Some new open sets in $\mu_N$ topological space

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Abstract
In this article we lay a stress on the open sets which have a enormous impact on the $\mu_N$ topological space. Several types of $\mu_N$-Open sets were contrived and their roles and natures were enunciated. Also Continuous functions on the $\mu_N$ topological spaces were asseverated.

Keywords
$\mu_N$-Semi open sets, $\mu_N$- Pre Open Sets, $\mu_N$- Open sets, $\mu_N$- $\alpha$ Open sets, $\mu_N$- $\beta$ open sets, $\mu_N$-Continuous.

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1. Introduction

In 1965, Zadeh[15] found out fuzzy set theory which plays a vital role in real life application in order to cope up with uncertainty. In 1968, Chang[3] invented fuzzy topology which gives an utterance in the field of topology. Keeping these two as aspirations, In 1983, K.Attanassov [1] explored intuitionistic fuzzy sets by giving attention to both membership and non-membership of the elements. Several notions of fuzzy sets and fuzzy topology were explored after the existence of intuitionistic fuzzy sets. In 1997, by marking Attanassov’s work as inspiration, Coker[5] worked with the Intuitionistic fuzzy sets by applying the concepts of fuzziness and got Intuitionistic fuzzy topological space which helped Attanassov to discover the interval valued intuitionistic fuzzy set on a universe $X$ as an object $A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X \}$.

F.Smarandache[6] focussed his views towards the degree of indeterminacy and which led into neutrosophic sets. Later on, A.A.Salama and S.A.Albowi[11] introduced the neutrosophic topological spaces with the help of neutrosophic sets and proceeding this A.A. Salama, F.Smarandache and Valeri Kromov[13] introduced the continuous functions in neutrosophic topological spaces. By setting all these works together as inspiration, In 2020 we[10] contrived $\mu_N$ Topological Space and their basic properties were discussed. In this discourse, we explore our thoughts towards various open sets in $\mu_N$ Topological Space which can be developed later and some of their basic properties were discussed. Also, $\mu_N$ continuous functions were introduced and also their features were contemplated.

2. Preliminaries

The concepts given here are used to brush up our memories regarding the basic concepts of $\mu_N$ Topological Space.

Definition 2.1. [11] Let $X$ be a non-empty fixed set. A Neutrosophic set [NS for short] $A$ is an object having the form $A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X \}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function, the degree of indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set $A$.

Remark 2.2. [11] A neutrosophic set $A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X \}$ can be identified to an ordered triple $A = \{< \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \}$ in $[-0,1]^+$ on $X$. 
Remark 2.3. [11] For the sake of simplicity, we shall use the symbol $A = \{< \mu(x), \sigma(x), \gamma(x)> \}$ for the neutrosophic set $A = \{< x, \mu(x), \sigma(x), \gamma(x)> : x \in X \}$.

Remark 2.4. Every intuitionistic fuzzy set $A$ is a non-empty set in $X$ is obviously on Neutrosophic sets having the form $A = \{< \mu_a(x), 1 - \mu_a(x) + \sigma_a(x), \gamma_a(x)> : x \in X \}$. In order to construct the tools for developing Neutrosophic Set and Neutrosophic topology, here we introduce the neutrosophic sets $0_N$ and $1_N$ in $X$ as follows:

- $0_N$ may be defined as follows:
  - $(0_1)0_N = \{< x, 0, 0, 1 > : x \in X \}$
  - $(0_2)0_N = \{< x, 0, 1, 1 > : x \in X \}$
  - $(0_3)0_N = \{< x, 0, 1, 0 > : x \in X \}$
  - $(0_4)0_N = \{< x, 0, 0, 0 > : x \in X \}$

- $1_N$ may be defined as follows:
  - $(1_1)1_N = \{< x, 1, 0, 0 > : x \in X \}$
  - $(1_2)1_N = \{< x, 1, 0, 1 > : x \in X \}$
  - $(1_3)1_N = \{< x, 1, 1, 0 > : x \in X \}$
  - $(1_4)1_N = \{< x, 1, 1, 1 > : x \in X \}$

Definition 2.5. [11] Let $A = \{< \mu_a, \sigma_a, \gamma_a > \}$ be a NS on $X$, then the complement of the set $A$ is $\overline{A}$ defined in three ways as follows:

- $(C_1)A = \{< x, 1 - \mu_a(x), 1 - \sigma_a(x), 1 - \gamma_a(x) > : x \in X \}$
- $(C_2)A = \{< x, \gamma_a(x), \sigma_a(x), \mu_a(x) > : x \in X \}$
- $(C_3)A = \{< x, \gamma_a(x), 1 - \sigma_a(x), \mu_a(x) > : x \in X \}$

Definition 2.6. [11] Let $X$ be a non-empty set and neutrosophic sets $A$ and $B$ in the form

$A = \{< x, \mu_a(x), \sigma_a(x), \gamma_a(x) > : x \in X \}$ and $B = \{< x, \mu_b(x), \sigma_b(x), \gamma_b(x) > : x \in X \}$. Then we may consider two possibilities for definitions for subsets($A \subseteq B$).

