Multi anti fuzzy graph

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**Abstract**

In this paper, the new concept multi anti fuzzy graph is introduced and discussed its related concepts and basic definitions with examples. Domination on multi anti fuzzy graph is defined and its domination number is established with example. Characterization of a minimal dominating set of a multi fuzzy graph is established.

**Keywords**

Multi anti fuzzy graph, degree, order and size of a multi anti fuzzy graph, dominating set, domination number.

**AMS Subject Classification**

03E72, 03B20, 03F55, 05C72, 05C07, 05C62.

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**1. Introduction**

In 1736, Euler first introduced the concept of graph theory. Graph theory is an extremely necessary implement to constitute multiple real world situations. Nowadays, graphs do not implement all the systems properly due to the unpredictability of the parameters of systems. In 1965, Zadeh [10] introduced the concept of fuzzy set theory and show the intention of the authors to generalize the classical notion of a set. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics. In 1975, Rosenfeld [6] introduced the concepts of fuzzy graphs, there after many research have generalized they notions graph theory. Fuzzy graph theory is advanced with large number of branches. In 2010, Sabu Sebastian, T.V. Ramakrishnan [7] introduced Multi fuzzy set. It is useful to characterize the problems in the fields of image processing, taste recognition, pattern recognition, decision making and approximation of vague data. In 2017, R. Muthuraj, A. Sasireka [2,3] introduced the concept of Anti Fuzzy Graphs and Domination on Anti Fuzzy Graph. In 2020, R. Muthuraj, S. Revathi [4] discussed the new concept of multi fuzzy graph. In this paper, the concept of multi anti fuzzy graph is the extension of an anti fuzzy graph. The dominating set of a multi anti fuzzy graph is introduced and the domination number is established.

**2. Multi anti fuzzy graph**

In this section, the concept of multi anti fuzzy graph is introduced and discussed its related concepts. Throughout this paper, denote the edge between two vertices \(u\) and \(v\) as \(uv\).

**Definition 2.1.** A Multi antifuzzy graph \(G_A = (\sigma, \mu)\) defined on the underlying crisp graph \(G^c = (V, E)\), where \(E \subseteq V \times V\), is a pair of functions \(\sigma : V \rightarrow [0, 1]\) and \(\mu : V \times V \rightarrow [0, 1]\). \(\mu\) is a symmetric fuzzy relation on \(V\) such that \(\mu(uv) \geq \max\{\sigma(u), \sigma(v)\}\) for all \(u, v \in V\).

**Definition 2.2.** Let \(X\) be a non-empty set. A multi anti fuzzy set \(A\) in \(X\) is defined as a set of ordered sequences: \(A = \{\{(x, \mu_1(x), \mu_2(x), \ldots, \mu_i(x), \ldots) : x \in X\} : \mu_i : X \rightarrow [0, 1]\) for all \(i\).

**Remark 2.3.** If the sequences of the membership functions have only \(k\)-terms (finite number of terms), \(k\) is called the...
Definition 2.4. A multi anti fuzzy graph (MAFG) of dimension \( m \) defined on the underlying crisp graph \( G = (V, E) \), where \( E \subseteq V \times V \), is denoted as \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) and \( \sigma_i : V \to [0, 1] \) and \( \mu_i : V \times V \to [0, 1] \). \( \mu_i \) is a symmetric fuzzy relation on \( \sigma_i \) such that \( \mu_i(uv) \geq \max \{ \sigma_i(u), \sigma_i(v) \} \), for all \( i = 1, 2, 3, \ldots, m, u, v \in V \) and \( uv \in E \).

Remark 2.5. (a) MAFG with dimension 1 is the usual anti fuzzy graph.

(b) MAFG with dimension 2 is different from intuitionistic anti fuzzy graph.

