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Stabilities of mixed type Quintic-Sextic functional equations in various normed spaces

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Abstract

In this paper, we introduce "Mixed Type Quintic - Sextic functional equations" and then provide their general solution, and prove generalized Ulam - Hyers stabilities in Banach spaces and Fuzzy normed spaces, by using both the direct Hyers - Ulam method and the alternative fixed point method.

Keywords

Quintic functional equation, sextic functional equation, mixed type quintic - sextic functional equation, generalized Ulam - Hyers stability, Banach space, Fuzzy Banach space, Hyers - Ulam method, alternative fixed point method.

AMS Subject Classification

39B52, 32B72, 32B82.

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1. Introduction

The stability problem for functional equation is originated from a question of S.M. Ulam [45] under group homomorphisms and positively answered for an additive functional equation on Banach spaces by D.H. Hyers [23] and T. Aoki [2]. It was further generalized and marvelous outcome has been obtained by number of authors one can refer [21, 35, 38, 42].

Over the last seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations were discussed. For more information on such problems the interested readers can refer the monographs of [1, 4, 5, 8, 18, 22, 24–26, 33, 36, 37, 41, 43, 48].

The general solution of **Quintic and Sextic functional equations**

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x) +5f(x-y)-f(x-2y)=120f(y) \qquad (1.1)$$

and

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y) -6f(x-2y)+f(x-3y)=720f(y) \qquad (1.2)$$

was introduced and investigated by T.Z. Xu et. al., [47] and establish the generalized Ulam - Hyers stability in quasi $\beta-$normed spaces via fixed point method .

In this paper, we introduce the **Mixed Type Quintic- Sextic functional equation** of the form

$$\begin{aligned} &E(w+4v)-5E(w+3v)-\frac{1}{2}\Big(E_s^q(w+3v)\Big)+10E(w+2v)\\ &+\frac{5}{2}\Big(E_s^q(w+2v)\Big)-10E(w+v)-5\Big(E_s^q(w+v)\Big)\\ &+5E(w)+5\Big(E_s^q(w)\Big)-E(w-v)\\ &-\frac{5}{2}\Big(E_s^q(w-v)\Big)+\Big(E_s^q(w-2v)\Big)=120E(v)+300\Big(E_s^q(v)\Big) \end{aligned} \qquad (1.3)$$

where $E_s^q(w) = \Big(E(w)+E(-w)\Big)$ which is different from (1.1) and (1.2). It is easy to verify that $E(w) = c_1w^5 + c_2w^6$ is the solution of the functional equation (1.3) for any positive constants c_1, c_2.

The main aim of this paper is to provide the general solution and generalized Ulam - Hyers stabilities of (1.3) in Banach spaces and fuzzy normed spaces, by using both the direct Hyers - Ulam method and the alternative fixed point method.

Now, we present the result due to Margolis, Diaz [28] and Radu [34] for fixed point theory.

Theorem 1.1. *[28, 34] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \longrightarrow \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FPC1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FPC2) The sequence $(T^n x)$ is convergent to a fixed point y^* of T
(FPC3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FPC4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

2. General Solution

In this section, we test the general solution of the functional equation (1.3). To prove the solution, we define $\mathscr{U}_1$ and $\mathscr{U}_2$ be real vector spaces.

Theorem 2.1. *For an odd mapping $E : \mathscr{U}_1 \longrightarrow \mathscr{U}_2$ fulfilling the functional equation (1.3) for all $w, v \in \mathscr{U}_1$, then E is quintic.*

Proof. Given $E : \mathscr{U}_1 \longrightarrow \mathscr{U}_2$ is an odd function. Using oddness of E in (1.3), one can obtain that

$$E(w+4v) - 5E(w+3v) + 10E(w+2v) - 10E(w+v) + 5E(w) - E(w-v) = 120E(v) \tag{2.1}$$

for all $w, v \in \mathscr{U}_1$. Now, interchanging (w, v) by $(0,0)$, $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (2.1) and using oddness of E, we arrive the subsequent equations

$$E(0) = 0$$
$$E(8w) - 5E(6w) + 10E(4w) - 129E(2w) = 0 \tag{2.2}$$
$$E(8w) - 5E(7w) + 10E(6w) - 10E(5w) + 5E(4w) - E(3w) - 120E(w) = 0 \tag{2.3}$$
$$E(7w) - 5E(6w) + 10E(5w) - 10E(4w) + 5E(3w) - E(2w) - 120E(w) = 0 \tag{2.4}$$
$$E(6w) - 5E(5w) + 10E(4w) - 10E(3w) + 5E(2w) - 121E(w) = 0 \tag{2.5}$$
$$E(5w) - 5E(4w) + 10E(3w) - 10E(2w) - 115E(w) = 0 \tag{2.6}$$
$$E(4w) - 5E(3w) + 10E(2w) - 129E(w) = 0 \tag{2.7}$$
$$E(3w) - 4E(2w) - 115E(w) = 0 \tag{2.8}$$

for all $w \in \mathscr{U}_1$. Subtracting (2.3) from (2.2), one can see that

$$5E(7w) - 15E(6w) + 10E(5w) + 5E(4w) + E(3w) - 129E(2w) + 120E(w) = 0 \tag{2.9}$$

for all $w \in \mathscr{U}_1$. Multiplying (2.4) by 5 and subtracting from (2.9), one can observe that

$$E(6w) - 40E(5w) + 55E(4w) - 24E(3w) - 1245E(2w) - 720E(w) = 0 \tag{2.10}$$

for all $w \in \mathscr{U}_1$. Multiplying (2.5) by 10 and subtracting from (2.10), one can find that

$$10E(5w) - 45E(4w) + 76E(3w) - 174E(2w) + 1930E(w) = 0 \tag{2.11}$$

for all $w \in \mathscr{U}_1$. Multiplying (2.6) by 10 and subtracting from (2.11), one can verify that

$$5E(4w) - 24E(3w) - 74E(2w) + 3080E(w) = 0 \tag{2.12}$$

for all $w \in \mathscr{U}_1$. Multiplying (2.7) by 5 and subtracting from (2.12), one can see that

$$E(3w) - 124E(2w) + 3725E(w) = 0 \tag{2.13}$$

for all $w \in \mathscr{U}_1$. Subtracting (2.8) from (2.13), one can arrive

$$120E(2w) - 3840E(w) = 0 \tag{2.14}$$

for all $w \in \mathscr{U}_1$. Thus it follows from (2.14), we achieve

$$120E(2w) = 3840E(w) \Longrightarrow E(2w) = 32E(w) \Longrightarrow E(2w) = 2^5E(w) \tag{2.15}$$

for all $w \in \mathscr{U}_1$. Hence E is quintic. □

Theorem 2.2. *For an even mapping $E : \mathscr{U}_1 \longrightarrow \mathscr{U}_2$ fulfilling the functional equation (1.3) for all $w, v \in \mathscr{U}_1$, then E is sextic.*

Proof. Given $E:\mathscr{U}_1\longrightarrow\mathscr{U}_2$ is an even function. Using evenness of E in (1.3), one can obtain that

$$E(w+4v)-6E(w+3v)+15E(w+2v)-20E(w+v)+15E(w)-6E(w-v)+E(w-2v)=720E(v) \tag{2.16}$$

for all $w,v\in\mathscr{U}_1$. Now, interchanging (w,v) by $(0,0)$, $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (2.16) and using evenness of E, we arrive the subsequent equations

$$E(0)=0$$
$$E(8w)-6E(6w)+16E(4w)-746E(2w)=0 \tag{2.17}$$
$$E(8w)-6E(7w)+15E(6w)-20E(5w)+15E(4w)-6E(3w)+E(2w)-720E(w)=0 \tag{2.18}$$
$$E(7w)-6E(6w)+15E(5w)-20E(4w)+15E(3w)-6E(2w)-719E(w)=0 \tag{2.19}$$
$$E(6w)-6E(5w)+15E(4w)-20E(3w)+15E(2w)-726E(w)=0 \tag{2.20}$$
$$E(5w)-6E(4w)+15E(3w)-20E(2w)-704E(w)=0 \tag{2.21}$$
$$E(4w)-6E(3w)+16E(2w)-746E(w)=0 \tag{2.22}$$
$$2E(3w)-12E(2w)-690E(w)=0 \tag{2.23}$$

for all $w\in\mathscr{U}_1$. Subtracting (2.18) from (2.17), one can see that

$$6E(7w)-21E(6w)+20E(5w)+E(4w)+6E(3w)-747E(2w)+720E(w)=0 \tag{2.24}$$

for all $w\in\mathscr{U}_1$. Multiplying (2.19) by 6 and subtracting from (2.24), one can observe that

$$15E(6w)-70E(5w)+121E(4w)-84E(3w)-711E(2w)+5034E(w)=0 \tag{2.25}$$

for all $w\in\mathscr{U}_1$. Multiplying (2.20) by 15 and subtracting from (2.25), one can find that

$$20E(5w)-104E(4w)+216E(3w)-936E(2w)+15924E(w)=0 \tag{2.26}$$

for all $w\in\mathscr{U}_1$. Multiplying (2.21) by 20 and subtracting from (2.26), one can verify that

$$16E(4w)-84E(3w)-536E(2w)+30004E(w)=0 \tag{2.27}$$

for all $w\in\mathscr{U}_1$. Multiplying (2.22) by 16 and subtracting from (2.27), one can see that

$$12E(3w)-792E(2w)+41940E(w)=0 \tag{2.28}$$

for all $w\in\mathscr{U}_1$. Multiplying (2.23) by 6 and subtracting from (2.28), one can arrive

$$720E(2w)-46080E(w)=0 \tag{2.29}$$

for all $w\in\mathscr{U}_1$. Thus it follows from (2.29), we achieve

$$720E(2w)=46080E(w)\Longrightarrow E(2w)=2^6E(w) \tag{2.30}$$

for all $w\in\mathscr{U}_1$. Hence E is sextic. □

Hereafter, through this article, we use the following notations:

- The functional equation can be taken as

$$\begin{aligned}\mathscr{E}(w,v)=&E(w+4v)-5E(w+3v)-\frac{1}{2}\left(E_s^q(w+3v)\right)\\&+10E(w+2v)+\frac{5}{2}\left(E_s^q(w+2v)\right)\\&-10E(w+v)-5\left(E_s^q(w+v)\right)\\&+5E(w)+5\left(E_s^q(w)\right)-E(w-v)\\&-\frac{5}{2}\left(E_s^q(w-v)\right)+\left(E_s^q(w-2v)\right)\\&-120E(v)-300\left(E_s^q(v)\right),\end{aligned}$$

  where $E_s^q(w)=\left(E(w)+E(-w)\right)$.

- Let $\alpha=\{-1,+1\}$.
- Define a constant ξ as

$$\xi_\psi=\begin{cases}2, & if\quad \psi=0;\\ \frac{1}{2}, & if\quad \psi=1.\end{cases}$$

3. Stability Results In Banach Space

In this section, we confirm the generalized Ulam - Hyers stability in Banach space using Hyers - Ulam method and the alternative fixed point method. In order to establish the stability results, let us take $\mathscr{W}_1$ be a normed space and $\mathscr{W}_2$ be a Banach space.

3.1 Hyers - Ulam Method

Theorem 3.1. *For an odd mapping $\mathscr{E}_q:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ fulfilling the functional inequality*

$$\left\|\mathscr{E}_q(w,v)\right\|\leq\mathscr{S}(w,v) \tag{3.1}$$

for all $w,v\in\mathscr{W}_1$. Then there exists one and only quintic function $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w)-E_q(w)\right\|\leq\frac{1}{2^5\cdot 5!}\times\sum_{\gamma=\frac{1-\alpha}{2}}^{\infty}\frac{1}{2^{5\gamma\alpha}}\mathscr{S}_5\left(2^{\gamma\alpha}w,2^{\gamma\alpha}w\right) \tag{3.2}$$

for all $w\in\mathscr{W}_1$. The mapping Q_5 is defined as

$$Q_5(w)=\lim_{\beta\to\infty}\frac{1}{2^{5\alpha\beta}}E_q\left(2^{\alpha\beta}w\right) \tag{3.3}$$

for all $w \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the condition

$$\lim_{\beta\to\infty} \frac{1}{2^{5\alpha\beta}} \mathscr{S}\left(2^{\alpha\beta} w, 2^{\alpha\beta} v\right) = 0 \tag{3.4}$$

for all $w, v \in \mathscr{W}_1$. The function $\mathscr{S}_5\left(2^{\gamma\alpha} w, 2^{\gamma\alpha} w\right)$ is defined by

$$\begin{aligned}\mathscr{S}_5\left(2^{\gamma\alpha} w, 2^{\gamma\alpha} w\right) = {} & \mathscr{S}(0, 2^{\gamma\alpha}\cdot 2w) + \mathscr{S}(2^{\gamma\alpha}\cdot 4w, 2^{\gamma\alpha} w)\\ & + 5\mathscr{S}(2^{\gamma\alpha}\cdot 3w, 2^{\gamma\alpha} w)\\ & + 10\mathscr{S}(2^{\gamma\alpha}\cdot 2w, 2^{\gamma\alpha} w)\\ & + 10\mathscr{S}(2^{\gamma\alpha} w, 2^{\gamma\alpha} w)\\ & + 5\mathscr{S}(0, 2^{\gamma\alpha} w)\\ & + \mathscr{S}(-2^{\gamma\alpha} w, 2^{\gamma\alpha} w)\end{aligned} \tag{3.5}$$

for all $w \in \mathscr{W}_1$.

Proof. Using oddness of E in (3.1), one can obtain that

$$\begin{aligned}&\left\|E_q(w+4v) - 5E_q(w+3v) + 10E_q(w+2v)\right.\\ &\quad - 10E_q(w+v) + 5E_q(w) - E_q(w-v)\\ &\quad \left. -120E_q(v)\right\| \le \mathscr{S}(w,v)\end{aligned} \tag{3.6}$$

for all $w, v \in \mathscr{W}_1$. Now, interchanging (w,v) by $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (3.6) and using oddness of E, we arrive the subsequent inequalities

$$\left\|E_q(8w) - 5E_q(6w) + 10E_q(4w) - 129E_q(2w)\right\| \le \mathscr{S}(0,2w) \tag{3.7}$$

$$\begin{aligned}&\left\|E_q(8w) - 5E_q(7w) + 10E_q(6w) - 10E_q(5w)\right.\\ &\left. +5E_q(4w) - E_q(3w) - 120E_q(w)\right\| \le \mathscr{S}(4w,w)\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\left\|E_q(7w) - 5E_q(6w) + 10E_q(5w) - 10E_q(4w)\right.\\ &\left. +5E_q(3w) - E_q(2w) - 120E_q(w)\right\| \le \mathscr{S}(3w,w)\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\left\|E_q(6w) - 5E_q(5w) + 10E_q(4w) - 10E_q(3w)\right.\\ &\quad\left. +5E_q(2w) - 121E_q(w)\right\| \le \mathscr{S}(2w,w)\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\left\|E_q(5w) - 5E_q(4w) + 10E_q(3w)\right.\\ &\quad\left. -10E_q(2w) - 115E_q(w)\right\| \le \mathscr{S}(w,w)\end{aligned} \tag{3.11}$$

$$\left\|E_q(4w) - 5E_q(3w) + 10E_q(2w) - 129E_q(w)\right\| \le \mathscr{S}(0,w) \tag{3.12}$$

$$\left\|E_q(3w) - 4E_q(2w) - 115E_q(w)\right\| \le \mathscr{S}(-w,w) \tag{3.13}$$

for all $w \in \mathscr{W}_1$. From (3.7) and (3.8), we have

$$\begin{aligned}&\left\|5E_q(7w) - 15E_q(6w) + 10E_q(5w) + 5E_q(4w)\right.\\ &\qquad\left. +E_q(3w) - 129E_q(2w) + 120E_q(w)\right\|\\ &\le \left\|E_q(8w) - 5E_q(6w) + 10E_q(4w) - 129E_q(2w)\right\|\\ &\quad + \left\|E_q(8w) - 5E_q(7w) + 10E_q(6w) - 10E_q(5w)\right.\\ &\qquad\left. +5E_q(4w) - E_q(3w) - 120E_q(w)\right\|\\ &\le \mathscr{S}(0,2w) + \mathscr{S}(4w,w)\end{aligned} \tag{3.14}$$

for all $w \in \mathscr{W}_1$. Multiplying (3.9) by 5, we see that

$$\begin{aligned}&\left\|5E_q(7w) - 25E_q(6w) + 50E_q(5w) - 50E_q(4w)\right.\\ &\quad\left. +25E_q(3w) - 5E_q(2w) - 600E_q(w)\right\| \le 5\mathscr{S}(3w,w)\end{aligned} \tag{3.15}$$

for all $w \in \mathscr{W}_1$. It follows from (3.14) and (3.15), we arrive

$$\begin{aligned}&\left\|10E_q(6w) - 40E_q(5w) + 55E_q(4w) - 24E_q(3w)\right.\\ &\quad\left. -124E_q(2w) - 720E_q(w)\right\|\\ &\quad \le \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w)\end{aligned} \tag{3.16}$$