- ($A \subseteq B$) if
  - $A \subseteq B$ then $\mu_a(x) \leq \mu_b(x)$, $\sigma_a(x) \leq \sigma_b(x)$, $\gamma_a(x) \geq \gamma_b(x)$, $\forall x \in X$
- ($B \subseteq A$) if
  - $B \subseteq A$ then $\mu_a(x) \leq \mu_b(x)$, $\sigma_a(x) \leq \sigma_b(x)$, $\gamma_a(x) \geq \gamma_b(x)$, $\forall x \in X$

Proposition 2.7. [11] For any neutrosophic set $A$, the following conditions holds: $0_N \subseteq A \subseteq 1_N$.

Definition 2.8. [11] Let $X$ be a non-empty set and $A = \{< x, \mu_a(x), \sigma_a(x), \gamma_a(x) > : x \in X \}$.

- $B = \{< x, \mu_b(x), \sigma_b(x), \gamma_b(x) > : x \in X \}$ are NSs.

Then $A \cap B$ may be defined as:

- $(I_1)A \cap B = \{< x, \mu_a(x) \wedge \mu_b(x), \sigma_a(x) \wedge \sigma_b(x), \gamma_a(x) \vee \gamma_b(x) > : x \in X \}$
- $(I_2)A \cap B = \{< x, \mu_a(x) \wedge \mu_b(x), \sigma_a(x) \wedge \sigma_b(x), \gamma_a(x) \vee \gamma_b(x) > : x \in X \}$

Definition 2.9. [10] A neutrosophic topology is a non-empty set $X$ is a family of neutrosophic subsets in $X$ satisfying the following axioms:

(1) $\bigcup_{\mu_a(x), \sigma_a(x), \gamma_a(x) > : x \in X \} \subseteq \bigcup_{\mu_b(x), \sigma_b(x), \gamma_b(x) > : x \in X \} \subseteq \bigcup_{\mu_c(x), \sigma_c(x), \gamma_c(x) > : x \in X \}$

Remark 2.10. [10] The elements of $\mu_n$ are $\mu_n$-open sets and their complement is called $\mu_n$ closed sets.

Definition 2.11. [10] Let $(X, \mu_n)$ be a $\mu_n$ TS and $A = \{< x, \mu_a(x), \sigma_a(x), \gamma_a(x) > : x \in X \}$ be a neutrosophic set in $X$. Then the $\mu_n$-Closure is the intersection of all $\mu_n$ closed sets containing $A$.

Definition 2.12. [10] Let $(X, \mu_n)$ be a $\mu_n$ TS and $A = \{< x, \mu_a(x), \sigma_a(x), \gamma_a(x) > : x \in X \}$ be a neutrosophic set in $X$. Then the $\mu_n$-Interior is the union of all $\mu_n$ open sets contained in $A$.

Definition 2.13. [13] A NS $A$ of a NTS $X$ is said to be

(i) a neutrosophic pre-open if $A \subseteq N\text{Int}(NCIA)$

(ii) a neutrosophic Semi-open if $A \subseteq NC\text{Int}(NCIA)$

(iii) a neutrosophic $\alpha$-open if $A \subseteq NC\text{Int}(NC\text{Int}(NCIA))$.

Definition 2.14. [13] Let $(X, \tau)$ and $(Y, \sigma)$ be neutrosophic topological spaces. Then a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called neutrosophic continuous (in short $N$-continuous) function if the inverse image of every neutrosophic open set in $(Y, \sigma)$ is neutrosophic open set in $(X, \tau)$.

3. Open sets in $\mu_n$ topological space

Definition 3.1. A neutrosophic set in a $\mu_n$ topological space is said to be $\mu_n$ Semi Open if $A \subseteq \mu_n\text{Cl}(\mu_n\text{Int}A)$.

Theorem 3.2. Every $\mu_n$-open set is $\mu_n$-Semi Open.