Definition 2.6. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then the cardinality of a MAFG \( G_A \) is denoted as \( |G_A| \) and is defined as

\[
|G_A| = \sum_{u \in V} \frac{1 + \sigma_1(u) + \sigma_2(u) + \ldots + \sigma_m(u)}{m} \\
+ \sum_{u \in E} \frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m}.
\]

Definition 2.7. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then the vertex cardinality of a MAFG \( G \) or the order of a MAFG is denoted as \( |V| \) or \( O(G) \) or \( p \) and is defined as

\[
|V| = \sum_{u \in V} \frac{1 + \sigma_1(u) + \sigma_2(u) + \ldots + \sigma_m(u)}{m}.
\]

Definition 2.8. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Let \( D \subseteq V \) Then the cardinality of \( D \) or the fuzzy cardinality of \( D \) of \( G \) is denoted as \( |D| \) and is defined as

\[
|D| = \sum_{u \in D} \frac{1 + \sigma_1(u) + \sigma_2(u) + \ldots + \sigma_m(u)}{m}.
\]

Definition 2.9. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then the edge cardinality of a MAFG \( G \) or the size of a MAFG \( G \) is denoted as \( |E| \) or \( S(G) \) or \( q \) and is defined as

\[
|E| = \sum_{u \in E} \frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m}.
\]

Definition 2.10. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then the degree of a vertex \( u \in V \) in \( G \) is defined as

\[
d^A_G(u) = \sum_{v \in V} \frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m}.
\]

Definition 2.11. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). The minimum degree of \( G \) is defined as

\[
\delta(G_A) = \min \{ d^A_G(u)/u \in V \}.
\]

Definition 2.12. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). The maximum degree of \( G \) is defined as

\[
\Delta(G_A) = \max \{ d^A_G(u)/u \in V \}.
\]

Definition 2.13. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). An edge \( uv = e \) is said to be an effective edge if \( \mu_i(uv) = \max \{ \sigma_i(u), \sigma_i(v) \} \), for all \( i = 1, 2, 3, \ldots, m \).

Definition 2.14. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then \( G_A \) is said to be a strong MAFG if \( \mu_i(uv) = \max \{ \sigma_i(u), \sigma_i(v) \} \), for all \( i = 1, 2, 3, \ldots, m \).

Definition 2.15. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then \( G_A \) is said to be a complete MAFG if

(i) Every vertex of \( V \) in \( G \) is adjacent to every other vertex of \( V \) in \( G \).

(ii) \( \mu_i(uv) = \max \{ \sigma_i(u), \sigma_i(v) \} \), for all \( i = 1, 2, 3, \ldots, m \).

Definition 2.16. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Then the effective degree of a vertex \( u \in V \) in \( G \) is defined as

\[
d^E_G(u) = \sum_{v \in V} \frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m}
\]

where, \( uv \) is an effective edge.

Definition 2.17. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). The minimum effective degree of \( G \) is defined as

\[
\delta^E(G_A) = \min \{ d^E_G(u)/u \in V \}.
\]

Definition 2.18. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). The maximum effective degree of \( G \) is defined as

\[
\Delta^E(G_A) = \max \{ d^E_G(u)/u \in V \}.
\]

Definition 2.19. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). The vertices \( u \) and \( v \) in \( V \) of \( G \) is said to be independent if \( \mu_i(uv) > \max \{ \sigma_i(u), \sigma_i(v) \} \), for some \( i = 1, 2, 3, \ldots, m \).

The subset \( S \) of \( V \) is said to be an independent set if for any two vertices \( u \) and \( v \) in \( S \subseteq V \) of \( G \) is independent.

Definition 2.20. Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Let \( u \in V \) of \( G \). Then the neighbors (neighborhood) of \( u \) or an open neighbors of \( u \) in \( V \) of \( G \) is denoted by \( N(u) \) and is defined as

\[
N(u) = \{ v \in V / \mu_i(uv) = \max \{ \sigma_i(u), \sigma_i(v) \} \}
\]

for all \( i = 1, 2, 3, \ldots, m \). The closed neighbor of \( u \) in \( V \) of \( G_A \) is denoted by \( N[u] \) and is defined as \( N[u] = N(u) \cup \{ u \} \).
Definition 2.21. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. The vertex $u \in V$ of $G_A$ is said to be isolated vertex if
\[
\mu_i(uv) > \max \left\{ \sigma_i(u), \sigma_i(v) \right\},
\]
for some $i = 1, 2, 3, \ldots, m$ and for all $v \in V - \{u\}$. In other words, The vertex $u \in V$ of $G_A$ is said to be isolated vertex if $N(u) = \emptyset$ or $|N(u)| = 0.$

Definition 2.22. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. The vertex $u \in V$ of $G_A$ is said to be pendant vertex if $|N(u)| = 1.$ An edge incident on the pendant vertex is called a pendant edge.