for all $w \in \mathscr{W}_1$. Multiplying (3.10) by 10, we find that

$$\begin{aligned}&\left\|10E_q(6w) - 50E_q(5w) + 100E_q(4w) - 100E_q(3w)\right.\\ &\quad\left. +50E_q(2w) - 1210E_q(w)\right\| \le 10\mathscr{S}(2w,w)\end{aligned} \tag{3.17}$$

for all $w \in \mathscr{W}_1$. It follows from (3.17) and (3.16), we obtain

$$\begin{aligned}&\left\|10E_q(5w) - 45E_q(4w) + 76E_q(3w)\right.\\ &\qquad\left. -174E_q(2w) + 1930E_q(w)\right\|\\ &\le \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w) + 10\mathscr{S}(2w,w)\end{aligned} \tag{3.18}$$

for all $w \in \mathscr{W}_1$. Multiplying (3.11) by 10, we see that

$$\begin{aligned}&\left\|10E_q(5w) - 50E_q(4w) + 100E_q(3w)\right.\\ &\quad\left. -100E_q(2w) - 1150E_q(w)\right\| \le 10\mathscr{S}(w,w)\end{aligned} \tag{3.19}$$

for all $w \in \mathscr{W}_1$. From (3.18) and (3.19), we get

$$\begin{aligned}&\left\|5E_q(4w) - 24E_q(3w) - 74E_q(2w) + 3080E_q(w)\right\|\\ &\quad \le \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w)\\ &\qquad + 10\mathscr{S}(2w,w) + 10\mathscr{S}(w,w)\end{aligned} \tag{3.20}$$

for all $w \in \mathscr{W}_1$. Multiplying (3.12) by 5, we have

$$\begin{aligned}&\left\|5E_q(4w) - 25E_q(3w) + 50E_q(2w) - 645E_q(w)\right\|\\ &\qquad \le 5\mathscr{S}(0,w)\end{aligned} \tag{3.21}$$

for all $w \in \mathscr{W}_1$. Combining (3.20) and (3.21), we arrive

$$\begin{aligned}&\left\|E_q(3w) - 124E_q(2w) + 3725E_q(w)\right\|\\ &\quad \le \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w)\\ &\qquad + 10\mathscr{S}(2w,w) + 10\mathscr{S}(w,w) + 5\mathscr{S}(0,w)\end{aligned} \tag{3.22}$$

for all $w \in \mathscr{W}_1$. It follows from (3.13) and (3.22), we achieve

$$\begin{aligned}&\left\|120E_q(2w) - 3840E_q(w)\right\|\\ &\le \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w) + 10\mathscr{S}(2w,w)\\ &\qquad + 10\mathscr{S}(w,w) + 5\mathscr{S}(0,w) + \mathscr{S}(-w,w)\end{aligned} \tag{3.23}$$

for all $w \in \mathscr{W}_1$. Let us take

$$\begin{aligned}\mathscr{S}_5(w,w) = {} & \mathscr{S}(0,2w) + \mathscr{S}(4w,w) + 5\mathscr{S}(3w,w)\\ & + 10\mathscr{S}(2w,w) + 10\mathscr{S}(w,w) + 5\mathscr{S}(0,w)\\ & + \mathscr{S}(-w,w)\end{aligned} \tag{3.24}$$

for all $w \in \mathscr{W}_1$. Using (3.24) in (3.24), we reach

$$\left\|120E_q(2w) - 3840E_q(w)\right\| \le \mathscr{S}_5(w,w) \tag{3.25}$$

for all $w \in \mathscr{W}_1$. It follows from (3.25) that

$$\left\|\frac{E_q(2w)}{2^5} - E_q(w)\right\| \le \frac{1}{2^5\cdot 5!} \times \mathscr{S}_5(w,w) \tag{3.26}$$

for all $w \in \mathscr{W}_1$. Changing w by $2w$ and multiply by $\frac{1}{2^5}$ in (3.26) and adding the resultant inequality to (3.26), one can obtain

$$\left\|\frac{E_q(2^2w)}{2^{10}} - E_q(w)\right\| \le \frac{1}{2^5\cdot 5!} \times \left[\mathscr{S}_5(w,w) + \frac{1}{2^5}\mathscr{S}_5(2w,2w)\right] \tag{3.27}$$

for all $w \in \mathscr{W}_1$. Generalized for a positive integer β, we have

$$\left\|\frac{E_q(2^\beta w)}{2^{5\beta}} - E_q(w)\right\| \le \frac{1}{2^5\cdot 5!} \times \sum_{\gamma=0}^{\beta-1} \frac{\mathscr{S}_5(2^\gamma w, 2^\gamma w)}{2^{5\gamma}} \tag{3.28}$$

for all $w \in \mathscr{W}_1$. By defining w by $2^\delta w$ and dividing by $2^{5\delta}$ in (3.28) and letting $\beta \to \infty$, it shows that the sequence

$$\left\{\frac{E_q(2^\beta w)}{2^{5\beta}}\right\}$$

is a Cauchy sequence. Since $\mathscr{W}_2$ is complete, this sequence converges to a point $Q_5(w)$ in $\mathscr{W}_2$. Thus, we define this function by

$$Q_5(w) = \lim_{\beta\to\infty} \frac{E_q(2^\beta w)}{2^{5\beta}} \tag{3.29}$$

for all $w \in \mathscr{W}_1$. Taking limit as β tends to ∞ in (3.28) and using (3.29), we arrive (3.2) for the case $\alpha = 1$.

To prove that the existing $Q_5(w)$ satisfies the functional equation (1.3), changing (w,v) by $(2^\beta w, 2^\beta v)$ and dividing by $2^{5\beta}$ in (3.1), we get

$$\frac{1}{2^{5\beta}}\left\|\mathscr{E}_q(2^\beta w, 2^\beta v)\right\| \le \frac{1}{2^{5\beta}} \times \mathscr{S}(2^\beta w, 2^\beta v) \tag{3.30}$$

for all $w,v \in \mathscr{W}_1$. Letting β tends to ∞ in (3.30) using (3.4), (3.29), we obtain

$$Q_5(w,v) = 0 \tag{3.31}$$

for all $w,v \in \mathscr{W}_1$. Thus Q_5 satisfies the functional equation (1.3).

It is easy to prove that the existence of Q_5 is unique. Indeed, let R_5 be an another quintic mapping satisfying (1.3) and (3.2). Now

$$Q_5(2^\delta w) = 2^{5\delta}Q_5(w) \qquad \textit{and} \qquad R_5(2^\delta w) = 2^{5\delta}R_5(w).$$

Thus,

$$\begin{aligned}\|Q_5(w) - R_5(w)\| &= \frac{1}{2^{5\delta}}\left\{\left\|Q_5(2^\delta w) - R_5(2^\delta w)\right\|\right\} \\ &\le \frac{1}{2^{5\delta}}\left\{\left\|Q_5(2^\delta w) - E_q(2^\delta w)\right\| + \left\|R_5(2^\delta w) - E_q(2^\delta w)\right\|\right\} \\ &\le \frac{1}{2^4\cdot 5!} \times \sum_{\gamma=0}^{\infty} \frac{1}{2^{5(\gamma+\delta)}}\mathscr{S}_5\left(2^{\gamma+\delta}w, 2^{\gamma+\delta}w\right)\end{aligned} \tag{3.32}$$

for all $w \in \mathscr{W}_1$. Letting δ tends to ∞ in (3.32), we have $Q_5(w) - R_5(w) = 0$ which implies $Q_5(w) = R_5(w)$ for all $w \in \mathscr{W}_1$. Hence the theorem holds for $\alpha = 1$.

On changing w by $\frac{w}{2}$ in (3.25), we get

$$\left\|E_q(w) - 2^5E_q\left(\frac{w}{2}\right)\right\| \le \frac{1}{5!} \times \mathscr{S}_5\left(\frac{w}{2}, \frac{w}{2}\right) \tag{3.33}$$

for all $w \in \mathscr{W}_1$. Replacing w by $\frac{w}{2}$ in (3.33) and multiplying by 2^5 and adding the resultant inequality to (3.33), we arrive

$$\left\|E_q(w) - 2^{10}E_q\left(\frac{w}{2^2}\right)\right\| \le \frac{1}{5!} \times \left[\mathscr{S}_5\left(\frac{w}{2}, \frac{w}{2}\right) + 2^5\mathscr{S}_5\left(\frac{w}{2^2}, \frac{w}{2^2}\right)\right] \tag{3.34}$$

for all $w \in \mathscr{W}_1$. Generalizing for a positive integer β, we see that

$$\begin{aligned}\left\|E_q(w) - 2^{5\beta}E_q\left(\frac{w}{2^\beta}\right)\right\| &\le \frac{1}{5!} \times \sum_{\gamma=1}^{\beta} 2^{5\beta-1}\mathscr{S}_5\left(\frac{w}{2^\beta}, \frac{w}{2^\beta}\right) \\ &= \frac{1}{2^5\cdot 5!} \times \sum_{\gamma=1}^{\beta} 2^{5\beta}\mathscr{S}_5\left(\frac{w}{2^\beta}, \frac{w}{2^\beta}\right)\end{aligned} \tag{3.35}$$

for all $w \in \mathscr{W}_1$. The rest of the proof is similar clues that of case $\alpha = 1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.1 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 3.2. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\left\|\mathscr{E}_q(w,v)\right\| \le \Phi \tag{3.36}$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w) - E_q(w)\right\| \le \frac{33\Phi}{5!|31|} \tag{3.37}$$

for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.1 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 3.3. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\left\|\mathscr{E}_q(w,v)\right\| \leq \begin{cases} \Phi\left\{||w||^{\phi}+||v||^{\phi}\right\}; \\ \Phi\left\{||w||^{\phi_1}+||v||^{\phi_2}\right\}; \end{cases} \tag{3.38}$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $\phi,\phi_1,\phi_2 \neq 5$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w)-E_q(w)\right\| \leq \begin{cases} \dfrac{\Gamma_{5T}\ \Phi||w||^{\phi}}{5!|2^5-2^{\phi}|}; \\ \dfrac{\Gamma_{5T1}\ \Phi||w||^{\phi_1}}{5!|2^5-2^{\phi_1}|}+\dfrac{\Gamma_{5T2}\ \Phi||w||^{\phi_2}}{5!|2^5-2^{\phi_2}|}; \end{cases} \tag{3.39}$$

where

$$\begin{aligned} \Gamma_{5T} &= \left(11\cdot 2^{\phi}+5\cdot 3^{\phi}+4^{\phi}+43\right); \\ \Gamma_{5T1} &= \left(10\cdot 2^{\phi_1}+5\cdot 3^{\phi_1}+4^{\phi_1}+32\right); \\ \Gamma_{5T2} &= \left(2^{\phi_2}+11\right); \end{aligned} \tag{3.40}$$

for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.1 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.4. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\left\|\mathscr{E}_q(w,v)\right\| \leq \begin{cases} \Phi\left\{||w||^{\phi}||v||^{\phi}+\left[||w||^{2\phi}+||v||^{2\phi}\right]\right\}; \\ \Phi\left\{||w||^{\phi_1}||v||^{\phi_2}+\left[||w||^{\phi_1+\phi_2}+||v||^{\phi_1+\phi_2}\right]\right\}; \end{cases} \tag{3.41}$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $2\phi,\phi_1+\phi_2 \neq 5$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w)-E_q(w)\right\| \leq \begin{cases} \dfrac{\Gamma_{5J}\ \Phi||w||^{\phi}}{5!|2^5-2^{2\phi}|}; \\ \dfrac{\Gamma_{5J1}\ \Phi||w||^{\phi_1+\phi_2}}{5!|2^5-2^{\phi_1+\phi_2}|}; \end{cases} \tag{3.42}$$

where

$$\begin{aligned} \Gamma_{5J} &= \left(10\cdot 2^{\phi}+11\cdot 2^{2\phi}+5(3^{\phi}+3^{2\phi})+4^{\phi}+4^{2\phi}+54\right); \\ \Gamma_{5J1} &= \Big(11\cdot 2^{\phi_1+\phi_2}+5\cdot 3^{\phi_1+\phi_2}+4^{\phi_1+\phi_2} \\ &\qquad +4^{\phi_1}+5\cdot 3^{\phi_1}+10\cdot 2^{\phi_1}+54\Big); \end{aligned} \tag{3.43}$$

for all $w \in \mathscr{W}_1$.

Theorem 3.5. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}_s(w,v)\| \leq \mathscr{S}(w,v) \tag{3.44}$$

for all $w,v \in \mathscr{W}_1$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w)-E_s(w)\| \leq \frac{1}{2^6\cdot 6!}\times\sum_{\gamma=\frac{1-\alpha}{2}}^{\infty}\frac{1}{2^{6\gamma\alpha}}\mathscr{S}_6\left(2^{\gamma\alpha}w,2^{\gamma\alpha}w\right) \tag{3.45}$$

for all $w \in \mathscr{W}_1$. The mapping Q_6 is defined as

$$Q_6(w) = \lim_{\beta\to\infty}\frac{1}{2^{6\alpha\beta}}E_s\left(2^{\alpha\beta}w\right) \tag{3.46}$$

for all $w \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the condition

$$\lim_{\beta\to\infty}\frac{1}{2^{6\alpha\beta}}\mathscr{S}\left(2^{\alpha\beta}w,2^{\alpha\beta}v\right) = 0 \tag{3.47}$$

for all $w,v \in \mathscr{W}_1$. The function $\mathscr{S}_6\left(2^{\gamma\alpha}w,2^{\gamma\alpha}w\right)$ is defined by

$$\begin{aligned} &\mathscr{S}_6\left(2^{\gamma\alpha}w,2^{\gamma\alpha}w\right) \\ &= \mathscr{S}(0,2^{\gamma\alpha}\cdot 2w)+\mathscr{S}(2^{\gamma\alpha}\cdot 4w,2^{\gamma\alpha}w) \\ &\quad +5\mathscr{S}(2^{\gamma\alpha}\cdot 3w,2^{\gamma\alpha}w)+10\mathscr{S}(2^{\gamma\alpha}\cdot 2w,2^{\gamma\alpha}w) \\ &\quad +10\mathscr{S}(2^{\gamma\alpha}w,2^{\gamma\alpha}w)+5\mathscr{S}(0,2^{\gamma\alpha}w) \\ &\quad +\mathscr{S}(-2^{\gamma\alpha}w,2^{\gamma\alpha}w) \end{aligned} \tag{3.48}$$

for all $w \in \mathscr{W}_1$.