Proof. Let $U$ be a $\mu_n$-open set in $X$ which implies us that $\mu_n\text{Int}U = U$. Hence, we get $U \subseteq \mu_n\text{Cl}(\mu_n\text{Int}U) \Rightarrow U = \mu_n\text{Cl}(\mu_n\text{Int}U)$. Thus, $U \subseteq \mu_n\text{Cl}(\mu_n\text{Int}U)$. Hence, it is $\mu_n$-Semi Open.

Remark 3.3. Converse of the theorem need not be true. It is furnished by the following example.

Let $X \equiv \{a\}; \mu_n = \{0, A, B, C\}$ where $A = < 0, 0, 0, 0 >$, $B = < 0, 1, 0, 0 >$, $C = < 0, 0, 0, 0 >$, $D = < 0, 3, 0, 0 >$, $E = < 0, 0, 0, 0 >$. Here, The $\mu_n$-Semi Open sets are $\{0_N, A, B, C, E\}$ which tell us that $E$ is $\mu_n$-Semi Open but it is not $\mu_n$-open set.

Definition 3.4. A neutrosophic set in a $\mu_n$ topological space is said to be $\mu_n$ Pre-Open if $A \subseteq \mu_n\text{Int}(\mu_n\text{Cl}A)$.

Theorem 3.5. Every $\mu_n$-open set is $\mu_n$-Pre Open.

Proof. We have $A \subseteq \mu_n\text{Cl}A$ which implies us obviously that $\mu_n\text{Int}A \subseteq \mu_n\text{Int}(\mu_n\text{Cl}A)$. As we all know that $\mu_n\text{Int}A \subseteq A$. Hence, we get $A \subseteq \mu_n\text{Int}(\mu_n\text{Cl}A)$.

Remark 3.6. The reversal concept of the above theorem need be necessarily true. The following problem gives us the crystal clear idea about it.
Example 3.7. Let $X = \{a, b\}, Y = \{u, v\}$ and $(X, \tau), (Y, \sigma)$ be the $\mu_N$ TS where $\tau = \{A, B, C, D, 0_N\}$ and $\sigma = \{D, E, 0_N\}$. $A = (0.7, 0.3, 0.8 < 0.5, 0.8, 0.9 >, B = (0.4, 0.9, 0.9 < 0.3, 0.9, 0.9 >, C = (0.5, 0.8, 0.7 < 0.5, 0.8, 0.8 >, D = (0.5, 0.8, 0.8 < 0.5, 0.8, 0.7 >, E = (0.3, 0.9, 0.9 < 0.4, 0.9, 0.9 >. Here, A, B, C, D, E, 0_N are $\mu_N$-Pre open sets of $(X, \tau)$. Particularly $E$ is $\mu_N$-Pre Open set of $(X, \tau)$ but it is not $\mu_N$-open in $(X, \tau)$.

Definition 3.8. A neutrosophic set in a $\mu_N$ topological space is said to be $\mu_N$-$\alpha$-open if $A \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\mu_N \text{Int}(A)))$.

Theorem 3.9. Every $\mu_N$-$\alpha$-open set is $\mu_N$-$\alpha$-Open.

Proof. Let $A$ be $\mu_N$-$\alpha$-open set which yields us that $\mu_N \text{Int}(A) = A$. We have $A \subseteq \mu_N \text{Cl}(A)$. From this we obtain $\mu_N \text{Int}(A) \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\mu_N \text{Int}(A))) \Rightarrow A \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\mu_N \text{Int}(A)))$. Hence, it is $\mu_N$-$\alpha$-Open.

Remark 3.10. Converse statement of the theorem need not be true. The Scenario is explained below.

Let $X = \{a\}$; $\mu_N = \{0_N, A, C\}$ where $A = (0.7, 0.8, 0.9 <, B = (0.3, 0.4, 0.6 >, C = (0.9, 0.7, 0.6 >. Here, The $\mu_N$-$\alpha$-Open sets are $\{0_N, A, B, C\}$. In this set $B$ is $\mu_N$-$\alpha$-Open set but it is not $\mu_N$-open set.

Theorem 3.11. Every $\mu_N$-$\alpha$-open set is $\mu_N$-$\beta$-Open.

Proof. Let $A$ be $\mu_N$-$\alpha$-open set then that $\mu_N \text{Int}(A) = A$. We have $A \subseteq \mu_N \text{Cl}(A)$ which gives us that $\mu_N \text{Int}(A) \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(A)) \Rightarrow A \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\mu_N \text{Int}(A)))$.

Now we have $A \subseteq \mu_N \text{Cl}(A)$ and $A \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(A))$. Both together gives us $A \subseteq \mu_N \text{Cl}(\mu_N \text{Int}(\mu_N \text{Cl}(A)))$.