Definition 2.23. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. Then the neighborhood degree of a vertex $u \in V$ in $G_A$ is defined as
\[
d^m_N(u) = \sum_{v \in N(u)} \frac{1 + \sigma_1(v) + \sigma_2(v) + \ldots + \sigma_m(v)}{m}.
\]

Definition 2.24. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. The minimum neighborhood degree of $G$ is defined as
\[
\delta^m_N(G_A) = \min \left\{ d^m_N(u) | u \in V \right\}.
\]

Definition 2.25. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. The maximum neighborhood degree of $G$ is defined as
\[
\Delta^m_N(G_A) = \max \left\{ d^m_N(u) | u \in V \right\}.
\]

Proposition 2.26. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. The sum of the degree of all vertices in a MAFG is equal to twice the sum of the membership values of all edges. That is
\[
\sum_{u \in V} d^m_N(u) = 2|E|.
\]

Proof. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. Let $u, v \in V$ of $G_A$ and $uv \in E$. Then, it is clear from the definition that, the edge $uv$ contribute the value
\[
\frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m}
\]
twice to the sum of degree of all vertices of $G_A$. That is,
\[
\sum_{u \in V} d^m_N(u) = 2 \times \frac{1 + \mu_1(uv) + \mu_2(uv) + \ldots + \mu_m(uv)}{m} = 2|E|
\]
\[
\Box
\]

Example 2.27. Consider $G_A = \left( (\sigma_1, \sigma_2, \sigma_3), (\mu_1, \mu_2, \mu_3) \right)$ be a MAFG of dimension 3. Here,
\[
V = \{a, b, c, d, e\}; E = \{ab, bc, cd, ca, d, ea\}
\]
All edges except $ac$ and $ed$ are effective edges.
The sets $\{a, c\}, \{b, c\}$ and $\{b, d\}$ are independent sets.

3. Domination on MA FG

In this section, the concept of domination on MAFG is defined and its domination number is obtained for MAFG with example. Throughout this section, domination on MAFG using effective edges is only considered.

Definition 3.1. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$. Let $u, v \in D \subseteq V$ in $G$. We say that $u$ dominates $v$ in $G$ if $\mu_i(uv) = \max \{\sigma_i(u), \sigma_i(v)\}$, for all $i = 1, 2, 3, \ldots; uv \in E$.

A subset $D$ of $V$ is said to be a dominating set in $G_A$ if for every $v \in V - D$ there exist $u \in D$ such that $u$ dominates $v$.

A dominating set $D$ of $V$ is said to be a minimal dominating set if no proper subset of $D$ is a dominating set of $G_A$.

The maximum fuzzy cardinality of a minimal dominating set in $G_A$ is called the domination number of $G_A$ and is denoted by $\gamma(G_A)$ or simply $\gamma_A$.

Example 3.2. Consider the MFG with dimension 3. The domination set is $D = \{a, d\}$ and $\gamma_A = 1.7$.

Remark 3.3. Let $G_A = \left( (\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m) \right)$ be a MAFG of dimension $m$

(a) For any $u, v \in V$, if $u$ dominates $v$ then $v$ dominates $u$ and hence domination is a symmetric relation on $V$.

(b) For any $u \in V, N(u)$ is precisely the set of all $v \in V$ which are dominated by $u$.

(c) If $\mu_i(uv) < \min \{\sigma_i(u), \sigma_i(v)\}$ for all $u, v \in V$, then obviously the only dominating set in $G$ is $V$.

(d) An isolated vertex does not dominate any other vertex in $G$. 
The following Theorem gives a characterization of minimal dominating sets in MAFG which is analogous to the results of A. Somasundaram and S. Somasundaram (1998) [8] in the case of a fuzzy graph.

**Theorem 3.4.** Let \( G_A = (\{\sigma_1, \sigma_2, \ldots, \sigma_m\}, (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). A dominating set \( D \) of \( G_A \) is a minimal dominating set if and only if for each \( u \in D \), one of the following two conditions holds.

(i) \( N(u) \cap D = \phi \).

(ii) There is a vertex \( v \in V - D \) such that \( N(v) \cap D = \{u\} \).

**Proof.** Let \( G_A = (\{\sigma_1, \sigma_2, \ldots, \sigma_m\}, (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \). Let \( D \) be a minimal dominating set of \( G \) and \( u \in D \). Then \( D_u \) is a minimal dominating set. Hence there exists \( v \in V - D_u \) such that \( v \) is not dominated by any element of \( D_u \).