Proof. Using evenness of E in (3.44), one can obtain that

$$\begin{aligned} &\|E_s(w+4v)-6E_s(w+3v)+15E_s(w+2v) \\ &\quad -20E_s(w+v)+15E_s(w)-6E_s(w-v) \\ &\qquad +E_s(w-2v)-720E_s(v)\| \leq \mathscr{S}(w,v) \end{aligned} \tag{3.49}$$

for all $w,v \in \mathscr{W}_1$. Now, interchanging (w,v) by $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (3.49) and using evenness of E, we arrive the subsequent inequalities

$$\begin{aligned} &\|E_s(8w)-6E_s(6w)+16E_s(4w)-746E_s(2w)\| \\ &\leq \mathscr{S}(0,2w) \end{aligned} \tag{3.50}$$

$$\begin{aligned} &\|E_s(8w)-6E_s(7w)+15E_s(6w)-20E_s(5w) \\ &\quad +15E_s(4w)-6E_s(3w)+E_s(2w)-720E_s(w)\| \\ &\leq \mathscr{S}(4w,w) \end{aligned} \tag{3.51}$$

$$\begin{aligned} &\|E_s(7w)-6E_s(6w)+15E_s(5w)-20E_s(4w) \\ &\quad +15E_s(3w)-6E_s(2w)-719E_s(w)\| \leq \mathscr{S}(3w,w) \end{aligned} \tag{3.52}$$

$$\begin{aligned} &\|E_s(6w)-6E_s(5w)+15E_s(4w)-20E_s(3w) \\ &\quad +15E_s(2w)-726E_s(w)\| \leq \mathscr{S}(2w,w) \end{aligned} \tag{3.53}$$

$$\|E_s(5w)-6E_s(4w)+15E_s(3w) -20E_s(2w)-704E_s(w)\| \leq \mathscr{S}(w,w) \quad (3.54)$$

$$\|E_s(4w)-6E_s(3w)+16E_s(2w)-746E_s(w)\| \leq \mathscr{S}(0,w) \quad (3.55)$$

$$\|2E_s(3w)-12E_s(2w)-690E_s(w)\| \leq \mathscr{S}(-w,w) \quad (3.56)$$

for all $w \in \mathscr{W}_1$. From (3.50) and (3.51), we have

$$\begin{aligned}\|6E_s(7w)&-21E_s(6w)+20E_s(5w)+E_s(4w) +6E_s(3w)-747E_s(2w)+720E_s(w)\| \\ &\leq \|E_s(8w)-6E_s(6w)+16E_s(4w)-746E_s(2w)\| \\ &+\|E_s(8w)-6E_s(7w)+15E_s(6w)-20E_s(5w) +15E_s(4w)-6E_s(3w)+E_s(2w)-720E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w) \end{aligned} \quad (3.57)$$

for all $w \in \mathscr{W}_1$. Multiplying (3.52) by 6, we see that

$$\|6E_s(7w)-36E_s(6w)+90E_s(5w)-120E_s(4w) +90E_s(3w)-36E_s(2w)-5514E_s(w)\| \leq 6\mathscr{S}(3w,w) \quad (3.58)$$

for all $w \in \mathscr{W}_1$. It follows from (3.57) and (3.58), we arrive

$$\begin{aligned}\|15E_s(6w)&-70E_s(5w)+121E_s(4w)-84E_s(3w) -711E_s(2w)+5034E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w)+6\mathscr{S}(3w,w)\end{aligned} \quad (3.59)$$

for all $w \in \mathscr{W}_1$. Multiplying (3.53) by 15, we find that

$$\|15E_s(6w)-90E_s(5w)+225E_s(4w)-300E_s(3w) +225E_s(2w)-10890E_s(w)\| \leq 15\mathscr{S}(2w,w) \quad (3.60)$$

for all $w \in \mathscr{W}_1$. Combining (3.60) and (3.59), we obtin

$$\begin{aligned}\|20E_s(5w)&-104E_s(4w)+216E_s(3w) -936E_s(2w)+15924E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w) +6\mathscr{S}(3w,w)+15\mathscr{S}(2w,w)\end{aligned} \quad (3.61)$$

for all $w \in \mathscr{W}_1$. Multiplying (3.54) by 20, we get

$$\|20E_s(5w)-120E_s(4w)+300E_s(3w) -400E_s(2w)-14080E_s(w)\| \leq 20\mathscr{S}(w,w) \quad (3.62)$$

for all $w \in \mathscr{W}_1$. It follows from (3.61) and (3.62), we have

$$\begin{aligned}\|16E_s(4w)&-84E_s(3w)-536E_s(2w)+30004E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w)+6\mathscr{S}(3w,w) +15\mathscr{S}(2w,w)+20\mathscr{S}(w,w)\end{aligned} \quad (3.63)$$

for all $w \in \mathscr{W}_1$. Multiplying (3.55) by 16, we see that

$$\|16E_s(4w)-96E_s(3w)+256E_s(2w)-11936E_s(w)\| \leq 16\mathscr{S}(0,w) \quad (3.64)$$

for all $w \in \mathscr{W}_1$. From (3.63) and (3.64), we arrive

$$\begin{aligned}\|12E_s(3w)&-792E_s(2w)+41940E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w)+6\mathscr{S}(3w,w) +15\mathscr{S}(2w,w)+20\mathscr{S}(w,w)+16\mathscr{S}(0,w)\end{aligned} \quad (3.65)$$

for all $w \in \mathscr{W}_1$. Multiplying (3.56) by 6 and it follows from (3.65), we achieve

$$\begin{aligned}\|720E_s(2w)&-46080E_s(w)\| \\ &\leq \mathscr{S}(0,2w)+\mathscr{S}(4w,w)+6\mathscr{S}(3w,w)+15\mathscr{S}(2w,w) +20\mathscr{S}(w,w)+16\mathscr{S}(0,w)+6\mathscr{S}(-w,w)\end{aligned} \quad (3.66)$$

for all $w \in \mathscr{W}_1$. Let us take

$$\begin{aligned}\mathscr{S}_6(w,w) = \mathscr{S}(0,2w)&+\mathscr{S}(4w,w)+6\mathscr{S}(3w,w) +15\mathscr{S}(2w,w)+20\mathscr{S}(w,w)+16\mathscr{S}(0,w) +6\mathscr{S}(-w,w)\end{aligned} \quad (3.67)$$

for all $w \in \mathscr{W}_1$. Using (3.67) in (3.67), we reach

$$\|720E_s(2w)-46080E_s(w)\| \leq \mathscr{S}_6(w,w) \quad (3.68)$$

for all $w \in \mathscr{W}_1$. It follows from (3.68) that

$$\left\|\frac{E_s(2w)}{2^6}-E_s(w)\right\| \leq \frac{1}{2^6\cdot 6!}\times \mathscr{S}_6(w,w) \quad (3.69)$$

for all $w \in \mathscr{W}_1$. Changing w by $2w$ and multiply by $\frac{1}{2^6}$ in (3.69) and adding the resultant inequality to (3.69), one can obtain

$$\left\|\frac{E_s(2^2w)}{2^{12}}-E_s(w)\right\| \leq \frac{1}{2^6\cdot 6!}\left[\mathscr{S}_6(w,w)+\frac{1}{2^6}\mathscr{S}_6(2w,2w)\right] \quad (3.70)$$

for all $w \in \mathscr{W}_1$. Generalized for a positive integer β, we have

$$\left\|\frac{E_s(2^\beta w)}{2^{6\beta}}-E_s(w)\right\| \leq \frac{1}{2^6\cdot 6!}\times \sum_{\gamma=0}^{\beta-1}\frac{\mathscr{S}_6(2^\gamma w,2^\gamma w)}{2^{6\gamma}} \quad (3.71)$$

for all $w \in \mathscr{W}_1$. The rest of the proof is similar lines to that of Theorem 3.1. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.5 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 3.6. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}_s(w,v)\| \leq \Phi \quad (3.72)$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w)-E_s(w)\| \leq \frac{65\Phi}{6!|63|} \quad (3.73)$$

for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.5 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 3.7. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}_s(w,v)\| \leq \begin{cases} \Phi\left\{||w||^{\phi}+||v||^{\phi}\right\}; \\ \Phi\left\{||w||^{\phi_1}+||v||^{\phi_2}\right\}; \end{cases} \tag{3.74}$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $\phi, \phi_1, \phi_2 \neq 6$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w)-E_s(w)\| \leq \begin{cases} \dfrac{\Gamma_{6T}\ \Phi||w||^{\phi}}{6!|2^6-2^{\phi}|}; \\ \dfrac{\Gamma_{6T1}\ \Phi||w||^{\phi_1}}{6!|2^6-2^{\phi_1}|}+\dfrac{\Gamma_{6T2}\ \Phi||w||^{\phi_2}}{6!|2^6-2^{\phi_2}|}; \end{cases} \tag{3.75}$$

where

$$\begin{aligned} \Gamma_{6T} &= \Big(16\cdot 2^{\phi}+6\cdot 3^{\phi}+4^{\phi}+91\Big); \\ \Gamma_{6T1} &= \Big(15\cdot 2^{\phi_1}+6\cdot 3^{\phi_1}+4^{\phi_1}+26\Big); \\ \Gamma_{6T2} &= \Big(2^{\phi_2}+65\Big); \end{aligned} \tag{3.76}$$

for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.5 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.8. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}_s(w,v)\| \leq \begin{cases} \Phi\left\{||w||^{\phi}||v||^{\phi}+\left[||w||^{2\phi}+||v||^{2\phi}\right]\right\}; \\ \Phi\left\{||w||^{\phi_1}||v||^{\phi_2}+\left[||w||^{\phi_1+\phi_2}+||v||^{\phi_1+\phi_2}\right]\right\}; \end{cases} \tag{3.77}$$

for all $w,v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $2\phi, \phi_1+\phi_2 \neq 6$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w)-E_s(w)\| \leq \begin{cases} \dfrac{\Gamma_{6J}\ \Phi||w||^{2\phi}}{6!|2^6-2^{2\phi}|}; \\ \dfrac{\Gamma_{6J1}\ \Phi||w||^{\phi_1+\phi_2}}{6!|2^6-2^{\phi_1+\phi_2}|}; \end{cases} \tag{3.78}$$

where

$$\begin{aligned} \Gamma_{6J} &= \Big(16\cdot 2^{\phi}+15\cdot 2^{2\phi}+6(3^{\phi}+3^{2\phi})+4^{\phi}+4^{2\phi}+117\Big); \\ \Gamma_{6J1} &= \Big(16\cdot 2^{\phi_1+\phi_2}+5(3^{\phi_1+\phi_2}+3^{\phi_1}) \\ &\qquad +4^{\phi_1+\phi_2}+4^{\phi_1}+15\cdot 2^{\phi_1}+90\Big); \end{aligned} \tag{3.79}$$

for all $w \in \mathscr{W}_1$.

Theorem 3.9. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}(w,v)\| \leq \mathscr{S}(w,v) \tag{3.80}$$

for all $w,v \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned} &\|E(w)-Q_5(w)-Q_6(w)\| \\ &\leq \frac{1}{2^6\cdot 5!}\sum_{\gamma=\frac{1-\alpha}{2}}^{\infty}\frac{1}{2^{5\gamma\alpha}}\left\{\mathscr{S}_5\Big(2^{\gamma\alpha}w,2^{\gamma\alpha}w\Big)\right. \\ &\qquad\left.+\ \mathscr{S}_5\Big(-2^{\gamma\alpha}w,-2^{\gamma\alpha}w\Big)\right\} \\ &+\frac{1}{2^7\cdot 6!}\sum_{\gamma=\frac{1-\alpha}{2}}^{\infty}\frac{1}{2^{6\gamma\alpha}}\left\{\mathscr{S}_6\Big(2^{\gamma\alpha}w,2^{\gamma\alpha}w\Big)\right. \\ &\qquad\left.+\ \mathscr{S}_6\Big(-2^{\gamma\alpha}w,-2^{\gamma\alpha}w\Big)\right\} \end{aligned} \tag{3.81}$$

for all $w \in \mathscr{W}_1$. The mappings Q_5 and Q_6 are defined in (3.3) and (3.46) for all $w \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the conditions (3.4) and (3.47) for all $w,v \in \mathscr{W}_1$. The functions $\mathscr{S}_5\Big(2^{\gamma\alpha}w,2^{\gamma\alpha}w\Big)$ and $\mathscr{S}_6\Big(2^{\gamma\alpha}w,2^{\gamma\alpha}w\Big)$ are given in (3.5) and (3.48) for all $w \in \mathscr{W}_1$.

Proof. We know that by definition of odd function, we have

$$E_O(w) = \frac{E_q(w)-E_q(-w)}{2} \tag{3.82}$$

for all $w \in \mathscr{W}_1$. It follows from (3.82) that

$$\begin{aligned} \|E_O(w,v)\| &\leq \frac{1}{2}\Big\{E_q(w,v)-E_q(-w,-v)\Big\} \\ &\leq \frac{1}{2}\Big\{\mathscr{S}(w,v)+\mathscr{S}(-w,-v)\Big\} \end{aligned} \tag{3.83}$$

for all $w \in \mathscr{W}_1$. Thus by Theorem 3.1, we arrive

$$\begin{aligned} &\|Q_5(w)-E_O(w)\| \\ &\leq \frac{1}{2^6\cdot 5!}\sum_{\gamma=\frac{1-\alpha}{2}}^{\infty}\frac{1}{2^{5\gamma\alpha}}\left\{\mathscr{S}_5\Big(2^{\gamma\alpha}w,2^{\gamma\alpha}w\Big)\right. \\ &\qquad\left.+\ \mathscr{S}_5\Big(-2^{\gamma\alpha}w,-2^{\gamma\alpha}w\Big)\right\} \end{aligned} \tag{3.84}$$

for all $w \in \mathscr{W}_1$. We know that by definition of even function, we have

$$E_E(w) = \frac{E_q(w)+E_q(-w)}{2} \tag{3.85}$$

for all $w \in \mathscr{W}_1$. It follows from (3.85) that

$$\begin{aligned} \|E_E(w,v)\| &\leq \frac{1}{2}\Big\{E_s(w,v)-E_s(-w,-v)\Big\} \\ &\leq \frac{1}{2}\Big\{\mathscr{S}(w,v)+\mathscr{S}(-w,-v)\Big\} \end{aligned} \tag{3.86}$$

for all $w \in \mathscr{W}_1$. Thus by Theorem 3.5, we arrive

$$\begin{aligned}&\|Q_6(w) - E_E(w)\| \\ &\leq \frac{1}{2^7 \cdot 6!} \sum_{\gamma=\frac{1-\alpha}{2}}^{\infty} \frac{1}{2^{6\gamma\alpha}} \left\{ \mathscr{S}_6\left(2^{\gamma\alpha}w, 2^{\gamma\alpha}w\right) \right. \\ &\qquad \left. + \mathscr{S}_6\left(-2^{\gamma\alpha}w, -2^{\gamma\alpha}w\right) \right\} \end{aligned} \tag{3.87}$$

for all $w \in \mathscr{W}_1$. Now, define a function

$$E(w) = E_O(w) + E_E(w) \tag{3.88}$$

for all $w \in \mathscr{W}_1$. Combining (3.88), (3.84) and (3.87), we derived our result. □

The following corollary is an immediate consequence of Theorem 3.9 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 3.10. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}(w,v)\| \leq \Phi \tag{3.89}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|E(w) - Q_5(w) - Q_6(w)\| \leq \left(\frac{33}{5!|31|} + \frac{65}{6!|63|}\right)\Phi \tag{3.90}$$

for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.9 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 3.11. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}(w,v)\| \leq \begin{cases} \Phi\left\{||w||^{\phi} + ||v||^{\phi}\right\}; \\ \Phi\left\{||w||^{\phi_1} + ||v||^{\phi_2}\right\}; \end{cases} \tag{3.91}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $\phi, \phi_1, \phi_2 \neq 5, 6$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\|E(w) - Q_5(w) - Q_6(w)\| \\ &\leq \begin{cases} \left(\frac{\Gamma_{5T}}{5!|2^5 - 2^{\phi}|} + \frac{\Gamma_{6T}}{6!|2^6 - 2^{\phi}|}\right)\Phi||w||^{\phi}; \\ \left\{\left(\frac{\Gamma_{5T1}}{5!|2^5-2^{\phi_1}|} + \frac{\Gamma_{6T1}}{6!|2^6-2^{\phi_1}|}\right)\Phi||w||^{\phi_1} \right. \\ \left. + \left(\frac{\Gamma_{6T2}}{6!|2^6-2^{\phi_2}|} + \frac{\Gamma_{5T2}}{5!|2^5-2^{\phi_2}|}\right)\Phi||w||^{\phi_2}\right\}; \end{cases}\end{aligned} \tag{3.92}$$

where $\Gamma_g s$ are defined in (3.40) and (3.76) respectively for all $w \in \mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 3.9 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.12. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}(w,v)\| \leq \begin{cases} \Phi\left\{||w||^{\phi}||v||^{\phi} + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \\ \Phi\left\{||w||^{\phi_1}||v||^{\phi_2} + \left[||w||^{\phi_1+\phi_2} + ||v||^{\phi_1+\phi_2}\right]\right\}; \end{cases} \tag{3.93}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $2\phi, \phi_1 + \phi_2 \neq 5, 6$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\|E(w) - Q_5(w) - Q_6(w)\| \\ &\leq \begin{cases} \left\{\left(\frac{\Gamma_{5J}}{5!|2^5-2^{2\phi}|} + \frac{\Gamma_{6J}}{6!|2^6-2^{2\phi}|}\right)\Phi||w||^{2\phi}\right\}; \\ \left\{\left(\frac{\Gamma_{5J1}}{5!|2^5-2^{\phi_1+\phi_2}|} + \frac{\Gamma_{6J1}}{6!|2^6-2^{\phi_1+\phi_2}|}\right)\Phi||w||^{\phi_1+\phi_2}\right\}; \end{cases}\end{aligned} \tag{3.94}$$

where $\Gamma'_g s$ are defined in (3.43) and (3.79) respectively for all $w \in \mathscr{W}_1$.

3.2 Alternative Fixed Point Method

Theorem 3.13. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality (3.1) for all $w, v \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0, \infty)$ is a function fulfilling the condition*

$$\lim_{\beta\to\infty} \frac{1}{\xi_\psi^{5\beta}} \mathscr{S}\left(\xi_\psi^{\beta}w, \xi_\psi^{\beta}v\right) = 0 \tag{3.95}$$

for all $w, v \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w) - E_q(w)\right\| \leq \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}} \mathscr{S}\left(w, w\right) \tag{3.96}$$

for all $w \in \mathscr{W}_1$. If $\mathscr{L} = \mathscr{L}(\psi)$, with the property

$$\frac{1}{\xi_\psi^5} \mathscr{S}_5\left(\xi_\psi w, \xi_\psi w\right) = \mathscr{L}\mathscr{S}_5\left(w, w\right) \tag{3.97}$$

with the condition that

$$\mathscr{S}_5\left(w, w\right) = \frac{1}{5!}\mathscr{S}_5\left(\frac{w}{2}, \frac{w}{2}\right), \tag{3.98}$$

where $\mathscr{S}_5\left(w, w\right)$ is defined in (3.24) for all $w \in \mathscr{W}_1$.