Remark 3.12. Converse of the above theorem need not be true which explained with the help of an example as below.

Let $X = \{a\}$; $\mu_N = \{0_N, A, B\}$ where $A = (0.3, 0.6, 0.9 >, B = (0.4, 0.6, 0.7 >, C = (0.5, 0.7, 0.8 >. Here, The $\mu_N$-$\beta$-Open sets are $\{0_N, A, B, C\}$. In this set $B$ is $\mu_N$-$\beta$-Open set but it is not $\mu_N$-open set.

Theorem 3.13. Every $\mu_N$-$\alpha$-open set is $\mu_N$-$\beta$-Open.

Proof. Let $A$ be a $\mu_N$-$\alpha$-open set which leads us to have $A \subseteq \mu_N \text{Int}(\mu_N \text{Cl}(\mu_N \text{Int}(A)))$. From this it can be easily derived as $A \subseteq \mu_N \text{Cl}(\mu_N \text{Int}(A))$. Hence, Every $\mu_N$-$\alpha$-open set is $\mu_N$ Semi-Open.

Remark 3.16. The reversal statement of the above theorem need not be true. Let $X = \{a\}, \mu_N = \{0_N, A, B, C\}, A <= (0.3, 0.3, 0.5 >, B = (0.1, 0.2, 0.3 >, C = < 0.3, 0.2, 0.3 >, D = (0.3, 0.6, 0.2 >, E = (0.3, 0.8, 0.5 >. The $\mu_N$ Semi-open sets are $\{0_N, A, B, C, E\}$ and $\mu_N$-$\alpha$-Open sets are $\{0_N, A, B, C\}$. Here, $E$ is $\mu_N$ Semi-Open set but not $\mu_N$-$\alpha$-open set.

Definition 3.17. A neutrosophic set in a $\mu_N$ topological space is said to be $\mu_N$-$\alpha$-Continuous if $f(A) = f(\mu_N \text{Int}(A)) \subseteq \mu_N \text{Int}(f(A))$.

4. $\mu_N$-Continuous Functions

Definition 4.1. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\mu_N$-Continuous function if the inverse image of $\mu_N$-closed sets in $(Y, \sigma)$ is $\mu_N$-closed in $(X, \tau)$.

Example 4.2. Let $X = \{a, b, c\}$ and $Y = \{u, v, w\}$ and $\tau = \{A, B, 0_N\}$, $\sigma = \{C, D, 0_N\}$ where $A = (0.5, 0.8, 0.9 >, B = (0.7, 0.8, 0.9 >, C = (0.7, 0.8, 0.9 >, D = (0.8, 0.9, 0.9 >. Here, we define a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = a, f(c) = c.$

Hence, we get $f^{-1}(A) = (0.5, 0.8, 0.9 >, f^{-1}(B) = (0.7, 0.8, 0.9 >$. Now, $f^{-1}(A) = (0.5, 0.8, 0.9 >$ and $f^{-1}(B) = (0.7, 0.8, 0.9 >$. Hence $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\mu_N$-Continuous.

Theorem 4.3. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\mu_N$-$\alpha$-Continuous function if and only if the inverse image of $\mu_N$-$\alpha$-open sets in $(Y, \sigma)$ is $\mu_N$-$\alpha$-open in $(X, \tau)$.

Proof. Essential condition: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\mu_N$-Continuos function and $U$ be a $\mu_N$-$\alpha$-open sets in $(Y, \sigma)$. Since $f$ is $\mu_N$-$\alpha$-Continous, $f^{-1}(Y - U) = X - f^{-1}(U)$ is $\mu_N$-$\alpha$-closed set in $(X, \tau)$ and hence $f^{-1}(U)$ is $\mu_N$-$\alpha$-open in $(X, \tau)$.

Sufficient Condition: Assume that $f^{-1}(V)$ is $\mu_N$-$\alpha$-open in $(X, \tau)$ for each $\mu_N$-$\alpha$-open set in $(Y, \sigma)$. Let $V$ be a $\mu_N$-$\alpha$-closed set in $(Y, \sigma)$ which yields that $Y - V$ is $\mu_N$-$\alpha$-open set in $(Y, \sigma)$. Now, $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\mu_N$-$\alpha$-closed set in $(X, \tau)$ which implies us that $f^{-1}(V)$ is $\mu_N$-$\alpha$-open in $(X, \tau)$. Hence, $f$ is $\mu_N$-Continous.
Theorem 4.4. Let \((X, \tau), (Y, \sigma), (Z, \rho)\) be three \(\mu_N\)-topological spaces. If \(f: (X, \tau) \rightarrow (Y, \sigma)\) and \(g: (Y, \sigma) \rightarrow (Z, \rho)\) are \(\mu_N\)-Continuous then \(g \circ f: (X, \tau) \rightarrow (Z, \rho)\) is \(\mu_N\)-Continuous.