**Case i:** If \( v = u \), then \( v = u \) is not dominated by any element of \( D_u \) and hence it is not dominated by any element of \( D \) and hence, \( N(u) \cap D = \phi \).

**Case ii:** If \( v \neq u \) then \( v \) dominates \( u \) as \( D \) is a minimal dominating set of \( G_A \) and for each \( u \in D \), one of the following two conditions holds.

(i) \( N(u) \cap D = \phi \).

(ii) There is a vertex \( v \in V - D \) such that \( N(v) \cap D = \{u\} \).

Suppose if \( D \) is not a minimal dominating set of \( G_A \) then \( D_1 \subset D \) is a dominating set of \( G \).

Consider an element \( u \in D \) and \( u \notin D_1 \). Then \( u \in V - D_1 \) and there exists \( w \in D_1 \) such that \( w \) dominates \( u \) and so \( w \in N(u) \). Also \( w \in D_1 \subset D \) and hence \( N(u) \cap D \neq \phi \).

Given \( D \) is not a minimal dominating set, then there is a vertex \( v \in V - D \) such that either \( v \) is dominated by more than one vertex of \( D \) or there exist an element \( u \in D \) such that \( u \) does not dominate any \( v \) for all \( v \in V - D \).

Case i: Let \( u, w \in D \) dominates \( v \) and \( u, w \in N(v) \). Then \( N(v) \cap D = \{u, w\} \). Case ii: Then for this \( u \in D, N(v) \cap D \neq \{u\} \) for all \( v \in V - D \).

Hence, conditions i and ii do not hold because of the assumption that \( D \) is not a minimal dominating set of \( G_A \). Hence \( D \) is a minimal dominating set of \( G_A \).

**Theorem 3.5.** Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \) without isolated vertices. Let \( D \) be the minimal dominating set of \( G_A \). Then \( V - D \) is a dominating set of \( G_A \).

**Proof.** Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a MAFG of dimension \( m \) without isolated vertices. Let \( D \) be the minimal dominating set of \( G_A \). Let \( u \in D \). Since \( G_A \) has no isolated vertices then \( v \in N(u) \)

**Case i:** If \( v \in V - D \), then every element of \( D \) is dominated by some element of \( V - D \). Hence, \( V - D \) is a dominating set of \( G_A \).

**Case ii:** If \( v \in D \) and \( D \) is a minimal dominating set, then there exists an element \( x \in V - D \) such that \( x \in N(u) \).

That is, for every element \( u \in D \), there exists an element \( x \in V - D \) such that \( x \) dominates \( u \). Hence \( V - D \) is a dominating set of \( G \).

**Theorem 3.6.** For any MAFG without isolated vertices, \( 2 \gamma \leq \frac{p}{2} \), where \( p \) is the order of MAFG.

**Proof.** Any MAFG without isolated vertices has two disjoint dominating sets and hence \( \gamma \leq \frac{p}{2} \), where \( p \) is the order of MAFG.

**Theorem 3.7.** Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a complete MAFG of dimension \( m \) without isolated vertices.
Then
\[ \gamma(G_A) = \gamma_A = \max \left\{ \frac{1 + \sigma_1(u) + \sigma_2(u) + \ldots + \sigma_m(u)}{m} : u \in V \right\}. \]

**Proof.** Let \( G_A = ((\sigma_1, \sigma_2, \ldots, \sigma_m), (\mu_1, \mu_2, \ldots, \mu_m)) \) be a complete MA FG of dimension \( m \) without isolated vertices. Then every vertex of \( V \) in \( G \) is adjacent to every other vertex of \( V \) in \( G \). Hence, the vertex with maximum fuzzy cardinality dominates all other vertices and hence
\[ \gamma(G_A) = \gamma_A = \max \left\{ \frac{1 + \sigma_1(u) + \sigma_2(u) + \ldots + \sigma_m(u)}{m} : u \in V \right\}. \]

4. Conclusion
The authors introduced the concept of multi anti fuzzy graph and its based definitions and the concept of domination on multi anti fuzzy graph. The applications of multi anti fuzzy graph and operations and other dominations on multi anti fuzzy graph will be reported in forthcoming papers.

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