Proof. Consider the set

$$\mathscr{B} = \left\{E_1 | E_1 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2, E_1(0) = 0\right\} \tag{3.99}$$

Let us introduce the generalized metric on (3.99) by

$$\inf\left\{\zeta : \|E_1(w) - E_2(w)\| \leq \zeta\mathscr{S}_5\left(w, w\right)\right\} \tag{3.100}$$

for all $w \in \mathscr{W}_1$. One can easy to see verify that (3.100) is complete with respect to the defined metric. Define a mapping $\Upsilon : \mathscr{B} \to \mathscr{B}$ by

$$\Upsilon E_q(w) = \frac{1}{\xi_\psi^5} E_q(\xi_\psi) \tag{3.101}$$

for all $w \in \mathscr{W}_1$. Now, for any $E_1, E_2 \in \mathscr{B}$, we arrive

$$\|E_1(w) - E_2(w)\| \le \zeta \mathscr{S}_5\Big(w,w\Big)$$
$$\Longrightarrow \left\| \frac{1}{\xi_\psi^5} E_1(\xi_\psi) - \frac{1}{\xi_\psi^5} E_2(\xi_\psi) \right\| \le \zeta \mathscr{S}_5\Big(w,w\Big)$$
$$\Longrightarrow \|\Upsilon E_1(w) - \Upsilon E_2(w)\| \le \mathscr{L} \zeta \mathscr{S}_5\Big(w,w\Big)$$

for all $w \in \mathscr{W}_1$. This implies that Υ is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. It follows from (3.26) that

$$\left\| \frac{E_q(2w)}{2^5} - E_q(w) \right\| \le \frac{1}{2^5 \cdot 5!} \mathscr{S}_5(w,w) \tag{3.102}$$

for all $w \in \mathscr{W}_1$.

For the case $\psi = 0$, it follows from (3.97), (3.98), (3.101) and (3.102),

$$\left\|\Upsilon E_q(w) - E_q(w)\right\| \le \mathscr{L} \mathscr{S}_5(w,w) = \mathscr{L}^{1-0} \mathscr{S}_5(w,w) \tag{3.103}$$

for all $w \in \mathscr{W}_1$.

Replacing w by $\frac{w}{2}$ in (3.102), we obtain

$$\left\|E_q(w) - 2^5 E_q\left(\frac{w}{2}\right)\right\| \le \frac{1}{5!} \mathscr{S}_5\left(\frac{w}{2}, \frac{w}{2}\right) \tag{3.104}$$

for all $w \in \mathscr{W}_1$.

For the case $\psi = 1$, it follows from (3.97), (3.98), (3.101) and (3.104),

$$\left\|E_q(w) - \Upsilon E_q(w)\right\| \le \mathscr{S}_5(w,w) = \mathscr{L}^{1-1} \mathscr{S}_5(w,w) \tag{3.105}$$

for all $w \in \mathscr{W}_1$. From (3.103) and (3.105), we see that

$$\left\|E_q(w) - \Upsilon E_q(w)\right\| \le \mathscr{L}^{1-\psi} \mathscr{S}_5(w,w) \tag{3.106}$$

for all $w \in \mathscr{W}_1$. Thus, condition (FPC1) of Theorem 1.1 holds. It follows from condition (FPC2) of Theorem 1.1 , that there exists a fixed point Q_5 of Υ in $\mathscr{L}$ such that

$$Q_5(w) = \lim_{\beta \to \infty} \frac{1}{\xi_\psi^5} E_q\Big(\xi_\psi w\Big) \tag{3.107}$$

for all $w \in \mathscr{W}_1$. To prove the existing Q_5 satisfies (1.3), the proof is similar to that of Theorem 3.1.

Again by condition (FPC3) of Theorem 1.1 , Q_5 is the unique fixed point of Υ in the set

$$\Delta = \Big\{Q_5 : \left\|E_q(w) - Q_5(w)\right\| \le \infty\Big\} \tag{3.108}$$

for all $w \in \mathscr{W}_1$. Finally, by condition (FPC4) of Theorem 1.1 , we get

$$\left\|E_q(w) - Q_5(w)\right\| \le \frac{1}{1-\mathscr{L}} \left\|E_q(w) - \Upsilon E_q(w)\right\|$$
$$\Longrightarrow \left\|E_q(w) - Q_5(w)\right\| \le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}} \mathscr{S}_5(w,w)$$

for all $w \in \mathscr{W}_1$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.13 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.14. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\left\|\mathscr{E}_q(w,v)\right\| \le \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \tag{3.109}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\left\|Q_5(w) - E_q(w)\right\| \le \begin{cases} \dfrac{33\Phi}{5!|31|}; \\ \dfrac{\Gamma_{5T}\ \Phi||w||^\phi}{5!|2^5 - 2^\phi|}; & \phi \neq 5 \\ \dfrac{\Gamma_{5J}\Phi||w||^{2\phi}}{5!|2^5 - 2^{2\phi}|}; & 2\phi \neq 5 \end{cases} \tag{3.110}$$

where $\Gamma_g s$ are defined in (3.40) and (3.43) respectively for all $w \in \mathscr{W}_1$.

Proof. If we take

$$\mathscr{S}(w,v) = \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \tag{3.111}$$

for all $w, v \in \mathscr{W}_1$. Changing (w,v) by $(\xi_\psi^\beta w, \xi_\psi^\beta v)$ and dividing by $\xi_\psi^{5\beta}$ in (3.111) and letting β tends to ∞, we see that (3.95) holds for all $w, v \in \mathscr{W}_1$.

It follows from (3.98), (3.111) and (3.24), one can find that

$$\begin{aligned} &\mathscr{S}\Big(w,w\Big) \\ &= \frac{1}{5!}\ \mathscr{S}\left(\frac{w}{2}, \frac{w}{2}\right) \\ &= \begin{cases} \frac{33\Phi}{5!}; \\ \frac{\left(11\cdot 2^\phi + 5\cdot 3^\phi + 4^\phi + 43\right)\Phi||w||^\phi}{5! 2^\phi}; \\ \frac{\left(10\cdot 2^\phi + 11\cdot 2^{2\phi} + 5(3^\phi + 3^{2\phi}) + 4^\phi + 4^{2\phi} + 54\right)\Phi||w||^{2\phi}}{5! 2^{2\phi}}; \end{cases} \end{aligned} \tag{3.112}$$

for all $w \in \mathscr{W}_1$.

Again, it follows from (3.97), (3.111) and (3.24), one can observe that

$$\frac{1}{\xi_\psi^5}\mathscr{S}_5\Big(\xi_\psi w, \xi_\psi w\Big) = \begin{cases} \frac{33\Phi}{5!\ \xi_\psi^5}; \\ \frac{\left(11\cdot 2^\phi+5\cdot 3^\phi+4^\phi+43\right)\Phi||w||^\phi}{5!\ \xi_\psi^{5-\phi}}; \\ \frac{\left(10\cdot 2^\phi+11\cdot 2^{2\phi}+5(3^\phi+3^{2\phi})+4^\phi+4^{2\phi}+54\right)\Phi||w||^{2\phi}}{5!\ \xi_\psi^{5-2\phi}}; \end{cases} = \begin{cases} \mathscr{L}\mathscr{S}_5\Big(w,w\Big); \\ \mathscr{L}\mathscr{S}_5\Big(w,w\Big); \\ \mathscr{L}\mathscr{S}_5\Big(w,w\Big); \end{cases} \tag{3.113}$$

for all $w \in \mathscr{W}_1$. Thus, the functional inequality (3.96) holds for the following cases.

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^5} = \frac{1}{2^5} = 2^{-5}$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{-5}\right)^{1-0}}{1-2^{-5}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{1}{31}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^5 = 2^5$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{5}\right)^{1-1}}{1-2^{5}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{1}{-31}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^{5-\phi}} = \frac{1}{2^{5-\phi}} = 2^{\phi-5} : \phi < 5$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{\phi-5}\right)^{1-0}}{1-2^{\phi-5}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{2^\phi}{2^5-2^\phi}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^{5-\phi} = 2^{5-\phi} : \phi > 5$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{5-\phi}\right)^{1-1}}{1-2^{5-\phi}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{2^\phi}{2^\phi-2^5}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^{5-2\phi}} = \frac{1}{2^{5-2\phi}} = 2^{2\phi-5} : 2\phi < 5$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{2\phi-5}\right)^{1-0}}{1-2^{2\phi-5}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{2^{2\phi}}{2^5-2^{2\phi}}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^{5-2\phi} = 2^{5-2\phi} : 2\phi > 5$$

$$\begin{aligned}\big\|Q_5(w) - E_q(w)\big\| &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{\left(2^{5-2\phi}\right)^{1-1}}{1-2^{5-2\phi}}\mathscr{S}_5\Big(w,w\Big) \\ &= \frac{2^{2\phi}}{2^{2\phi}-2^5}\mathscr{S}_5\Big(w,w\Big)\end{aligned}$$

□

Theorem 3.15. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality (3.44) for all $w, v \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the condition*

$$\lim_{\beta\to\infty}\frac{1}{\xi_\psi^{6\beta}}\mathscr{S}\Big(\xi_\psi^\beta w, \xi_\psi^\beta v\Big) = 0 \tag{3.114}$$

for all $w, v \in \mathscr{W}_1$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w) - E_s(w)\| \le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\mathscr{S}_6\Big(w,w\Big) \tag{3.115}$$

for all $w \in \mathscr{W}_1$. If $\mathscr{L} = \mathscr{L}(\psi)$, with the property

$$\frac{1}{\xi_\psi^6}\mathscr{S}_6\Big(\xi_\psi w, \xi_\psi w\Big) = \mathscr{L}\mathscr{S}_6\Big(w,w\Big) \tag{3.116}$$

with the condition that

$$\mathscr{S}_6\Big(w,w\Big) = \frac{1}{6!}\ \mathscr{S}_6\Big(\frac{w}{2},\frac{w}{2}\Big), \tag{3.117}$$

where $\mathscr{S}_6\Big(w,w\Big)$ is defined in (3.67) for all $w \in \mathscr{W}_1$.

Proof. Consider the set

$$\mathscr{B} = \Big\{E_2 | E_2 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2, E_2(0) = 0\Big\} \tag{3.118}$$

Let us introduce the generalized metric on (3.118) by

$$\inf\Big\{\zeta : \|E_1(w) - E_2(w)\| \le \zeta \mathscr{S}_6\Big(w,w\Big)\Big\} \tag{3.119}$$

for all $w \in \mathscr{W}_1$. One can easy to see verify that (3.119) is complete with respect to the defined metric. Define a mapping $\Upsilon : \mathscr{B} \to \mathscr{B}$ by

$$\Upsilon E_s(w) = \frac{1}{\xi_\psi^6} E_s(\xi_\psi) \tag{3.120}$$

for all $w \in \mathscr{W}_1$. Now, for any $E_1, E_2 \in \mathscr{B}$, we arrive

$$\begin{aligned} &\|E_1(w) - E_2(w)\| \le \zeta \mathscr{S}_6\Big(w,w\Big) \\ &\Longrightarrow \left\| \frac{1}{\xi_\psi^6} E_1(\xi_\psi) - \frac{1}{\xi_\psi^6} E_2(\xi_\psi) \right\| \le \zeta \mathscr{S}_6\Big(w,w\Big) \\ &\Longrightarrow \|\Upsilon E_1(w) - \Upsilon E_2(w)\| \le \mathscr{L} \zeta \mathscr{S}_6\Big(w,w\Big) \end{aligned}$$

for all $w \in \mathscr{W}_1$. This implies that Υ is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. The rest of the proof is similar lines to that of Theorem 3.13. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.15 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.16. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}_s(w,v)\| \le \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \tag{3.121}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\|Q_6(w) - E_s(w)\| \le \begin{cases} \dfrac{65\Phi}{6!|63|}; \\ \dfrac{\Gamma_{6T}\ \Phi||w||^\phi}{6!|2^6 - 2^\phi|}; & \phi \neq 6 \\ \dfrac{\Gamma_{6J}\ \Phi||w||^{2\phi}}{6!|2^6 - 2^{2\phi}|}; & 2\phi \neq 6 \end{cases} \tag{3.122}$$

where $\Gamma_g' s$ are defined in (3.76) and (3.79) respectively for all $w \in \mathscr{W}_1$.

Proof. The proof of the corollary is similar clues and ideas of Corollary 3.14. Hence the details of the proof are omitted. □

Theorem 3.17. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality (3.80) for all $w, v \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the conditions (3.95) and (3.114) for all $w, v \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality*

$$\begin{aligned} &\|E(w) - Q_5(w) - Q_6(w)\| \\ &\le \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\Big\{\mathscr{S}\Big(w,w\Big) + \mathscr{S}\Big(-w,-w\Big)\Big\} \end{aligned} \tag{3.123}$$

for all $w \in \mathscr{W}_1$. If $\mathscr{L} = \mathscr{L}(\psi)$, with the properties and conditions (3.97), (3.116) (3.98), (3.117) for all $w \in \mathscr{W}_1$.

Proof. From (3.83) of Theorem 3.9 and by Theorem 3.13, we obtain

$$\|Q_5(w) - E_O(w)\| \le \frac{1}{2} \times \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\Big\{\mathscr{S}\Big(w,w\Big) + \mathscr{S}\Big(-w,-w\Big)\Big\} \tag{3.124}$$

for all $w \in \mathscr{W}_1$. Also, from (3.85) of Theorem 3.9 and by Theorem 3.15, we obtain

$$\|Q_6(w) - E_s E(w)\| \le \frac{1}{2} \times \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\Big\{\mathscr{S}\Big(w,w\Big) + \mathscr{S}\Big(-w,-w\Big)\Big\} \tag{3.125}$$

for all $w \in \mathscr{W}_1$. Finally, from (3.88) of of Theorem 3.9 and (3.124), (3.125), we prove our desired result. □

The following corollary is an immediate consequence of Theorem 3.17 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 3.18. *For an mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\|\mathscr{E}(w,v)\| \le \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \tag{3.126}$$

for all $w, v \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned} &\|E(w) - Q_5(w) - Q_6(w)\| \\ &\le \begin{cases} \left(\frac{33}{5!|31|} + \frac{65}{6!|63|}\right) 2\Phi; \\ \left(\frac{\Gamma_{5T}}{5!|2^5 - 2^\phi|} + \frac{\Gamma_{6T} 2\Phi||w||^\phi}{6!|2^6 - 2^\phi|}\right); & \phi \neq 5,6 \\ \left(\frac{\Gamma_{5J}}{5!|2^5 - 2^{2\phi}|} + \frac{\Gamma_{6J}}{6!|2^6 - 2^{2\phi}|}\right) 2\Phi||w||^{2\phi}; & 2\phi \neq 5,6 \end{cases} \end{aligned} \tag{3.127}$$

where $\Gamma_g' s$ are defined in (3.40), (3.43), (3.76) and (3.79) respectively for all $w \in \mathscr{W}_1$.

4. Stability Results In Fuzzy Banach Space

In this section, we confirm the generalized Ulam - Hyers stability in Fuzzy Banach space using Hyers and Radus methods.

Fuzzy theory was initiated by Zadeh [49] in 1965. Currently, this theory is a powerful tool for modeling uncertainty and vagueness in miscellaneous problems arising in the field of science and engineering. We use the definition of fuzzy normed spaces given in [11] and [29–32].

4.1 Definitions on Fuzzy Banach Spaces

Definition 4.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s,t \in \mathbb{R}$,*
$(FNS1)$ $N(x,c) = 0$ *for* $c \leq 0$;
$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;
$(FNS3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
$(FNS4)$ $N(x+y,s+t) \geq min\{N(x,s),N(y,t)\}$;
$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;
$(FNS6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 4.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \ x \in X, \\ 0, & t \leq 0, \ x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 4.3. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 4.4. *A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

Definition 4.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 4.6. *A mapping $f : X \longrightarrow Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

The stability of a quiet number of functional equations in Fuzzy Banach space were inspected in [3, 6, 7, 9, 10, 29–32]. In order to establish results, we need the following assumptions. Let $\mathscr{W}_1$ be a linear space, $\mathscr{W}_2$ be a Fuzzy Banach space and $\mathscr{W}_3$ be a Fuzzy normed space.