Proof. Let \(U\) be any \(\mu_N\)-open set in \((Z, \rho)\). Since \(g\) is \(\mu_N\)-Continuous, \(g^{-1}(U)\) is \(\mu_N\)-open and hence it is \(\mu_N\)-open in \((Y, \sigma)\). Also, since \(f\) is \(\mu_N\)-Continuous, \(f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)\) is \(\mu_N\)-open. Hence, \(g \circ f\) is \(\mu_N\)-Continuous.

Definition 4.5. A mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\mu_N\)-Semi Continuous function if the inverse image of \(\mu_N\)-closed sets in \((Y, \sigma)\) is \(\mu_N\)-Semi closed sets in \((X, \tau)\).

Theorem 4.6. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Semi Continuous function if and only if the inverse image of \(\mu_N\)-open sets in \((Y, \sigma)\) is \(\mu_N\)-Semi open in \((X, \tau)\). Proof is similar to the proof of theorem 4.4.

Theorem 4.7. Every \(\mu_N\)-Continuous is \(\mu_N\)-Semi Continuous.

Proof. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Continuous mapping. Let \(A\) be a \(\mu_N\)-open set in \((Y, \sigma)\). By hypothesis we get \(f^{-1}(A)\) is \(\mu_N\)-open in \((X, \tau)\). Since, every \(\mu_N\)-open set is \(\mu_N\)-Semi Open, we get \(f^{-1}(A)\) is \(\mu_N\)-Semi open in \((X, \tau)\). Thus we obtain that \(f\) is \(\mu_N\)-Semi Continuous.

Remark 4.8. Converse of theorem 4.7 is not true. This condition can be explored by the following example.

Example 4.9. Let \(X = \{a, b\}\) and \(Y = \{u, v\}, \tau = \{A, B, C, D, 0_N\}\) and \(\sigma = \{D, E, 0_N\}\) where \(A = \{a, b, 0_N\}\). Then, \(N\)-Continuous function need not be \(\mu_N\)-Continuous.

We define a mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) such that \(f(a) = u\) and \(f(b) = v\). Hence \(f^{-1}(D) = D\) and \(f^{-1}(E) = E\) where \(D\) and \(E\) are \(\mu_N\)-Open sets of \((X, \tau)\) but \(E\) is not \(\mu_N\)-Open in \((X, \tau)\). Thus we conclude that Every \(\mu_N\)-Continuous function need not be \(\mu_N\)-Continuous.

Definition 4.10. A mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\mu_N\)-Pre Continuous function if the inverse image of \(\mu_N\)-closed sets in \((Y, \sigma)\) is \(\mu_N\)-Pre closed sets in \((X, \tau)\).

Theorem 4.11. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Pre Continuous function if and only if the inverse image of \(\mu_N\)-open sets in \((Y, \sigma)\) is \(\mu_N\)-Pre open in \((X, \tau)\). Proof is similar to the proof of theorem 4.4.

Theorem 4.12. Every \(\mu_N\)-Continuous is \(\mu_N\)-Pre Continuous.

Proof. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Continuous mapping. Let \(A\) be a \(\mu_N\)-open set in \((Y, \sigma)\). Since \(f\) is \(\mu_N\)-Continuous we get \(f^{-1}(A)\) is \(\mu_N\)-open in \((X, \tau)\). Thus, \(f\) is \(\mu_N\)-Pre Continuous.

Remark 4.13. The reverse statement of the above theorem need not be true. The Scenario will be emulated in the forthcoming example. Let \(X = \{a, b\}\) and \(Y = \{U, V\}, \tau = \{A, B, C, D, 0_N\}\) and \(\sigma = \{D, E, 0_N\}\) where \(A = \{a, b, 0_N\}\).

Definition 4.14. A mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\mu_N\)-\(\alpha\) Continuous function if the inverse image of \(\mu_N\)-closed sets in \((Y, \sigma)\) is \(\mu_N\)-\(\alpha\) closed sets in \((X, \tau)\).

Theorem 4.15. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-\(\alpha\) Continuous function if and only if the inverse image of \(\mu_N\)-open sets in \((Y, \sigma)\) is \(\mu_N\)-\(\alpha\) open in \((X, \tau)\). Proof is similar to the proof of theorem 4.4.

