right| < |1 - \alpha|.
\]

Hence, completed the proof of theorem 6. \( \square \)

**Corollary 4.7.** Let a function of the form (1) be in the class A satisfying inequality (12)

\[
(I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z
\]

is univalent in D

**Proof.**

\[
\left( (I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z \right)' = (I_{a,b,c} \psi(z))^{\gamma - 1} - t!L_\gamma + 1.
\]

From theorem (6), we have

\[
\Re \left\{ \left( (I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z \right)' \right\} > 0, \quad (z \in U)
\]

Therefore, by [2] Alexander-Noshiro-Warschawski theorem the

\[
(I_{a,b,c} \psi(z))^{\gamma - 1} + (1 - t!L_\gamma)z
\]

is univalent in D. \( \square \)

\[\text{References}\]