4.2 Hyers - Ulam Method

Theorem 4.7. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}_q(w,v),z\right) \geq \mathscr{N}'\left(\mathscr{S}(w,v),z\right) \tag{4.1}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_5(w) - E_q(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5(w,w), \frac{5!|2^5 - \varepsilon|\, z}{33}\right) \tag{4.2}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. The mapping Q_5 is defined as

$$\lim_{\beta\to\infty} \mathscr{N}\left(Q_5(w) - \frac{1}{2^{5\alpha\beta}} E_q\left(2^{\alpha\beta} w\right), z\right) = 1 \tag{4.3}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow \mathscr{W}_3$ is a function fulfilling the condition

$$\lim_{\beta\to\infty} \mathscr{N}'\left(\mathscr{S}\left(2^{\alpha\beta} w, 2^{\alpha\beta} v\right), 2^{5\alpha\beta} z\right) = 1 \tag{4.4}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ with the condition that

$$\mathscr{N}'\left(\mathscr{S}\left(2^{\alpha\beta} w, 2^{\alpha\beta} w\right), z\right) = \mathscr{N}'\left(\varepsilon^{\alpha\beta} \mathscr{S}\left(w,w\right), z\right) \tag{4.5}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ for some $\varepsilon > 0$ with $0 < \left(\frac{\varepsilon}{2^5}\right)^\alpha < 1$. The function $\mathscr{S}_5\left(w,w\right)$ is defined by

$$\begin{aligned} \mathscr{S}_5(w,w) = \min\Big\{ & \mathscr{N}'\left(\mathscr{S}(0,2w),z\right) + \mathscr{N}'\left(\mathscr{S}(4w,w),z\right) \\ & + \mathscr{N}'\left(\mathscr{S}(3w,w),z\right) + \mathscr{N}'\left(\mathscr{S}(2w,w),z\right) \\ & + \mathscr{N}'\left(\mathscr{S}(w,w),z\right) + \mathscr{N}'\left(\mathscr{S}(0,w),z\right) \\ & + \mathscr{N}'\left(\mathscr{S}(-w,w),z\right) \Big\} \end{aligned} \tag{4.6}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. Using oddness of E in (4.1), one can obtain that

$$\begin{aligned} & \mathscr{N}\left(E_q(w+4v) - 5E_q(w+3v) + 10E_q(w+2v)\right. \\ & \quad \left. -10E_q(w+v) + 5E_q(w) - E_q(w-v) - 120E_q(v), z\right) \\ & \geq \mathscr{N}'\left(\mathscr{S}(w,v),z\right) \end{aligned} \tag{4.7}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now, interchanging (w,v) by $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (4.7) and using oddness of E_q, we arrive the subsequent inequalities

$$\mathcal{N}\left(E_q(8w)-5E_q(6w)+10E_q(4w)-129E_q(2w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(0,2w),z\right) \tag{4.8}$$

$$\mathcal{N}\left(E_q(8w)-5E_q(7w)+10E_q(6w)-10E_q(5w) +5E_q(4w)-E_q(3w)-120E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(4w,w),z\right) \tag{4.9}$$

$$\mathcal{N}\left(E_q(7w)-5E_q(6w)+10E_q(5w)-10E_q(4w) +5E_q(3w)-E_q(2w)-120E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(3w,w),z\right) \tag{4.10}$$

$$\mathcal{N}\left(E_q(6w)-5E_q(5w)+10E_q(4w)-10E_q(3w) +5E_q(2w)-121E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(2w,w),z\right) \tag{4.11}$$

$$\mathcal{N}\left(E_q(5w)-5E_q(4w)+10E_q(3w)-10E_q(2w) -115E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(w,w),z\right) \tag{4.12}$$

$$\mathcal{N}\left(E_q(4w)-5E_q(3w)+10E_q(2w) -129E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(0,w),z\right) \tag{4.13}$$

$$\mathcal{N}\left(E_q(3w)-4E_q(2w)-115E_q(w),z\right) \geq \mathcal{N}'\left(\mathcal{S}(-w,w),z\right) \tag{4.14}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.8) and (4.9), we get

$$\begin{aligned}
&\mathcal{N}\left(5E_q(7w)-15E_q(6w)+10E_q(5w)+5E_q(4w) +E_q(3w)-129E_q(2w)+120E_q(w),z+z\right)\\
&\geq \min\Big\{\mathcal{N}\left(E_q(8w)-5E_q(6w)+10E_q(4w) -129E_q(2w),z\right)\\
&\quad +\mathcal{N}\left(E_q(8w)-5E_q(7w)+10E_q(6w)-10E_q(5w) +5E_q(4w)-E_q(3w)-120E_q(w),z\right)\Big\}\\
&\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right)\Big\}
\end{aligned} \tag{4.15}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.10) by 5, we see that

$$\mathcal{N}\left(5E_q(7w)-25E_q(6w)+50E_q(5w)-50E_q(4w) +25E_q(3w)-5E_q(2w)-600E_q(w),5z\right) \geq \mathcal{N}'\left(\mathcal{S}(3w,w),z\right) \tag{4.16}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. It follows from (4.15) and (4.16), we arrive

$$\begin{aligned}
&\mathcal{N}\left(10E_q(6w)-40E_q(5w)+55E_q(4w)-24E_q(3w) -124E_q(2w)720E_q(w),z+z+5z\right)\\
&\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right) +\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)\Big\}
\end{aligned} \tag{4.17}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.11) by 10, we have

$$\mathcal{N}\left(10E_q(6w)-50E_q(5w)+100E_q(4w)-100E_q(3w) +50E_q(2w)-1210E_q(w),10z\right) \geq \mathcal{N}'\left(\mathcal{S}(2w,w),z\right) \tag{4.18}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Combining (4.18) and (4.17), we obtain

$$\begin{aligned}
&\mathcal{N}\left(10E_q(5w)-45E_q(4w)+76E_q(3w)-174E_q(2w) +1930E_q(w),z+z+5z+10z\right)\\
&\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right) +\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right)\Big\}
\end{aligned} \tag{4.19}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.12) by 10, we find that

$$\mathcal{N}\left(10E_q(5w)-50E_q(4w)+100E_q(3w)-100E_q(2w) -1150E_q(w),10z\right) \geq \mathcal{N}'\left(\mathcal{S}(w,w),z\right) \tag{4.20}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. It follows from (4.19) and (4.20), we arrive

$$\begin{aligned}
&\mathcal{N}\left(5E_q(4w)-24E_q(3w)-74E_q(2w)+3080E_q(w) ,z+z+5z+10z+10z\right)\\
&\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right) +\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right) +\mathcal{N}'\left(\mathcal{S}(w,w),z\right)\Big\}
\end{aligned} \tag{4.21}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.13) by 5, we see that

$$\mathcal{N}\left(5E_q(4w)-25E_q(3w)+50E_q(2w)-645E_q(w),5z\right) \geq \mathcal{N}'\left(\mathcal{S}(0,w),z\right) \tag{4.22}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.21) and (4.22), we have

$$\begin{aligned}
&\mathcal{N}\left(E_q(3w)-124E_q(2w)+3725E_q(w) ,z+z+5z+10z+10z+5z\right)\\
&\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right) +\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right)\\
&\quad +\mathcal{N}'\left(\mathcal{S}(w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(0,w),z\right)\Big\}
\end{aligned} \tag{4.23}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. It follows from (4.14) and

(4.23), we arrive

$$\begin{aligned}
&\mathcal{N}\left(120E_q(2w)-3840E_q(w),\right.\\
&\qquad\left. z+z+5z+10z+10z+5z+z\right)\\
&\geq \min\Big\{\mathcal{N}'\left(\mathscr{S}(0,2w),z\right)+\mathcal{N}'\left(\mathscr{S}(4w,w),z\right)\\
&\qquad +\mathcal{N}'\left(\mathscr{S}(3w,w),z\right)+\mathcal{N}'\left(\mathscr{S}(2w,w),z\right)\\
&\qquad +\mathcal{N}'\left(\mathscr{S}(w,w),z\right)+\mathcal{N}'\left(\mathscr{S}(0,w),z\right)\\
&\qquad +\mathcal{N}'\left(\mathscr{S}(-w,w),z\right)\Big\}\\
&=\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right)
\end{aligned} \tag{4.24}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. The above equation can be written as (4.24) that

$$\mathcal{N}\left(120E_q(2w)-3840E_q(w),33z\right)\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right) \tag{4.25}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Using (FNS3) in (4.25), one can get that

$$\mathcal{N}\left(\frac{E_q(2w)}{2^5}-E_q(w),\frac{33z}{2^5\cdot 5!}\right)\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right) \tag{4.26}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Changing w by $2^\beta w$ and using (4.5), (FNS3) in (4.26), we arrive

$$\begin{aligned}
&\mathcal{N}\left(\frac{E_q(2^{\beta+1}w)}{2^{5(\beta+1)}}-\frac{E_q(2^\beta w)}{2^{5\beta}},\frac{33z}{2^5 2^{5\beta}\cdot 5!}\right)\\
&\geq\mathcal{N}'\left(\mathscr{S}_5(2^\beta w,2^\beta w),z\right)\\
&=\mathcal{N}'\left(\varepsilon^\beta\mathscr{S}_5(w,w),z\right)\\
&=\mathcal{N}'\left(\mathscr{S}_5(w,w),\frac{z}{\varepsilon^\beta}\right)
\end{aligned} \tag{4.27}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Switching z by $\varepsilon^\beta z$ in (4.27)

$$\begin{aligned}
&\mathcal{N}\left(\frac{E_q(2^{\beta+1}w)}{2^{5(\beta+1)}}-\frac{E_q(2^\beta w)}{2^{5\beta}},\frac{\varepsilon^\beta}{2^{5\beta}}\ \frac{33z}{2^5\cdot 5!}\right)\\
&\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right)
\end{aligned} \tag{4.28}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. One can easy to verify that

$$\frac{E_q(2^\beta w)}{2^{5\beta}}-E_q(w)=\sum_{\gamma=0}^{\beta-1}\frac{E_q(2^{\gamma+1}w)}{2^{5(\gamma+1)}}-\frac{E_q(2^\gamma w)}{2^{5\gamma}}. \tag{4.29}$$

From (4.28) and (4.29), we reach

$$\begin{aligned}
&\mathcal{N}\left(\frac{E_q(2^\beta w)}{2^{5\beta}}-E_q(w),\sum_{\gamma=0}^{\beta-1}\frac{\varepsilon^\gamma}{2^{5\gamma}}\ \frac{33z}{2^5\cdot 5!}\right)\\
&\geq\min\bigcup_{\gamma=0}^{\beta-1}\left\{\mathcal{N}\left(\frac{E_q(2^{\gamma+1}w)}{2^{5(\gamma+1)}}-\frac{E_q(2^\gamma w)}{2^{5\gamma}},\frac{\varepsilon^\gamma}{2^{5\gamma}}\ \frac{z}{2^5\cdot 5!}\right)\right\}\\
&\geq\min\bigcup_{\gamma=0}^{\beta-1}\left\{\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right)\right\}=\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right)
\end{aligned} \tag{4.30}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Interchanging w by $2^\delta w$ in (4.30) and using (4.5), (FNS3) and then switching z by $\varepsilon^\delta z$, we achieve

$$\begin{aligned}
&\mathcal{N}\left(\frac{E_q(2^{\beta+\delta}w)}{2^{5\beta+\delta}}-\frac{E_q(2^\delta w)}{2^{5\delta}},\sum_{\gamma=0}^{\beta-1}\left(\frac{\varepsilon}{2^5}\right)^{\gamma+\delta}\ \frac{33z}{2^5\cdot 5!}\right)\\
&\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),z\right)
\end{aligned} \tag{4.31}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ for $\beta>\delta\geq 0$. With the help of (FNS3), (4.31) can be remodified as

$$\begin{aligned}
&\mathcal{N}\left(\frac{E_q(2^{\beta+\delta}w)}{2^{5\beta+\delta}}-\frac{E_q(2^\delta w)}{2^{5\delta}},z\right)\\
&\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),\frac{z}{\sum_{\gamma=\delta}^{\delta+\beta-1}\left(\frac{\varepsilon}{2^5}\right)^\gamma\ \frac{33}{2^5\cdot 5!}}\right)
\end{aligned} \tag{4.32}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

By data $\left(\frac{\varepsilon}{2^5}\right)^\alpha<1$ and since $\sum_{\gamma=0}^{\beta-1}\left(\frac{\varepsilon}{2^5}\right)^\gamma<\infty$, by the Cauchy criterion for convergence and (FNS5) implies that

$$\left\{\frac{E_q(2^\beta w)}{2^{5\beta}}\right\}$$

is a Cauchy sequence in $\mathscr{W}_2$. Since $\mathscr{W}_2$ is a fuzzy Banach space, this sequence converges to some point $Q_5\in\mathscr{W}_2$. Thus, define the mapping $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ by

$$\lim_{\beta\to\infty}\mathcal{N}\left(Q_5(w)-\frac{1}{2^{5\beta}}E_q\left(2^\beta w\right),z\right)=1 \tag{4.33}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Setting $\delta=0$ and approaching β tends to ∞ in (4.32), we reach

$$\mathcal{N}\left(Q_5(w)-E_q(w),z\right)\geq\mathcal{N}'\left(\mathscr{S}_5(w,w),\frac{z(2^5-\varepsilon)5!}{33}\right) \tag{4.34}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

To prove that the existing $Q_5(w)$ satisfies the functional equation (1.3), changing (w,v) by $(2^\beta w,2^\beta v)$ and dividing by $2^{5\beta}$ in (4.1), we get

$$\begin{aligned}
\mathcal{N}\left(\mathscr{E}_q(w,v),z\right)&=\mathcal{N}\left(\frac{1}{2^{5\beta}}\ \mathscr{E}_q(2^\beta w,2^\beta v),z\right)\\
&\geq\mathcal{N}'\left(\mathscr{S}(2^\beta w,2^\beta v),2^{5\beta}z\right)
\end{aligned} \tag{4.35}$$

for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now,

$$\begin{aligned}
&\mathscr{N}\Bigg(Q_5(w+4v) - 5Q_5(w+3v) - \frac{1}{2}\Big(Q_{5^q_s}(w+3v)\Big)\\
&+10Q_5(w+2v) + \frac{5}{2}\Big(Q_{5^q_s}(w+2v)\Big) - 10Q_5(w+v)\\
&-5\Big(Q_{5^q_s}(w+v)\Big) + 5Q_5(w) + 5\Big(Q_{5^q_s}(w)\Big)\\
&-Q_5(w-v) - \frac{5}{2}\Big(Q_{5^q_s}(w-v)\Big) + \Big(Q_{5^q_s}(w-2v)\Big)\\
&-120Q_5(v) - 300\Big(Q_{5^q_s}(v)\Big), z\Big)\\
&\geq \min\Bigg\{\mathscr{N}\Bigg(Q_5(w+4v) - \frac{1}{2^{5\beta}}E_q(2^\beta(w+4v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-5Q_5(w+3v) + 5\frac{1}{2^{5\beta}}E_q(2^\beta(w+3v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-\frac{1}{2}Q_{5^q_s}(w+3v) + \frac{1}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w+3v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(10Q_5(w+2v) - 10\frac{1}{2^{5\beta}}E_q(2^\beta(w+2v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(\frac{5}{2}Q_{5^q_s}(w+2v) - \frac{5}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w+2v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-10Q_5(w+v) + 10\frac{1}{2^{5\beta}}E_q(2^\beta(w+v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-5Q_{5^q_s}(w+v) + 5\frac{1}{2^{5\beta}}E_q(2^\beta(w+v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(+5Q_5(w) - 5\frac{1}{2^{5\beta}}E_q(2^\beta(w)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(5Q_{5^q_s}(w) - 5\frac{1}{2^{5\beta}}E_q(2^\beta(w)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-Q_5(w-v) + \frac{1}{2^{5\beta}}E_q(2^\beta(w-v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-\frac{5}{2}Q_{5^q_s}(w-v) + \frac{5}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w-v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(Q_{5^q_s}(w-2v) - \frac{1}{2^{5\beta}}E_q(2^\beta(w-2v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-120Q_5(v) + 120\frac{1}{2^{5\beta}}E_q(2^\beta(v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(-300Q_{5^q_s}(v) + 300\frac{1}{2^{5\beta}}E_q(2^\beta(v)), \frac{z}{15}\Bigg),\\
&\mathscr{N}\Bigg(\frac{1}{2^{5\beta}}E_q(2^\beta(w+4v)) - 5\frac{1}{2^{5\beta}}E_q(2^\beta(w+3v))\\
&-\frac{1}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w+3v)) + 10\frac{1}{2^{5\beta}}E_q(2^\beta(w+2v))\\
&+\frac{5}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w+2v)) - 10\frac{1}{2^{5\beta}}E_q(2^\beta(w+v))\\
&-5\frac{1}{2^{5\beta}}E_q(2^\beta(w+v)) + 5\frac{1}{2^{5\beta}}E_q(2^\beta(w))\\
&+5\frac{1}{2^{5\beta}}E_q(2^\beta(w)) - \frac{1}{2^{5\beta}}E_q(2^\beta(w-v))\\
&-\frac{5}{2}\frac{1}{2^{5\beta}}E_q(2^\beta(w-v)) + \frac{1}{2^{5\beta}}E_q(2^\beta(w-2v))\\
&-120\frac{1}{2^{5\beta}}E_q(2^\beta(v)) - 300\frac{1}{2^{5\beta}}E_q(2^\beta(v))\Bigg), \frac{z}{15}\Bigg\}
\end{aligned} \tag{4.36}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. With the help of (4.33) and (4.35) in (4.36)