Theorem 4.16. Every \(\mu_N\)-Continuous is \(\mu_N\)-\(\alpha\) Continuous.

Proof. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Continuous mapping. Let \(A\) be a \(\mu_N\)-open set in \((Y, \sigma)\). Since \(f\) is \(\mu_N\)-Continuous we get \(f^{-1}(A)\) is \(\mu_N\)-open in \((X, \tau)\). Hence, \(f\) is \(\mu_N\)-\(\alpha\) Continuous.

Remark 4.17. Converse of the above theorem need not be true. It is explained by the example given below.

Example 4.18. Let \(X = \{a, b\}\) and \(Y = \{U, V\}, \tau = \{A, B, C, D, 0_N\}\) and \(\sigma = \{D, E, 0_N\}\) where \(A = \{a, b, 0_N\}\). Then, \(\mu_N\)-Continuous function need not be \(\mu_N\)-\(\alpha\) Continuous.

We define a mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) such that \(f(a) = u\) and \(f(b) = v\). Hence \(f^{-1}(D) = D\) and \(f^{-1}(E) = E\) where \(D\) and \(E\) are \(\mu_N\)-\(\alpha\) Open sets of \((X, \tau)\) but \(E\) is not \(\mu_N\)-Open in \((X, \tau)\). Thus we conclude that Every \(\mu_N\)-\(\alpha\) Continuous function need not be \(\mu_N\)-\(\alpha\) Continuous.

Definition 4.19. A mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\mu_N\)-\(\beta\) Continuous function if the inverse image of \(\mu_N\)-closed sets in \((Y, \sigma)\) is \(\mu_N\)-\(\beta\) closed sets in \((X, \tau)\).

Theorem 4.20. A mapping \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\mu_N\)-\(\beta\) Continuous function if and only if the inverse image of \(\mu_N\)-open sets in \((Y, \sigma)\) is \(\mu_N\)-\(\beta\) open in \((X, \tau)\). Proof is similar to the proof of theorem 4.4.

Theorem 4.21. Every \(\mu_N\)-Continuous is \(\mu_N\)-\(\beta\) Continuous.

Proof. Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a \(\mu_N\)-Continuous mapping. Let \(A\) be a \(\mu_N\)-open set in \((Y, \sigma)\). The inverse image of \(A\) is \(\mu_N\)-open set in \((X, \tau)\). We know that every \(\mu_N\)-open set is \(\mu_N\)-\(\beta\) Open set in \((X, \tau)\). Hence, \(f\) is \(\mu_N\)-\(\beta\) Continuous.

Remark 4.22. Converse of the above theorem need not be true. The Scenario is given in the preceding example.
Example 4.23. Let $X = \{a,b\}, Y = \{U,V\}$, $\tau = \{A,B,C,D, 0_Y\}$ and $\sigma = \{C,E,0_X\}$ be two $\mu_N$-topological spaces where $A = \{<0.6,0.4,0.8 > < 0.8,0.6,0.9 >, B = \{<0.6,0.3,0.8 > < 0.9,0.2,0.7 >, C = \{<0.5,0.4,0.9 > < 0.7,0.8,0.9 >, D = \{<0.4,0.6,0.9 >  < 0.6,0.8,0.9 >, E = \{<0.3,0.7,0.9 > < 0.5,0.9 , 0.9 >$. We define a mapping $f(a) = u$ and $f(b) = v$. Hence we get $f^{-1}(C) = C$ and $f^{-1}(E) = E$ where $C$ and $E$ are $\mu_N - \beta$ Open sets of $(X, \tau)$ but the inverse image of $E \in \sigma$ is $E$ which is not $\mu_N$- Open in $(X, \tau)$. Hence, Every $\mu_N - \beta$ Continuous need not be $\mu_N$-Continuous.

Theorem 4.24. Every $\mu_N - \alpha$ Continuous is $\mu_N$- Semi Continuous.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mu_N - \alpha$ Continuous mapping. Let $A$ be a $\mu_N - \alpha$ open set in $X$. By hypothesis we get $f^{-1}(A) = \mu_N - \alpha$ open in $(X, \tau)$. Since, every $\mu_N - \alpha$ open set is $\mu_N$-Semi Open we get $f^{-1}(A)$ is $\mu_N$-Semi open in $(X, \tau)$. Thus we obtain that $f$ is $\mu_N$-Semi Continuous.

Remark 4.25. The reversal statement of the above theorem need not be true. It can be delineated below with the help of an example.