$$\begin{aligned}
&\mathscr{N}\Bigg(Q_5(w+4v) - 5Q_5(w+3v) - \frac{1}{2}\Big(Q_{5^q_s}(w+3v)\Big)\\
&+10Q_5(w+2v) + \frac{5}{2}\Big(Q_{5^q_s}(w+2v)\Big) - 10Q_5(w+v)\\
&-5\Big(Q_{5^q_s}(w+v)\Big) + 5Q_5(w) + 5\Big(Q_{5^q_s}(w)\Big)\\
&-Q_5(w-v) - \frac{5}{2}\Big(Q_{5^q_s}(w-v)\Big) + \Big(Q_{5^q_s}(w-2v)\Big)\\
&-120Q_5(v) - 300\Big(Q_{5^q_s}(v)\Big), z\Big)\\
&\geq \min\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\
&\qquad \mathscr{N}'\Big(\mathscr{S}(2^\beta w, 2^\beta v), 2^{5\beta}z\Big)\Big\}
\end{aligned} \tag{4.37}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Approaching β tends to ∞ (4.37) and applying (4.3), we obtain

$$\begin{aligned}
&\mathscr{N}\Bigg(Q_5(w+4v) - 5Q_5(w+3v) - \frac{1}{2}\Big(Q_{5^q_s}(w+3v)\Big)\\
&+10Q_5(w+2v) + \frac{5}{2}\Big(Q_{5^q_s}(w+2v)\Big) - 10Q_5(w+v)\\
&-5\Big(Q_{5^q_s}(w+v)\Big) + 5Q_5(w) + 5\Big(Q_{5^q_s}(w)\Big)\\
&-Q_5(w-v) - \frac{5}{2}\Big(Q_{5^q_s}(w-v)\Big) + \Big(Q_{5^q_s}(w-2v)\Big)\\
&-120Q_5(v) - 300\Big(Q_{5^q_s}(v)\Big), z\Big) = 1
\end{aligned} \tag{4.38}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Using (FNS2) in (4.38) we see that

$$\begin{aligned}
&Q_5(w+4v) - 5Q_5(w+3v) - \frac{1}{2}\Big(Q_{5^q_s}(w+3v)\Big)\\
&+10Q_5(w+2v) + \frac{5}{2}\Big(Q_{5^q_s}(w+2v)\Big) - 10Q_5(w+v)\\
&-5\Big(Q_{5^q_s}(w+v)\Big) + 5Q_5(w) + 5\Big(Q_{5^q_s}(w)\Big)\\
&-Q_5(w-v) - \frac{5}{2}\Big(Q_{5^q_s}(w-v)\Big) + \Big(Q_{5^q_s}(w-2v)\Big)\\
&= 120Q_5(v) + 300\Big(Q_{5^q_s}(v)\Big)
\end{aligned}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ which shows that $Q_5(w)$ satisfies the functional equation (1.3).

It is easy to prove that the existence of Q_5 is unique. Indeed, let R_5 be an another quintic mapping satisfying (1.3) and (4.2). Now

$$Q_5(2^\delta w) = 2^{5\delta}Q_5(w) \qquad \textit{and} \qquad R_5(2^\delta w) = 2^{5\delta}R_5(w).$$

Thus,

$$\begin{aligned}
&\mathcal{N}\left(Q_5(w)-R_5(w),z\right)\\
&=\mathcal{N}\left(Q_5(2^\delta w)-R_5(2^\delta w),2^{5\delta}z\right)\\
&\geq \mathcal{N}'\left(\mathcal{S}_5(2^\delta w,2^\delta w),\ 5!(2^5-\varepsilon)\ \frac{2^{5\delta}\ z}{66}\right)\\
&=\mathcal{N}'\left(\mathcal{S}_5(w,w),\ 5!(2^5-\varepsilon)\ \frac{2^{5\delta}\ z}{\varepsilon^\delta\ 66}\right) \qquad (4.39)
\end{aligned}$$

for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$. Since

$$\lim_{\delta\to\infty}\mathcal{N}'\left(\mathcal{S}_5(w,w),\ 5!(2^5-\varepsilon)\ \frac{2^{5\delta}\ z}{\varepsilon^\delta\ 66}\right)=1 \qquad (4.40)$$

because

$$\lim_{\delta\to\infty}5!(2^5-\varepsilon)\ \frac{2^{5\delta}z}{\varepsilon^\delta\ 66}=\infty$$

for all $z\in\mathcal{W}_1$. Letting δ tends to ∞ in (4.39), using (4.40) and (FNS2), we have $Q_5(w)-R_5(w)=0$ which implies $Q_5(w)=R_5(w)$ for all $w\in\mathcal{W}_1$. Hence the theorem holds for $\alpha=1$.

On changing w by $\frac{w}{2}$ in (4.25), we get

$$\mathcal{N}\left(E_q(w)-2^5E\left(\frac{w}{2}\right),\frac{z}{5!}\right)\geq\mathcal{S}_5\left(\frac{w}{2},\frac{w}{2}\right) \qquad (4.41)$$

for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$. The rest of the proof is similar clues that of case $\alpha=1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 4.7 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 4.8. *For an odd mapping $\mathcal{E}_q:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_q(w,v),z\right)\geq\mathcal{N}'\left(\Phi,z\right) \qquad (4.42)$$

for all $w,v\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$ where $\Phi>0$ is a constant. Then there exists one and only quintic function $Q_5:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathcal{N}\left(Q_5(w)-E_q(w),z\right)\geq\mathcal{N}'\left(\Phi,\frac{5!|31|z}{33}\right) \qquad (4.43)$$

for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$.

The following corollary is an immediate consequence of Theorem 4.7 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 4.9. *For an odd mapping $\mathcal{E}_q:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_q(w,v),z\right)\geq\left\{\begin{array}{l}\mathcal{N}'\left(\Phi\left\{|w|^\phi+|v|^\phi\right\},z\right);\\ \mathcal{N}'\left(\Phi\left\{|w|^{\phi_1}+|v|^{\phi_2}\right\},z\right);\end{array}\right. \qquad (4.44)$$

for all $w,v\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$ where $\Phi>0$ is a constant and $\phi,\phi_1,\phi_2\neq5$. Then there exists one and only quintic function $Q_5:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}
&\mathcal{N}\left(Q_5(w)-E_q(w),z\right)\\
&\geq\left\{\begin{array}{l}\mathcal{N}'\left(\Gamma_{5T}\ \Phi|w|^\phi,\frac{5!|2^5-2^\phi|}{33}\right);\\ \mathcal{N}'\left(\Gamma_{5T1}\ \Phi|w|^{\phi_1}+\Gamma_{5T2}\Phi|w|^{\phi_2},\frac{5!|2^5-2^\phi|}{33}\right);\end{array}\right. \qquad (4.45)
\end{aligned}$$

where Γ_g's are defined in (3.40) respectively for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$.

The following corollary is an immediate consequence of Theorem 4.7 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.10. *For an odd mapping $\mathcal{E}_q:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_q(w,v),z\right)\geq\left\{\begin{array}{l}\mathcal{N}'\left(\Phi\left\{|w|^\phi|v|^\phi+\left[|w|^{2\phi}+|v|^{2\phi}\right]\right\},z\right);\\ \mathcal{N}'\left(\Phi\left\{|w|^{\phi_1}|v|^{\phi_2}+\left[|w|^{\phi_1+\phi_2}+|v|^{\phi_1+\phi_2}\right]\right\},z\right);\end{array}\right. \qquad (4.46)$$

for all $w,v\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$ where $\Phi>0$ is a constant and $2\phi,\phi_1+\phi_2\neq5$. Then there exists one and only quintic function $Q_5:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}
&\mathcal{N}\left(Q_5(w)-E_q(w),z\right)\\
&\geq\left\{\begin{array}{l}\mathcal{N}'\left(\Gamma_{5J}\ \Phi|w|^\phi,\frac{5!|2^5-2^{2\phi}|}{33}\right);\\ \mathcal{N}'\left(\Gamma_{5J1}\ \Phi|w|^{\phi_1+\phi_2},\frac{5!|2^5-2^{\phi_1+\phi_2}|\ z}{33}\right);\end{array}\right. \qquad (4.47)
\end{aligned}$$

where Γ_g's are defined in (3.43) respectively for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$.

Theorem 4.11. *For an even mapping $\mathcal{E}_s:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_s(w,v),z\right)\geq\mathcal{N}'\left(\mathcal{S}(w,v),z\right) \qquad (4.48)$$

for all $w,v\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$. Then there exists one and only sextic function $Q_6:\mathcal{W}_1\longrightarrow\mathcal{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathcal{N}\left(Q_6(w)-E_s(w),z\right)\geq\mathcal{N}'\left(\mathcal{S}_6(w,w),\ \frac{6!|2^6-\varepsilon|\ z}{65}\right) \qquad (4.49)$$

for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$. The mapping Q_6 is defined as

$$\lim_{\beta\to\infty}\mathcal{N}\left(Q_6(w)-\frac{1}{2^{6\alpha\beta}}E_s\left(2^{\alpha\beta}w\right),z\right)=1 \qquad (4.50)$$

for all $w\in\mathcal{W}_1$ and all $z\in\mathcal{W}_1$, where $\mathcal{S}:\mathcal{W}_1^2\longrightarrow\mathcal{W}_3$ is a function fulfilling the condition

$$\lim_{\beta\to\infty}\mathcal{N}'\left(\mathcal{S}\left(2^{\alpha\beta}w,2^{\alpha\beta}v\right),2^{6\alpha\beta}z\right)=1 \qquad (4.51)$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ with the condition that

$$\mathcal{N}'\left(\mathscr{S}\left(2^{\alpha\beta}w, 2^{\alpha\beta}w\right), z\right) = \mathcal{N}'\left(\varepsilon^{\alpha\beta}\mathscr{S}\left(w,w\right), z\right) \quad (4.52)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ for some $\varepsilon > 0$ with $0 < \left(\frac{\varepsilon}{2^6}\right)^{\alpha} < 1$. The function $\mathscr{S}_6\left(w,w\right)$ is defined by

$$\begin{aligned}\mathscr{S}_6(w,w) = \min\{&\mathcal{N}'(\mathscr{S}(0,2w),z) + \mathcal{N}'(\mathscr{S}(4w,w),z)\\ &+\mathcal{N}'(\mathscr{S}(3w,w),z) + \mathcal{N}'(\mathscr{S}(2w,w),z)\\ &+\mathcal{N}'(\mathscr{S}(w,w),z) + \mathcal{N}'(\mathscr{S}(0,w),z)\\ &+\mathcal{N}'(\mathscr{S}(-w,w),z)\}\end{aligned} \quad (4.53)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. Using evenness of E in (4.48), one can obtain that

$$\begin{aligned}&\mathcal{N}(E_s(w+4v) - 6E_s(w+3v) + 15E_s(w+2v) - 20E_s(w+v)\\ &+15E_s(w) - 6E_s(w-v) + E_s(w-2v) - 720E_s(v), z)\\ &\geq \mathcal{N}'(\mathscr{S}(w,v),z)\end{aligned} \quad (4.54)$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now, interchanging (w,v) by $(0,2w)$, $(4w,w)$, $(3w,w)$, $(2w,w)$, (w,w), $(0,w)$ and $(-w,w)$ in (4.54) and using evenness of E_s, we arrive the subsequent inequalities

$$\begin{aligned}&\mathcal{N}(E_s(8w) - 6E_s(6w) + 16E_s(4w) - 746E_s(2w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(0,2w),z)\end{aligned} \quad (4.55)$$

$$\begin{aligned}&\mathcal{N}(E_s(8w) - 6E_s(7w) + 15E_s(6w) - 20E_s(5w)\\ &+15E_s(4w) - 6E_s(3w) + E_s(2w) - 720E_s(w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(4w,w),z)\end{aligned} \quad (4.56)$$

$$\begin{aligned}&\mathcal{N}(E_s(7w) - 6E_s(6w) + 15E_s(5w) - 20E_s(4w)\\ &+15E_s(3w) - 6E_s(2w) - 719E_s(w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(3w,w),z)\end{aligned} \quad (4.57)$$

$$\begin{aligned}&\mathcal{N}(E_s(6w) - 6E_s(5w) + 15E_s(4w) - 20E_s(3w)\\ &+15E_s(2w) - 726E_s(w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(2w,w),z)\end{aligned} \quad (4.58)$$

$$\begin{aligned}&\mathcal{N}(E_s(5w) - 6E_s(4w) + 15E_s(3w) - 20E_s(2w)\\ &-704E_s(w), z) \geq \mathcal{N}'(\mathscr{S}(w,w),z)\end{aligned} \quad (4.59)$$

$$\begin{aligned}&\mathcal{N}(E_s(4w) - 6E_s(3w) + 16E_s(2w) - 746E_s(w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(0,w),z)\end{aligned} \quad (4.60)$$

$$\begin{aligned}&\mathcal{N}(2E_s(3w) - 12E_s(2w) - 690E_s(w), z)\\ &\geq \mathcal{N}'(\mathscr{S}(-w,w),z)\end{aligned} \quad (4.61)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.55) and (4.56), we have

$$\begin{aligned}&\mathcal{N}(6E_s(7w) - 21E_s(6w) + 20E_s(5w) + E_s(4w)\\ &\qquad +6E_s(3w) - 747E_s(2w) + 720E_s(w), z+z)\\ &\geq \min\Big\{\mathcal{N}(E_s(8w) - 6E_s(6w) + 16E_s(4w)\\ &\qquad -746E_s(2w), z)\\ &+\mathcal{N}(E_s(8w) - 6E_s(7w) + 15E_s(6w) - 20E_s(5w)\\ &\qquad +15E_s(4w) - 6E_s(3w) + E_s(2w) - 720E_s(w), z)\Big\}\\ &\geq \min\Big\{\mathcal{N}'(\mathscr{S}(0,2w),z) + \mathcal{N}'(\mathscr{S}(4w,w),z)\Big\}\end{aligned} \quad (4.62)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.57) by 6, we see that

$$\begin{aligned}&\mathcal{N}(6E_s(7w) - 36E_s(6w) + 90E_s(5w) - 120E_s(4w)\\ &\qquad +90E_s(3w) - 36E_s(2w) - 5514E_s(w), 6z)\\ &\geq \mathcal{N}'(\mathscr{S}(3w,w),z)\end{aligned} \quad (4.63)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. It follows from (4.62) and (4.63), we arrive

$$\begin{aligned}&\mathcal{N}(15E_s(6w) - 70E_s(5w) + 121E_s(4w) - 84E_s(3w)\\ &\qquad -711E_s(2w) + 5034E_s(w), z+z+6z)\\ &\geq \min\Big\{\mathcal{N}'(\mathscr{S}(0,2w),z) + \mathcal{N}'(\mathscr{S}(4w,w),z)\\ &\qquad + \mathcal{N}'(\mathscr{S}(3w,w),z)\Big\}\end{aligned} \quad (4.64)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.58) by 15, we get

$$\begin{aligned}&\mathcal{N}(15E_s(6w) - 90E_s(5w) + 225E_s(4w) - 300E_s(3w)\\ &\qquad +225E_s(2w) - 10890E_s(w), 15z)\\ &\geq \mathcal{N}'(\mathscr{S}(2w,w),z)\end{aligned} \quad (4.65)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Combining (4.65) and (4.64), we obtain

$$\begin{aligned}&\mathcal{N}(20E_s(5w) - 104E_s(4w) + 216E_s(3w) - 936E_s(2w)\\ &\qquad +15924E_s(w), z+z+6z+15z)\\ &\geq \min\Big\{\mathcal{N}'(\mathscr{S}(0,2w),z) + \mathcal{N}'(\mathscr{S}(4w,w),z)\\ &\qquad + \mathcal{N}'(\mathscr{S}(3w,w),z) + \mathcal{N}'(\mathscr{S}(2w,w),z)\Big\}\end{aligned} \quad (4.66)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Multiplying (4.59) by 20, we find that

$$\begin{aligned}&\mathcal{N}(20E_s(5w) - 120E_s(4w) + 300E_s(3w) - 400E_s(2w)\\ &\qquad -14080E_s(w), 20z) \geq \mathcal{N}'(\mathscr{S}(w,w),z)\end{aligned} \quad (4.67)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.66) and (4.67), we arrive

$$\mathcal{N}\left(16E_s(4w)-84E_s(3w)-536E_s(2w)+30004E_s(w),z+z+6z+15z+20z\right)$$
$$\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(w,w),z\right)\Big\} \quad (4.68)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Multiplying (4.60) by 16, we see that

$$\mathcal{N}\left(16E_s(4w)-96E_s(3w)+256E_s(2w)-11936E_s(w),16z\right)\geq\mathcal{N}'\left(\mathcal{S}(0,w),z\right) \quad (4.69)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. It follows from (4.68) and (4.69), we arrive

$$\mathcal{N}\left(12E_s(3w)-792E_s(2w)+41940E_s(w),z+z+6z+15z+20z+16z\right)$$
$$\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(0,w),z\right)\Big\} \quad (4.70)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Multiplying (4.61) by 6 and it follows from and (4.70), we arrive