Example 4.26. Let $X = \{a,b\}, Y = \{U,V\}, \tau = \{A,B,C,D, 0_Y\}$ and $\sigma = \{C,E,0_X\}$ be two $\mu_N$-topological spaces where $A < 0.6,0.4,0.8 > 0.8,0.6,0.9 >, B = < 0.6,0.3,0.8 >, 0.9,0.2,0.7 >, C = < 0.5,0.4,0.9 > 0.7,0.8,0.9 >, D = < 0.4,0.6,0.9 > 0.6,0.8,0.9 >, E = < 0.3,0.7,0.9 > < 0.5,0.9 , 0.9 >. We define a mapping $f(a) = u$ and $f(b) = v$. Hence we get $f^{-1}(C) = C$ and $f^{-1}(E) = E$ where $C$ and $E$ are $\mu_N$-Semi Open sets of $(X, \tau)$ but the inverse image of $E \in \sigma$ is $E$ which is not $\mu_N$- Open in $(X, \tau)$. Hence, every $\mu_N$-Semi Continuous need not be $\mu_N$-Continuous.

Theorem 4.27. Every $\mu_N$-Pre Continuous is $\mu_N - \beta$ Continuous.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mu_N$-Pre Continuous mapping. Let $A$ be a $\mu_N$-Pre open set in $X$. By hypothesis we get $f^{-1}(A)$ is $\mu_N$- Pre open in $(X, \tau)$. Since, every $\mu_N$-Pre open set is $\mu_N - \beta$ Open we get $f^{-1}(A)$ is $\mu_N - \beta$ Open in $(X, \tau)$. Thus we obtain that $f$ is $\mu_N - \beta$ Continuous.

Remark 4.28. The reversal statement of the above theorem need not be true. It can be explained with the help of an example as below:

Example 4.29. Let $X = \{a,b\}, Y = \{U,V\}, \tau = \{A,B,C,D, 0_Y\}$ and $\sigma = \{C,E,0_X\}$ be two $\mu_N$-topological spaces where $A = \{<0.6,0.4,0.8 > 0.8,0.6,0.9 >, B = \{<0.6,0.3,0.8 > 0.9,0.2,0.7 >, C = \{<0.5,0.4,0.9 > 0.7,0.8,0.9 >, D = \{<0.4,0.6,0.9 > 0.6,0.8,0.9 >, E = \{<0.3,0.7,0.9 > < 0.5,0.9 , 0.9 >. We define a mapping $f(a) = u$ and $f(b) = v$. Hence we get $f^{-1}(C) = C$ and $f^{-1}(E) = E$ where $C$ and $E$ are $\mu_N - \beta$ Open sets of $(X, \tau)$ but the inverse image of $E \in \sigma$ is $E$ which is not $\mu_N$-Pre Open in $(X, \tau)$. Hence, every $\mu_N - \beta$ Continuous need not be $\mu_N$-Continuous.

Theorem 4.30. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is $\mu_N$-Semi Continuous and $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$- Continuous then $g \circ f : (X, \tau) \to (Z, \rho)$ is $\mu_N$-Semi Continuous.

Proof. Let $V$ be any $\mu_N$- open in $(Z, \rho)$. Since, $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$-Continuous, $g^{-1}(V) = \mu_N$-open in $(Y, \sigma)$. Since, $f$ is $\mu_N$-Semi Continuous, $f^{-1}(g^{-1}(U))$ is $\mu_N$- Semi open, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\mu_N$-Semi Continuous.

Composition of two $\mu_N$-Semi Continuous need not be $\mu_N$-Semi Continuous. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. Let us assume $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ are two $\mu_N$-Semi Continuous functions. Let $V$ be any $\mu_N$-open in $(Z, \rho)$. Since, $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$-Semi Continuous, $g^{-1}(V)$ is $\mu_N$-Semi open in $(Y, \sigma)$. We know every $\mu_N$-Semi open sets need not be $\mu_N$-open. Since $f$ is $\mu_N$-Semi Continuous functions, we have to get the inverse image of $\mu_N$-open sets in $(Y, \sigma)$ must be $\mu_N$-Semi open in $(X, \tau)$. But here all the element of $(Y, \sigma)$ are $\mu_N$-Semi open. So we cannot explore a $\mu_N$-Semi Continuous function.

Theorem 4.31. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is $\mu_N$-Pre Continuous and $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$- Continuous then $g \circ f : (X, \tau) \to (Z, \rho)$ is $\mu_N$-Pre Continuous.

Proof. Let $V$ be any $\mu_N$-open in $(Z, \rho)$. Since, $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$-Continuous, $g^{-1}(V)$ is $\mu_N$-open in $(Y, \sigma)$. Since, $f$ is $\mu_N$-Pre Continuous, $f^{-1}(g^{-1}(U))$ is $\mu_N$-Pre open in $(X, \tau)$, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\mu_N$-Pre Continuous.

Composition of two $\mu_N$-Pre Continuous need not be $\mu_N$-Pre Continuous. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. Let us assume $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be two $\mu_N$-Pre Continuous functions. Let $V$ be any $\mu_N$-open in $(Z, \rho)$. Since, $g : (Y, \sigma) \to (Z, \rho)$ is $\mu_N$-Pre Continuous, $g^{-1}(V)$ is $\mu_N$-open in $(Y, \sigma)$. We know every $\mu_N$-Pre open sets need not be $\mu_N$-open. Since $f$ is $\mu_N$-Pre Continuous functions, we have to get the inverse image of $\mu_N$-open sets in $(Y, \sigma)$ must be $\mu_N$-Pre open in $(X, \tau)$. But here all the element of $(Y, \sigma)$ are $\mu_N$-Pre open, every $\mu_N$-Pre open sets need not be $\mu_N$-open So we cannot explore a $\mu_N$-Pre Continuous function.
Composition of two $\mu_N - \alpha$ Continuous need not be $\mu_N - \alpha$ Continuous. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. Let us assume $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \rho)$ be two $\mu_N - \alpha$ Continuous functions. Let $V$ be any $\mu_N$ open in $(Z, \rho)$. Since, $g: (Y, \sigma) \to (Z, \rho)$ is $\mu_N - \alpha$ Continuous, $g^{-1}(V)$ is $\mu_N - \alpha$ open in $(Y, \sigma)$. Since $f$ is $\mu_N - \alpha$ Continuous functions, we have to get the inverse image of $\mu_N$- open sets in $(Y, \sigma)$ must be $\mu_N - \alpha$ open in $(X, \tau)$. But here all the element of $(Y, \sigma)$ are $\mu_N - \alpha$ open, every $\mu_N - \alpha$ open sets need not be $\mu_N$- open So we cannot explore a $\mu_N - \alpha$ Continuous function.

Theorem 4.33. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. If $f: (X, \tau) \to (Y, \sigma)$ is $\mu_N - \beta$ Continuous and $g: (Y, \sigma) \to (Z, \rho)$ is $\mu_N - \beta$ Continuous then $g \circ f: (X, \tau) \to (Z, \rho)$ is $\mu_N - \beta$ Continuous.

Proof. Let $V$ be any $\mu_N$- open in $(Z, \rho)$. Since, $g: (Y, \sigma) \to (Z, \rho)$ is $\mu_N - \beta$ Continuous, $g^{-1}(V)$ is $\mu_N - \beta$ open in $(Y, \sigma)$. Since, $f$ is $\mu_N - \beta$ Continuous, $f^{-1}(g^{-1}(U))$ is $\mu_N - \beta$ open in $(X, \tau)$. $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\mu_N - \beta$ Continuous.

Composition of two $\mu_N - \beta$ Continuous need not be $\mu_N - \beta$ Continuous. Let $(X, \tau), (Y, \sigma), (Z, \rho)$ be three $\mu_N$ topological spaces. Let us assume $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \rho)$ be two $\mu_N - \beta$ Continuous functions. Let $V$ be any $\mu_N$- open in $(Z, \rho)$. Since, $g: (Y, \sigma) \to (Z, \rho)$ is $\mu_N - \beta$ Continuous, $g^{-1}(V)$ is $\mu_N - \beta$ open in $(Y, \sigma)$. Since $f$ is $\mu_N - \beta$ Continuous functions, we have to get the inverse image of $\mu_N$- open sets in $(Y, \sigma)$ must be $\mu_N - \beta$ open in $(X, \tau)$. But here all the element of $(Y, \sigma)$ are $\mu_N - \beta$ open, every $\mu_N - \beta$ open sets need not be $\mu_N$- open So we cannot explore a $\mu_N - \beta$ Continuous function.

5. Conclusion

In this paper, we have explored some new open sets in $\mu_N$ topological spaces and their features were investigated. The continuous functions of $\mu_N$ topological spaces and the composite functions of $\mu_N$ topological spaces were discovered and their features were discussed. We made a comparison on continuous functions of $\mu_N$ topological spaces with different types of $\mu_N$ Continuous functions and their nature were contemplated. Subsequently, we build up our research towards $\mu_N$- compact, $\mu_N$- connected and so on.

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References