$$\mathcal{N}\left(720E_s(2w)-46080E_s(w),z+z+6z+15z+20z+16z+6z\right)$$
$$\geq \min\Big\{\mathcal{N}'\left(\mathcal{S}(0,2w),z\right)+\mathcal{N}'\left(\mathcal{S}(4w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(3w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(2w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(w,w),z\right)+\mathcal{N}'\left(\mathcal{S}(0,w),z\right)+\mathcal{N}'\left(\mathcal{S}(-w,w),z\right)\Big\}$$
$$=\mathcal{N}'\left(\mathcal{S}_6(w,w),z\right) \quad (4.71)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. The above equation can be written as

$$\mathcal{N}\left(720E_s(2w)-46080E_s(w),65z\right)\geq\mathcal{N}'\left(\mathcal{S}_6(w,w),z\right) \quad (4.72)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Using (FNS3) in (4.72), one can get that

$$\mathcal{N}\left(\frac{E_s(2w)}{2^6}-E_s(w),\frac{65z}{2^6\cdot 6!}\right)\geq\mathcal{N}'\left(\mathcal{S}_6(w,w),z\right) \quad (4.73)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Changing w by $2^\beta w$ and using (4.52), (FNS3) in (4.73), we arrive

$$\mathcal{N}\left(\frac{E_s(2^{\beta+1}w)}{2^{6(\beta+1)}}-\frac{E_s(2^\beta w)}{2^{6\beta}},\frac{65z}{2^6 2^{6\beta}\cdot 6!}\right)\geq\mathcal{N}'\left(\mathcal{S}_6(2^\beta w,2^\beta w),z\right)=\mathcal{N}'\left(\varepsilon^\beta\mathcal{S}_6(w,w),z\right)=\mathcal{N}'\left(\mathcal{S}_6(w,w),\frac{z}{\varepsilon^\beta}\right) \quad (4.74)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Switching z by $\varepsilon^\beta z$ in (4.74)

$$\mathcal{N}\left(\frac{E_s(2^{\beta+1}w)}{2^{6(\beta+1)}}-\frac{E_s(2^\beta w)}{2^{6\beta}},\frac{\varepsilon^\beta}{2^{6\beta}}\ \frac{65z}{2^6\cdot 6!}\right)\geq\mathcal{N}'\left(\mathcal{S}_6(w,w),z\right) \quad (4.75)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. The rest of the proof is similar clues that of Theorem 4.7. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 4.11 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 4.12. *For an even mapping $\mathcal{E}_s:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_s(w,v),z\right)\geq\mathcal{N}'\left(\Phi,z\right) \quad (4.76)$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ where $\Phi>0$ is a constant. Then there exists one and only sextic function $Q_6:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathcal{N}\left(Q_6(w)-E_s(w),z\right)\geq\mathcal{N}'\left(\Phi,\frac{6!|63|z}{65}\right) \quad (4.77)$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 4.11 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 4.13. *For an even mapping $\mathcal{E}_s:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ fulfilling the functional inequality*

$$\mathcal{N}\left(\mathcal{E}_s(w,v),z\right)\geq\begin{cases}\mathcal{N}'\left(\Phi\left\{|w|^\phi+|v|^\phi\right\},z\right);\\ \mathcal{N}'\left(\Phi\left\{|w|^{\phi_1}+|v|^{\phi_2}\right\},z\right);\end{cases} \quad (4.78)$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ where $\Phi>0$ is a constant and $\phi,\phi_1,\phi_2\neq 6$. Then there exists one and only sextic function $Q_6:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathcal{N}\left(Q_6(w)-E_s(w),z\right)\geq\begin{cases}\mathcal{N}'\left(\Gamma_{6T}\ \Phi|w|^\phi,\frac{6!|2^6-2^\phi|}{65}\right);\\ \mathcal{N}'\left(\Gamma_{6T1}\ \Phi|w|^{\phi_1}+\Gamma_{6T2}\ \Phi|w|^{\phi_2},\frac{6!|2^6-2^\phi|}{65}\right);\end{cases} \quad (4.79)$$

where $\Gamma_g's$ are defined in (3.76) respectively for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 4.11 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.14. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}_s(w,v),z\right) \geq \begin{cases} \mathscr{N}'\left(\Phi\left\{|w|^{\phi}|v|^{\phi} + \left[|w|^{2\phi} + |v|^{2\phi}\right]\right\},z\right); \\ \mathscr{N}'\left(\Phi\left\{|w|^{\phi_1}|v|^{\phi_2} + \left[|w|^{\phi_1+\phi_2} + |v|^{\phi_1+\phi_2}\right]\right\},z\right); \end{cases} \tag{4.80}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ where $\Phi > 0$ is a constant and $2\phi, \phi_1+\phi_2 \neq 6$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_6(w) - E_s(w),z\right) \geq \begin{cases} \mathscr{N}'\left(\Gamma_{6J}\ \Phi|w|^{\phi}, \frac{6!|2^6-2^{2\phi}|}{65}\right); \\ \mathscr{N}'\left(\Gamma_{6J1}\ \Phi|w|^{\phi_1+\phi_2}, \frac{6!|2^6-2^{\phi_1+\phi_2}|\ z}{65}\right); \end{cases} \tag{4.81}$$

where $\Gamma_g's$ are defined in (3.79) respectively for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Theorem 4.15. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}(w,v),z\right) \geq \mathscr{N}'\left(\mathscr{S}(w,v),z\right) \tag{4.82}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and a one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned} &\mathscr{N}\left(E(w) - Q_5(w) - Q_6(w),z\right) \\ &\geq \min\left\{\mathscr{N}'\left(\mathscr{S}_5(w,w),\ \frac{5!|2^5-\varepsilon|\ z}{33}\right),\right. \\ &\qquad \mathscr{N}'\left(\mathscr{S}_5(-w,-w),\ \frac{5!|2^5-\varepsilon|\ z}{33}\right), \\ &\qquad \mathscr{N}'\left(\mathscr{S}_6(w,w),\ \frac{6!|2^6-\varepsilon|\ z}{65}\right), \\ &\qquad \left.\mathscr{N}'\left(\mathscr{S}_6(-w,-w),\ \frac{6!|2^6-\varepsilon|\ z}{65}\right)\right\} \end{aligned} \tag{4.83}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. The mappings Q_5 , Q_6 are defined in (4.3), (4.50) for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow \mathscr{W}_3$ is a function fulfilling the conditions (4.4), (4.51) for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ with the conditions that (4.5), (4.52) for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ for some $\varepsilon > 0$ with $0 < \left(\frac{\varepsilon}{2^5}\right)^{\alpha} < 1$, $0 < \left(\frac{\varepsilon}{2^6}\right)^{\alpha} < 1$. The functions $\mathscr{S}_5\Big(w,w\Big)$, $\mathscr{S}_6\Big(w,w\Big)$ are defined in (4.6), (4.53) for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. We know that by definition of odd function, we have

$$E_O(w) = \frac{E_q(w) - E_q(-w)}{2} \tag{4.84}$$

for all $w \in \mathscr{W}_1$. It follows from (4.84) that

$$\begin{aligned} &\mathscr{N}\left(E_O(u,v),z\right) \\ &= \mathscr{N}\left(\frac{1}{2}\Big\{E_q(w,v) - E_q(-w,-v)\Big\},z\right) \\ &= \mathscr{N}\left(\Big\{E_q(w,v) - E_q(-w,-v)\Big\},2z\right) \\ &\geq \min\Big\{\mathscr{N}\left(E_q(w,v),z\right), \mathscr{N}\left(E_q(-w,-v),z\right)\Big\} \end{aligned} \tag{4.85}$$

for all $w \in \mathscr{W}_1$. Thus by Theorem 4.7, we arrive

$$\begin{aligned} &\mathscr{N}\left(E_s(w) - Q_5(w),z\right) \\ &\geq \min\left\{\mathscr{N}'\left(\mathscr{S}_5(w,w),\ \frac{5!|2^5-\varepsilon|\ z}{33}\right),\right. \\ &\qquad \left.\mathscr{N}'\left(\mathscr{S}_5(-w,-w),\ \frac{5!|2^5-\varepsilon|\ z}{33}\right)\right\} \end{aligned} \tag{4.86}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

We know that by definition of even function, we have

$$E_E(w) = \frac{E_q(w) + E_q(-w)}{2} \tag{4.87}$$

for all $w \in \mathscr{W}_1$. It follows from (4.87) that

$$\begin{aligned} &\mathscr{N}\left(E_E(u,v),z\right) \\ &= \mathscr{N}\left(\frac{1}{2}\Big\{E_s(w,v) + E_s(-w,-v)\Big\},z\right) \\ &= \mathscr{N}\left(\Big\{E_s(w,v) + E_s(-w,-v)\Big\},2z\right) \\ &\geq \min\Big\{\mathscr{N}\left(E_s(w,v),z\right), \mathscr{N}\left(E_s(-w,-v),z\right)\Big\} \end{aligned} \tag{4.88}$$

for all $w \in \mathscr{W}_1$. Thus by Theorem 4.11, we arrive

$$\begin{aligned} &\mathscr{N}\left(E_s(w) - Q_6(w),z\right) \\ &\geq \min\left\{\mathscr{N}'\left(\mathscr{S}_6(w,w),\ \frac{6!|2^6-\varepsilon|\ z}{65}\right),\right. \\ &\qquad \left.\mathscr{N}'\left(\mathscr{S}_6(-w,-w),\ \frac{6!|2^6-\varepsilon|\ z}{65}\right)\right\} \end{aligned} \tag{4.89}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now, define a function

$$E(w) = E_O(w) + E_E(w) \tag{4.90}$$

for all $w \in \mathscr{W}_1$. Combining (4.86), (4.89) and (4.90), we reach our result. □

The following corollary is an immediate consequence of Theorem 4.15 regarding the Ulam - Hyers stability [23] of the functional equation (1.3).

Corollary 4.16. *For a mapping $\mathscr{E}:\mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}(w,v),z\right) \geq \mathscr{N}'\left(\Phi,z\right) \tag{4.91}$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ where $\Phi>0$ is a constant. Then there exists one and only quintic function $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ and a one and only sextic function $Q_6:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\mathscr{N}\left(E(w)-Q_5(w)-Q_6(w),z\right)\\&\geq \min\left\{\mathscr{N}'\left(\Phi,\frac{5!|31|z}{33}\right),\mathscr{N}'\left(\Phi,\frac{6!|63|z}{65}\right)\right\}\end{aligned} \tag{4.92}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 4.15 regarding the Ulam - Hyers - THRassias stability [38] of the functional equation (1.3).

Corollary 4.17. *For a mapping $\mathscr{E}:\mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}(w,v),z\right) \geq \left\{\begin{array}{l}\mathscr{N}'\left(\Phi\left\{|w|^{\phi}+|v|^{\phi}\right\},z\right);\\ \mathscr{N}'\left(\Phi\left\{|w|^{\phi_1}+|v|^{\phi_2}\right\},z\right);\end{array}\right. \tag{4.93}$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ where $\Phi>0$ is a constant and $\phi,\phi_1,\phi_2\neq 6$. Then there exists one and only quintic function $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ and a one and only sextic function $Q_6:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\mathscr{N}\left(E(w)-Q_5(w)-Q_6(w),z\right)\\&\geq\left\{\begin{array}{l}\min\left\{\mathscr{N}'\left(\Gamma_{5T}\ \Phi|w|^{\phi},\frac{5!|2^5-2^{\phi}|}{33}\right),\right.\\ \qquad\left.\mathscr{N}'\left(\Gamma_{6T}\ \Phi|w|^{\phi},\frac{6!|2^6-2^{\phi}|}{65}\right)\right\};\\ \min\left\{\mathscr{N}'\left(\Gamma_{5T1}\ \Phi|w|^{\phi_1}+\Gamma_{5T2}\ \Phi|w|^{\phi_2},\frac{5!|2^5-2^{\phi}|}{33}\right),\right.\\ \qquad\left.\mathscr{N}'\left(\Gamma_{6T1}\Phi|w|^{\phi_1}+\Gamma_{6T2}\Phi|w|^{\phi_2},\frac{6!|2^6-2^{\phi}|}{65}\right)\right\};\end{array}\right.\end{aligned} \tag{4.94}$$

where $\Gamma_g's$ are defined in (3.40) and (3.76) respectively for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

The following corollary is an immediate consequence of Theorem 4.15 regarding the Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.18. *For a mapping $\mathscr{E}:\mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\begin{aligned}&\mathscr{N}\left(\mathscr{E}(w,v),z\right)\\&\geq\left\{\begin{array}{l}\mathscr{N}'\left(\Phi\left\{|w|^{\phi}|v|^{\phi}+\left[|w|^{2\phi}+|v|^{2\phi}\right]\right\},z\right);\\ \mathscr{N}'\left(\Phi\left\{|w|^{\phi_1}|v|^{\phi_2}+\left[|w|^{\phi_1+\phi_2}+|v|^{\phi_1+\phi_2}\right]\right\},z\right);\end{array}\right.\end{aligned} \tag{4.95}$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$ where $\Phi>0$ is a constant and $2\phi,\phi_1+\phi_2\neq 6$. Then there exists one and only quintic function $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ and a one and only sextic function $Q_6:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\mathscr{N}\left(E(w)-Q_5(w)-Q_6(w),z\right)\\&\geq\left\{\begin{array}{l}\min\left\{\mathscr{N}'\left(\Gamma_{5J}\ \Phi|w|^{\phi},\frac{5!|2^5-2^{2\phi}|}{33}\right),\right.\\ \qquad\left.\mathscr{N}'\left(\Gamma_{6J}\ \Phi|w|^{\phi},\frac{6!|2^6-2^{2\phi}|}{65}\right)\right\};\\ \min\left\{\mathscr{N}'\left(\Gamma_{5J1}\ \Phi|w|^{\phi_1+\phi_2},\frac{5!|2^5-2^{\phi_1+\phi_2}|\ z}{33}\right),\right.\\ \qquad\left.\mathscr{N}'\left(\Gamma_{6J1}\ \Phi|w|^{\phi_1+\phi_2},\frac{6!|2^6-2^{\phi_1+\phi_2}|\ z}{65}\right)\right\};\end{array}\right.\end{aligned} \tag{4.96}$$

where $\Gamma_g's$ are defined in (3.43) and (3.79) respectively for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

4.3 Alternative Fixed Point Method

Theorem 4.19. *For an odd mapping $\mathscr{E}_q:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ fulfilling the functional inequality (4.1) for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$, where $\mathscr{S}:\mathscr{W}_1^2\longrightarrow[0,\infty)$ is a function fulfilling the condition*

$$\lim_{\beta\to\infty}\mathscr{N}'\left(\mathscr{S}\left(\xi_\psi^\beta w,\xi_\psi^\beta v\right),\xi_\psi^{5\beta}z\right)=1 \tag{4.97}$$

for all $w,v\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. Then there exists one and only quintic function $Q_5:\mathscr{W}_1\longrightarrow\mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_5(w)-E_q(w),z\right)\geq\mathscr{N}'\left(\mathscr{S}_5(w,w),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \tag{4.98}$$

for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$. If $\mathscr{L}=\mathscr{L}(\psi)$, with the property

$$\mathscr{N}'\left(\frac{1}{\xi_\psi^5}\mathscr{S}_5\left(\xi_\psi w,\xi_\psi w\right),z\right)=N'\left(L\mathscr{S}_5\left(w,w\right),z\right) \tag{4.99}$$

with the condition that

$$\mathscr{S}_5\left(w,w\right)=\frac{33}{5!}\ \mathscr{S}_5\left(\frac{w}{2},\frac{w}{2}\right), \tag{4.100}$$

where $\mathscr{S}_5\left(w,w\right)$ is defined in (4.24) for all $w\in\mathscr{W}_1$ and all $z\in\mathscr{W}_1$.

Proof. Consider the set

$$\mathscr{B}=\left\{E_1|E_1:\mathscr{W}_1\longrightarrow\mathscr{W}_2,E_1(0)=0\right\} \tag{4.101}$$

Let us introduce the generalized metric on (4.101) by

$$\inf\left\{\zeta:\mathscr{N}\left(E_1(w)-E_2(w),z\right)\geq\mathscr{N}'\left(\mathscr{S}_5\left(w,w\right),\zeta\ z\right)\right\} \tag{4.102}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. One can easy to see verify that (4.102) is complete with respect to the defined metric. Define a mapping $\Upsilon : \mathscr{B} \to \mathscr{B}$ by

$$\Upsilon E_q(w) = \frac{1}{\xi_\psi^5} E_q(\xi_\psi) \tag{4.103}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now, for any $E_1, E_2 \in \mathscr{B}$, we arrive

$$\mathscr{N}\left(E_1(w) - E_2(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5\left(w,w\right), \zeta\, z\right)$$
$$\mathscr{N}\left(\frac{1}{\xi_\psi^5}E_1(\xi_\psi) - \frac{1}{\xi_\psi^5}E_2(\xi_\psi), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5\left(w,w\right), \zeta\, \xi_\psi^5\, z\right)$$
$$\mathscr{N}\left(\Upsilon E_1(w) - \Upsilon E_2(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5\left(w,w\right), \mathscr{L}\zeta\, z\right)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. This implies that Υ is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. It follows from (4.26) that

$$\mathscr{N}\left(\frac{E_q(2w)}{2^5} - E_q(w), \frac{33z}{2^5 \cdot 5!}\right) \geq \mathscr{N}'\left(\mathscr{S}_5(w,w), z\right) \tag{4.104}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

For the case $\psi = 0$, it follows from (4.99), (4.100), (4.103), (FNS3) and (4.104),

$$\mathscr{N}\left(\Upsilon E_q(w) - E_q(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5(w,w), \mathscr{L}z\right) = \mathscr{N}'\left(\mathscr{S}_5(w,w), \mathscr{L}^{1-0}z\right) \tag{4.105}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Replacing w by $\frac{w}{2}$ in (4.104), we obtain

$$\mathscr{N}\left(E_q(w) - 2^5 E_q\left(\frac{w}{2}\right), \frac{33z}{5!}\right) \geq \mathscr{N}'\left(\mathscr{S}_5\left(\frac{w}{2}, \frac{w}{2}\right), z\right) \tag{4.106}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

For the case $\psi = 1$, it follows from (4.99), (4.100), (4.103), (FNS3) and (4.106),

$$\mathscr{N}\left(E_q(w) - \Upsilon E_q(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5(w,w), 1 \cdot z\right) = \mathscr{N}'\left(\mathscr{S}_5(w,w), \mathscr{L}^{1-1}z\right) \tag{4.107}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.105) and (4.107), we see that

$$\mathscr{N}\left(E_q(w) - \Upsilon E_q(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_5(w,w), \mathscr{L}^{1-\psi}z\right) \tag{4.108}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Thus, condition (FPC1) of Theorem 1.1 holds. It follows from condition (FPC2) of Theorem 1.1 , that there exists a fixed point Q_5 of Υ in $\mathscr{L}$ such that

$$\lim_{\beta \to \infty} \mathscr{N}\left(Q_5(w) - \frac{1}{\xi_\psi^5} E_q\left(\xi_\psi w\right), z\right) = 1 \tag{4.109}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. To prove the existing Q_5 satisfies (1.3), the proof is similar to that of Theorem 4.7.

Again by condition (FPC3) of Theorem 1.1 , Q_5 is the unique fixed point of Υ in the set

$$\Delta = \left\{Q_5 : \mathscr{N}\left(E_q(w) - Q_5(w), z\right) \geq \infty\right\} \tag{4.110}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Finally, by condition (FPC4) of Theorem 1.1 , we get

$$\mathscr{N}\left(E_q(w) - Q_5(w), z\right) \geq \mathscr{N}\left(E_q(w) - \Upsilon E_q(w), \left(\frac{1}{1-\mathscr{L}}\right) z\right) = \mathscr{N}'\left(\mathscr{S}_5(w,w), \left(\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}\right) z\right)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.19 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.20. *For an odd mapping $\mathscr{E}_q : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}_q(w,v), z\right) \geq \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \tag{4.111}$$

for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_5(w) - E_q(w), z\right) \geq \begin{cases} \mathscr{N}'\left(\Phi, \frac{33}{5!|31|}\, z\right); \\ \mathscr{N}'\left(\Gamma_{5T}\, \Phi||w||^\phi, \frac{33}{5!|2^\phi - 2^5|} z\right); & \phi \neq 5 \\ \mathscr{N}'\left(\Gamma_{5J}\, \Phi||w||^{2\phi}, \frac{33}{5!|2^{2\phi} - 2^5|} z\right); & 2\phi \neq 5 \end{cases} \tag{4.112}$$

where $\Gamma_g's$ are defined in (3.40) and (3.43) respectively for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. If we take

$$\mathscr{N}'\left(\mathscr{S}(w,v), z\right) = \begin{cases} \mathscr{N}'\left(\Phi, z\right); \\ \mathscr{N}'\left(\Phi\left\{||w||^\phi + ||v||^\phi\right\}, z\right); \\ \mathscr{N}'\left(\Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}, z\right); \end{cases}$$

(4.113)

for all $w, v \in \mathscr{W}_1$. Changing (w, v) by $(\xi_\psi^\beta w, \xi_\psi^\beta v)$ and dividing by $\xi_\psi^{5\beta}$ in (4.113) and letting β tends to ∞, we see that (4.97) holds for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

It follows from (4.100), (4.113) and (4.24), one can find that

$$\mathscr{S}\Big(w,w\Big) = \frac{33}{5!}\,\mathscr{S}\Big(\frac{w}{2},\frac{w}{2}\Big) = \begin{cases} \frac{33\Phi}{5!}; \\ \frac{33\left(11\cdot 2^\phi+5\cdot 3^\phi+4^\phi+43\right)\Phi||w||^\phi}{5!2^\phi}; \\ \frac{33\left(10\cdot 2^\phi+11\cdot 2^{2\phi}+5(3^\phi+3^{2\phi})+4^\phi+4^{2\phi}+54\right)\Phi||w||^{2\phi}}{5!2^{2\phi}}; \end{cases} \quad (4.114)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Again, it follows from (4.99), (4.113) and (4.24), one can observe that

$$\mathscr{N}'\left(\frac{1}{\xi_\psi^5}\mathscr{S}_5\Big(\xi_\psi w, \xi_\psi w\Big), z\right) = \begin{cases} \mathscr{N}'\Big(\mathscr{S}_5\Big(w,w\Big),\mathscr{L}z\Big); \\ \mathscr{N}'\Big(\mathscr{S}_5\Big(w,w\Big),\mathscr{L}z\Big); \\ \mathscr{N}'\Big(\mathscr{S}_5\Big(w,w\Big),\mathscr{L}z\Big); \end{cases} \quad (4.115)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Thus, the functional inequality (4.98) holds for the following cases.

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^5 = 2^5$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^5\right)^{1-1}}{1-2^5}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{1}{-31}\,z\right)\end{aligned}$$

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^5} = \frac{1}{2^5} = 2^{-5}$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^{-5}\right)^{1-0}}{1-2^{-5}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{1}{31}z\right)\end{aligned}$$

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^{5-\phi} = 2^{5-\phi} : \phi > 5$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^{5-\phi}\right)^{1-1}}{1-2^{5-\phi}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{2^\phi}{2^\phi-2^5}z\right)\end{aligned}$$

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^{5-\phi}} = \frac{1}{2^{5-\phi}} = 2^{\phi-5} : \phi < 5$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^{\phi-5}\right)^{1-0}}{1-2^{\phi-5}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{2^\phi}{2^5-2^\phi}z\right)\end{aligned}$$

$$For \quad \psi = 1 : \mathscr{L} = \xi_\psi^{5-2\phi} = 2^{5-2\phi} : 2\phi > 5$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^{5-2\phi}\right)^{1-1}}{1-2^{5-2\phi}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{2^{2\phi}}{2^{2\phi}-2^5}z\right)\end{aligned}$$

$$For \quad \psi = 0 : \mathscr{L} = \frac{1}{\xi_\psi^{5-2\phi}} = \frac{1}{2^{5-2\phi}} = 2^{2\phi-5} : 2\phi < 5$$

$$\begin{aligned}\mathscr{N}\left(Q_5(w)-E_q(w),z\right) &\geq \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{\left(2^{2\phi-5}\right)^{1-0}}{1-2^{2\phi-5}}z\right) \\ &= \mathscr{N}'\left(\mathscr{S}_5\Big(w,w\Big),\frac{2^{2\phi}}{2^5-2^{2\phi}}z\right)\end{aligned}$$

□

Theorem 4.21. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality (4.1) for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0, \infty)$ is a function fulfilling the condition*

$$\lim_{\beta\to\infty} \mathscr{N}'\left(\mathscr{S}\Big(\xi_\psi^\beta w, \xi_\psi^\beta v\Big), \xi_\psi^{6\beta} z\right) = 1 \quad (4.116)$$

for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_6(w) - E_s(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6(w,w), \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}} z\right) \quad (4.117)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. If $\mathscr{L} = \mathscr{L}(\psi)$, with the property

$$\mathscr{N}'\left(\frac{1}{\xi_\psi^6}\mathscr{S}_6\Big(\xi_\psi w, \xi_\psi w\Big), z\right) = N'\left(L\mathscr{S}_6\Big(w,w\Big), z\right) \quad (4.118)$$

with the condition that

$$\mathscr{S}_6\Big(w,w\Big) = \frac{65}{6!}\,\mathscr{S}_6\Big(\frac{w}{2},\frac{w}{2}\Big), \quad (4.119)$$

where $\mathscr{S}_6\Big(w,w\Big)$ is defined in (4.24) for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. Consider the set

$$\mathscr{B} = \Big\{E_2 | E_2 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2, E_2(0) = 0\Big\} \quad (4.120)$$

Let us introduce the generalized metric on (4.120) by

$$\inf\Big\{\zeta : \mathscr{N}\left(E_1(w) - E_2(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6\Big(w,w\Big), \zeta\, z\right)\Big\} \quad (4.121)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. One can easy to see verify that (4.121) is complete with respect to the defined metric. Define a mapping $\Upsilon : \mathscr{B} \to \mathscr{B}$ by

$$\Upsilon E_s(w) = \frac{1}{\xi_\psi^6} E_s(\xi_\psi) \quad (4.122)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Now, for any $E_1, E_2 \in \mathscr{B}$, we arrive

$$\mathscr{N}\left(E_1(w) - E_2(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6\Big(w,w\Big), \zeta\, z\right)$$

$$\mathscr{N}\left(\frac{1}{\xi_\psi^6}E_1(\xi_\psi) - \frac{1}{\xi_\psi^6}E_2(\xi_\psi), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6\Big(w,w\Big), \zeta\, \xi_\psi^5\, z\right)$$

$$\mathscr{N}\left(\Upsilon E_1(w) - \Upsilon E_2(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6\Big(w,w\Big), \mathscr{L}\zeta\, z\right)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. This implies that Υ is a strictly contractive mapping on $\mathscr{B}$ with Lipschitz constant $\mathscr{L}$. It follows from (4.26) that

$$\mathscr{N}\left(\frac{E_s(2w)}{2^6} - E_s(w), \frac{65z}{2^6 \cdot 6!}\right) \geq \mathscr{N}'\left(\mathscr{S}_6(w,w), z\right) \quad (4.123)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

For the case $\psi = 0$, it follows from (4.118), (4.119), (4.122), (FNS3) and (4.123),

$$\mathscr{N}\left(\Upsilon E_s(w) - E_s(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6(w,w), \mathscr{L}z\right) = \mathscr{N}'\left(\mathscr{S}_6(w,w), \mathscr{L}^{1-0}z\right) \quad (4.124)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Replacing w by $\frac{w}{2}$ in (4.123), we obtain

$$\mathscr{N}\left(E_s(w) - 2^6 E_s\left(\frac{w}{2}\right), \frac{65z}{6!}\right) \geq \mathscr{N}'\left(\mathscr{S}_6\Big(\frac{w}{2},\frac{w}{2}\Big), z\right) \quad (4.125)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

For the case $\psi = 1$, it follows from (4.118), (4.119), (4.122), (FNS3) and (4.125),

$$\mathscr{N}\left(E_s(w) - \Upsilon E_s(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6(w,w), 1 \cdot z\right) = \mathscr{N}'\left(\mathscr{S}_6(w,w), \mathscr{L}^{1-1}z\right) \quad (4.126)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. From (4.124) and (4.126), we see that

$$\mathscr{N}\left(E_s(w) - \Upsilon E_s(w), z\right) \geq \mathscr{N}'\left(\mathscr{S}_6(w,w), \mathscr{L}^{1-\psi}z\right) \quad (4.127)$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Thus, condition (FPC1) of Theorem 1.1 holds. The rest of the proof is similar lines to that of Theorem 4.19. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.21 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.22. *For an even mapping $\mathscr{E}_s : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}_s(w,v), z\right) \geq \begin{cases} \Phi; \\ \Phi\left\{||w||^\phi + ||v||^\phi\right\}; \\ \Phi\left\{||w||^\phi ||v||^\phi + \left[||w||^{2\phi} + ||v||^{2\phi}\right]\right\}; \end{cases} \quad (4.128)$$

for all $w, v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\mathscr{N}\left(Q_6(w) - E_s(w), z\right) \geq \begin{cases} \mathscr{N}'\left(\Phi, \frac{65}{6!|63|}\, z\right); \\ \mathscr{N}'\left(\Gamma_{6T}\, \Phi ||w||^\phi, \frac{65}{6!|2^\phi - 2^6|} z\right); & \phi \neq 6 \\ \mathscr{N}'\left(\Gamma_{6J}\, \Phi ||w||^{2\phi}, \frac{65}{6!|2^{2\phi} - 2^6|} z\right); & 2\phi \neq 6 \end{cases} \quad (4.129)$$

where $\Gamma_g's$ are defined in (3.76) and (3.79) respectively for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. The proof of the corollary is similar clues and ideas of Corollary 4.20. Hence the details of the proof are omitted. □

Theorem 4.23. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality (4.82) for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$, where $\mathscr{S} : \mathscr{W}_1^2 \longrightarrow [0,\infty)$ is a function fulfilling the condition (4.97), (4.116) for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and a one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality*

$$\begin{aligned}&\mathscr{N}\left(E(w)-Q_5(w)-Q_6(w),z\right)\\ &\geq \min\left\{\mathscr{N}'\left(\mathscr{S}_5(w,w),\ \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right),\right.\\ &\qquad \mathscr{N}'\left(\mathscr{S}_5(-w,-w),\ \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right),\\ &\qquad \mathscr{N}'\left(\mathscr{S}_6(w,w),\ \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right),\\ &\qquad \left.\mathscr{N}'\left(\mathscr{S}_6(-w,-w),\ \frac{\mathscr{L}^{1-\psi}}{1-\mathscr{L}}z\right)\right\}\end{aligned} \tag{4.130}$$

for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$. If $\mathscr{L} = \mathscr{L}(\psi)$, with the properties and conditions (4.99), (4.118) (4.100), (4.119) where $\mathscr{S}_5\Big(w,w\Big)$, $\mathscr{S}_6\Big(w,w\Big)$ are defined in (4.24), (4.71)for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

Proof. The proof of the the proof is similar lines to that of Theorem 4.15. □

The following corollary is an immediate consequence of Theorem 4.23 regarding the Ulam - Hyers stability [23], Ulam - Hyers - THRassias stability [38] and Ulam - Hyers - JMRassias stability [42] of the functional equation (1.3).

Corollary 4.24. *For a mapping $\mathscr{E} : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ fulfilling the functional inequality*

$$\mathscr{N}\left(\mathscr{E}_s(w,v),z\right) \geq \begin{cases} \Phi; \\ \Phi\left\{||w||^{\phi}+||v||^{\phi}\right\}; \\ \Phi\left\{||w||^{\phi}||v||^{\phi}+\left[||w||^{2\phi}+||v||^{2\phi}\right]\right\}; \end{cases} \tag{4.131}$$

for all $w,v \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$ where $\Phi > 0$ and ϕ is a constant. Then there exists one and only quintic function $Q_5 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ and a one and only sextic function $Q_6 : \mathscr{W}_1 \longrightarrow \mathscr{W}_2$ which satisfying the functional equation (1.3) and the functional inequality

$$\begin{aligned}&\mathscr{N}\left(E(w)-Q_5(w)-Q_6(w),z\right)\\ &\geq \begin{cases} \min\left\{\mathscr{N}'\left(\Phi,\frac{33}{5!|31|}\,z\right),\mathscr{N}'\left(\Phi,\frac{65}{6!|63|}\,z\right)\right\}; \\ \min\left\{\mathscr{N}'\left(\Gamma_{5T}\,\Phi||w||^{\phi},\frac{33}{5!|2^{\phi}-2^5|}z\right),\right. \\ \qquad \left.\mathscr{N}'\left(\Gamma_{6T}\Phi||w||^{\phi},\frac{65}{6!|2^{\phi}-2^6|}z\right)\right\}; & \phi \neq 5,6 \\ \min\left\{\mathscr{N}'\left(\Gamma_{5J}\,\Phi||w||^{2\phi},\frac{33}{5!|2^{2\phi}-2^5|}z\right),\right. \\ \qquad \left.\mathscr{N}'\left(\Gamma_{6J}\,\Phi||w||^{2\phi},\frac{65}{6!|2^{2\phi}-2^6|}z\right)\right\}; & 2\phi \neq 6 \end{cases}\end{aligned} \tag{4.132}$$

where $\Gamma_g's$ are defined in (3.40), (3.76), (3.43) and (3.79) respectively for all $w \in \mathscr{W}_1$ and all $z \in \mathscr{W}_1$.

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